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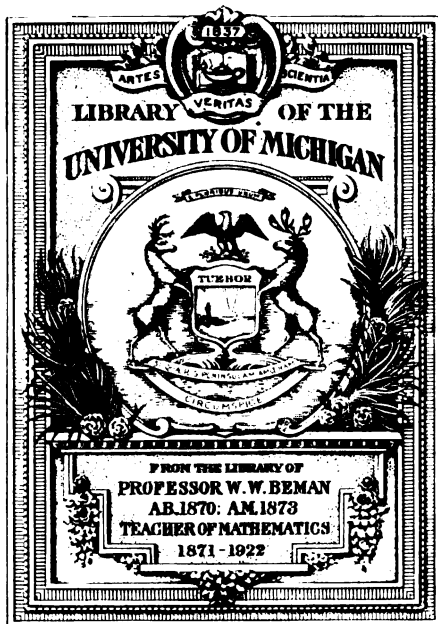
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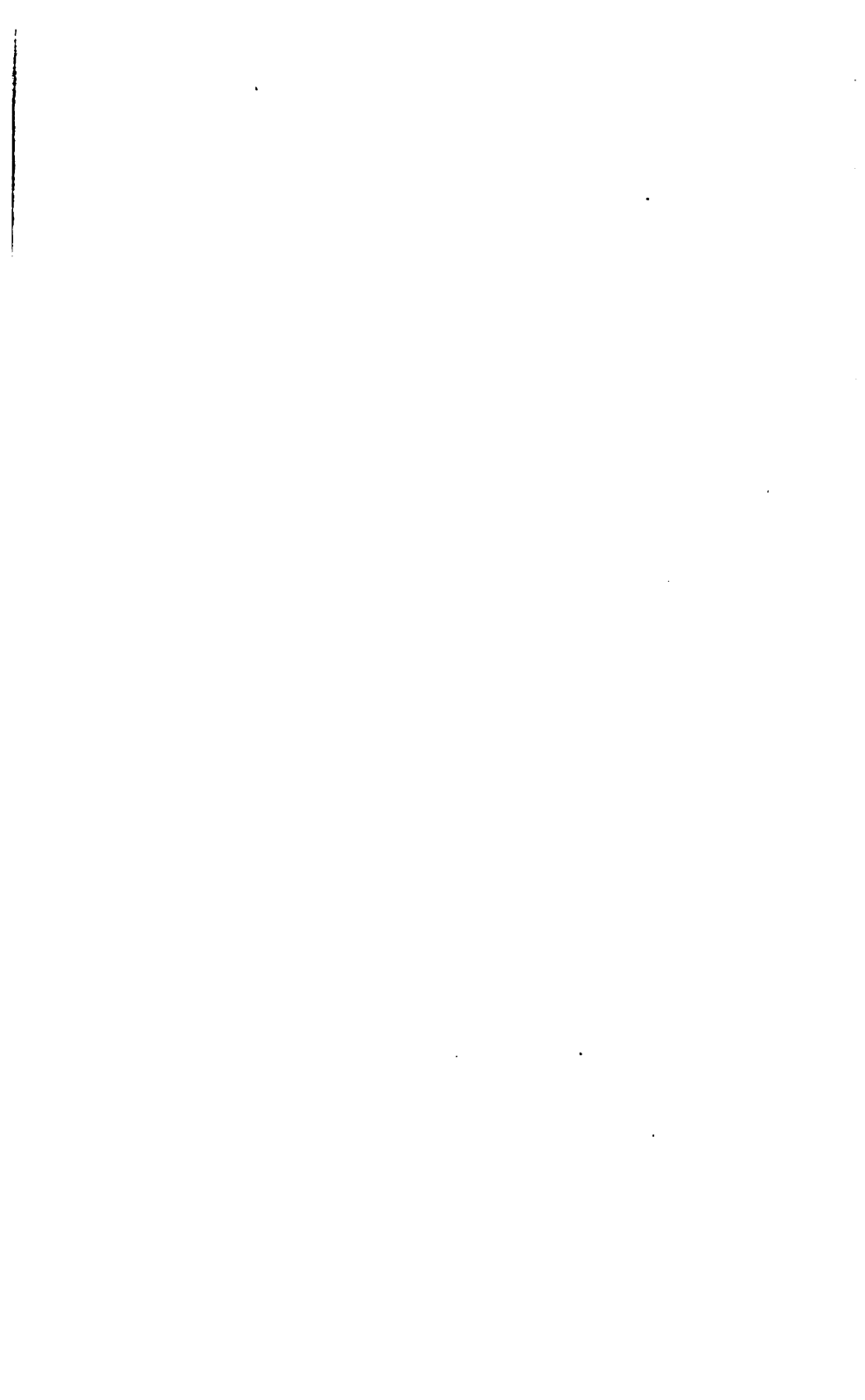
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MATHEMATICS

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GEOMETRICAL PROBLEMS

DEDUCIBLE
FROM THE FIRST SIX BOOKS OF EUCLID,
ARRANGED AND SOLVED:

TO WHICH IS ADDED
AN APPENDIX
CONTAINING THE ELEMENTS OF
Plane Trigonometry.

FOR THE USE OF THE YOUNGER STUDENTS.



By MILES BLAND, B.D.

FELLOW AND TUTOR OF ST. JOHN'S COLLEGE, CAMBRIDGE.

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1821

W. W. Beman

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ADVERTISEMENT.

THE following pages contain a collection of Problems, which are for the most part an easy application of the Elements of Euclid. They are arranged in what seemed to be the most natural order : The 1st section comprises such as contain the properties of straight lines and angles; the 2nd straight lines and circles : the 3rd straight lines and triangles; and the 4th parallelograms, squares and polygons. The 5th section contains those which require lines to be drawn in certain directions, but which involve properties of rectangles or squares, or such others as were excluded from the three first. The 6th comprises those by which figures are described, and also inscribed in or circumscribed about each other. The 7th comprehends such as contain the properties of triangles described in or about circles; the 8th those which contain the squares or rectangles of lines connected with circles; and the 9th the construction of triangles. To these is added an Appendix, intended to contain so much of the Elements of Plane Trigonometry, as is necessary for understanding those parts of Natural Philosophy which are the common subjects of Lectures in the Uni-

versity. The Reader who wishes for farther information, is referred to Mr. Woodhouse's treatise, or that of Cagnoli, to the latter of which are appended extensive Tables of trigonometrical formulæ.

From this performance the only credit expected is that of having endeavoured to place principles in a clear light, and to render a service to the younger students by setting before them a series of Problems, on the solution of which they are recommended to exercise their own ingenuity; for which purpose a table of Contents has been prefixed.



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SECTION I. Page 1.

1. **FROM** a given point to draw the shortest line possible to a given straight line.
2. If a perpendicular be drawn bisecting a given straight line, any point in this perpendicular is at equal distances, and any point without the perpendicular is at unequal distances, from the extremities of the line.
3. Through a given point, to draw a straight line which shall make equal angles with two straight lines given in position.
4. From two given points, to draw two equal straight lines, which shall meet in the same point of a line given in position.
5. From two given points on the same side of a line given in position to draw two lines which shall meet in that line, and make equal angles with it.
6. From two given points on the same side of a line given in position, to draw two lines which shall meet in a point in this line, so that their sum shall be less than the sum of any two lines drawn from the same points and terminated at any other point in the same line.
7. Of all straight lines which can be drawn from a given point to an indefinite straight line, that which is nearer to the perpendicular is less than the more remote. And from the same point there cannot be drawn more than two straight lines equal to each other, viz. one on each side of the perpendicular.
8. Through a given point, to draw a straight line so that the parts of it intercepted between that point and perpendiculars drawn from two other given points may have a given ratio.

9. From a given point between two indefinite straight lines given in position, to draw a line which shall be terminated by the given lines, and bisected in the given point.

10. From a given point without two indefinite straight lines given in position, to draw a line such that the parts intercepted by the point and the lines may have a given ratio.

11. From a given point, to draw a straight line which shall cut off from lines containing a given angle, segments that shall have a given ratio.

12. If from a given point any number of straight lines be drawn to a straight line given in position; to determine the locus of the points of section, which divide them in a given ratio.

13. A straight line being drawn parallel to one of the lines containing a given angle, and produced to meet the other; through a given point within the angle to draw a line cutting the other three, so that the part intercepted between the two parallel lines may have a given ratio to the part intercepted between the given point and the other line.

14. Two parallel lines being given in position; to draw a third such, that if from any point in it lines be drawn at given angles to the parallel lines, the intercepted parts may have a given ratio.

15. If three straight lines drawn from the same point and in the same direction be in continued proportion, and from that point also a line equal to the mean proportional be inclined at any angle; the lines joining the extremity of this line and of the proportionals will contain equal angles.

16. To trisect a right angle.

17. To trisect a given finite straight line.

18. To divide a given straight line into any number of equal parts.

COR. To divide a straight line into any number of parts having a given ratio.

19. To divide a given finite straight line harmonically.

20. If a given finite straight line be harmonically divided, and from its extremities and the points of division lines be drawn to meet

in any point, so that those from the extremities of the second proportional may be perpendicular to each other; the line drawn from the extremity of this proportional will bisect the angle formed by the lines drawn from the extremities of the other two.

21. If a straight line be drawn through any point in the line bisecting a given angle, and produced to cut the sides containing that angle, as also a line drawn from the angle perpendicular to the bisecting line; it will be harmonically divided.

22. If from a given point there be drawn three straight lines forming angles less than right angles, and from another given point without them a line be drawn intersecting the others, so as to be harmonically divided; then will all lines drawn from that point meeting the three lines be harmonically divided.

23. If a straight line be divided into two equal and also into two unequal parts, and be produced, so that the part produced may have to the whole line so produced, the same ratio that the unequal segments of the line have to each other; then shall the distances of the point of unequal section from one extremity of the given line, from its middle point, from the extremity of the part produced, and from the other extremity of the given line, be proportionals.

24. Three points being given; to determine another, through which if any straight line be drawn, perpendiculars upon it from two of the former shall together be equal to the perpendicular from the third.

25. From a given point in one of two straight lines given in position, to draw a line to cut the other, so that if from the point of intersection a perpendicular be let fall upon the former, the segment intercepted between it and the given point together with the first drawn line may be equal to a given line.

26. One of the lines which contain a given angle is also given. To determine a point in it such that if from thence to the indefinite line there be drawn a line having a given ratio to that segment of it which is adjacent to the given angle; the line so drawn and the other segment of the given line may together be equal to another given line.

27. Two straight lines and a point in each are given in position; to determine the position of another point in each, so that the straight

line joining these latter points may be equal to a given line, and their respective distances from the former points in a given ratio.

28. If a straight line be divided into any two parts and produced, so that the segments may have the same ratio that the whole line produced has to the part produced, and from the extremities of the given line perpendiculars be erected; then any line drawn through the point of section, meeting these perpendiculars, will be divided at that point into parts which have the same ratio, that those lines have which are drawn from the extremity of the produced line to the points of intersection with the perpendicular.

29. From two given points to draw two straight lines which shall contain a given angle, and meet two lines given in position, so that the parts intercepted between those points and the lines may have a given ratio.

30. The length of one of two lines which contain a given angle being given; to draw, from a given point without them, a straight line which shall cut the given line produced, so that the part produced may be in a given ratio to the part cut off from the indefinite line.

31. From two given straight lines to cut off two parts which may have a given ratio; so that the ratio of the remaining parts may also be equal to the ratio of two other given lines.

32. Three lines being given in position; to determine a point in one of them, from which if two lines be drawn at given angles to the other two, the two lines so drawn may together be equal to a given line.

33. If from a given point two straight lines be drawn including a given angle and having a given ratio, and one of them be always terminated by a straight line given in position; to determine the locus of the extremity of the other.

34. If from two given points straight lines be drawn containing a given angle, and from each of them segments be cut off having a given ratio; and the extremities of the segments of the lines drawn from one of the points be in a straight line given in position; to determine the locus of the extremities of the segments of lines drawn from the other.

SECTION II. Page 24.

1. IF a straight line be drawn to touch a circle, and be parallel to a chord; the point of contact will be the middle point of the arc cut off by that chord.

COR. 1. Parallel lines placed in a circle cut off equal parts of the circumference.

COR. 2. The two straight lines in a circle which join the extremities of two parallel chords are equal to each other.

2. If from a point without a circle two straight lines be drawn to the concave part of the circumference, making equal angles with the line joining the same point and the centre, the parts of these lines which are intercepted within the circle are equal.

3. Of all straight lines which can be drawn from two given points to meet on the convex circumference of a given circle; the sum of those two will be the least, which make equal angles with the tangent at the point of concurrence.

4. If a circle be described on the radius of another circle; any straight line drawn from the point where they meet to the outer circumference, is bisected by the interior one.

5. If two circles cut each other, and from either point of intersection diameters be drawn; the extremities of these diameters and the other point of intersection shall be in the same straight line.

6. If two circles cut each other, the straight line joining their two points of intersection is bisected at right angles by the straight line joining their centres.

7. To draw a straight line which shall touch two given circles.

8. If a line touching two circles cut another line joining their centres, the segments of the latter will be to each other, as the diameters of the circles.

9. If a straight line touch the interior of two concentric circles, and be placed in the outer; it will be bisected at the point of contact.

10. If any number of equal straight lines be placed in a circle; to determine the locus of their points of bisection.

11. If from a point in the circumference of a circle any number of chords be drawn; the locus of their points of bisection will be a circle.

12. If on the radius of a given semicircle, another semicircle be described, and from the extremity of the diameters any lines be drawn cutting the circumferences, and produced, so that the part produced may always have a given ratio to the part intercepted between the two circumferences; to determine the locus of the extremities of these lines.

13. If from a given point without a given circle straight lines be drawn and terminated by the circumference; to determine the locus of the points which divide them in a given ratio.

14. Having given the radius of a circle; to determine its centre when the circle touches two given lines which are not parallel.

15. Through three given points which are not in the same straight line, a circle may be described; but no other circle can pass through the same points.

16. From two given points on the same side of a line given in position, to draw two straight lines which shall contain a given angle, and be terminated in that line.

17. If from the extremities of any chord in a circle perpendiculars be drawn, meeting a diameter; the points of intersection are equally distant from the centre.

18. If from the extremities of the diameter of a semicircle perpendiculars be let fall on any line cutting the semicircle; the parts intercepted between those perpendiculars and the circumference are equal.

19. In a given circle to place a straight line parallel to a given straight line, and having a given ratio to it.

20. Through a given point, either within or without a given circle, to draw a straight line, the part of which intercepted by the circle shall be equal to a given line, not greater than the diameter of the circle.

21. From a given point in the diameter of a semicircle produced, to draw a line cutting the semicircle, so that lines drawn from the

points of intersection to the extremities of the diameter, cutting each other, may have a given ratio.

22. From the circumference of a given circle, to draw to a straight line given in position, a line which shall be equal and parallel to a given straight line.

23. The bases of two given circular segments being in the same straight line; to determine a point in it such, that a line being drawn through it making a given angle, the part intercepted between the circumferences of the circles may be equal to a given line.

24. If two chords of a given circle intersect each other, the angle of their inclination is equal to half the angle at the centre which stands on an arc equal to the sum or difference of the arcs intercepted between them, according as they meet within or without the circle.

25. If from a point without two circles which do not meet each other, two lines be drawn to their centres, which have the same ratio that their radii have; the angle contained by tangents drawn from that point towards the same parts will be equal to the angle contained by lines drawn to the centres.

26. To determine the Arithmetic, Geometric and Harmonic means between two given straight lines. ✓

27. If on each side of any point in a circle any number of equal arcs be taken, and the extremities of each pair joined: the sum of the chords so drawn will be equal to the last chord produced to meet a line drawn from the given point through the extremity of the first arc.

28. If the circumference of a semicircle be divided into an odd number of equal parts, and through the points which are equally distant from the diameter lines be drawn; the segments of these lines intercepted between radii drawn to the extremities of the most remote, will together be equal to a radius of the circle.

29. If from the extremities and the point of bisection of any arc of a circle, lines be drawn to any point in the opposite circumference; the sum of those drawn from the extremities will have to that from the point of bisection, the same ratio that the line joining the extremities has to that joining one of them and the point of bisection.

30. If two equal circles cut each other, and from either point of intersection a circle be described cutting them; the points where this circle cuts them, and the other point of intersection of the equal circles are in the same straight line.

31. If two equal circles cut each other, and from either point of intersection a line be drawn meeting the circumferences; the part of it intercepted between the circumferences will be bisected by the circle whose diameter is the common chord of the equal circles.

32. If two circles touch each other externally or internally; any straight line drawn through the point of contact will cut off similar segments.

33. If two circles touch each other externally or internally; two straight lines drawn through the point of contact will intercept arcs, the chords of which are parallel.

34. If two circles touch each other externally or internally; any two straight lines drawn through the point of contact, and terminated both ways by the circumference, will be cut proportionally by the circumference.

35. If two circles touch each other externally, and parallel diameters be drawn; the straight line joining the extremities of these diameters will pass through the point of contact.

36. If two circles touch each other and also touch a straight line; the part of the line between the points of contact is a mean proportional between the diameters of the circles.

37. If two circles touch each other externally, and the line joining their centres be produced to the circumferences; and from its middle point as a centre with any radius whatever a circle be described, and any line placed in it passing through the point of contact; the parts of the line intercepted between the circumference of this circle and each of the others will be equal.

38. If from the point of contact of two circles which touch each other internally, any number of lines be drawn; and through the points, where these intersect the circumferences, lines be drawn from any other point in each circumference, and produced to meet; the angles formed by these lines will be equal.

39. If two circles touch each other internally, and any two per-

pendiculars to their common diameter be produced to cut the circumferences; the lines joining the points of intersection and the point of contact are proportional.

40. If three circles, whose diameters are in continued proportion, touch each other internally, and from the extremity of the least diameter passing through the point of contact a perpendicular be drawn, meeting the circumferences of the other two circles; this diameter and the lines joining the points of intersection and contact are in continued proportion.

41. If a common tangent be drawn to any number of circles which touch each other internally, and from any point in this tangent as a centre, a circle be described cutting the others, and from this centre lines be drawn through the intersections of the circles respectively; the segments of them within each circle will be equal.

42. If from any point in the diameter of a circle produced, a tangent be drawn; a perpendicular from the point of contact to the diameter will divide it into segments which have the same ratio that the distances of the point without the circle from each extremity of the diameter, have to each other.

43. If from the extremity of the diameter of a given semicircle a straight line be drawn in it, equal to the radius, and from the centre a perpendicular let fall upon it and produced to the circumference; it will be a mean proportional between the lines drawn from the point of intersection with the circumference to the extremities of the diameter.

44. If from the extremity of the diameter of a circle, two lines be drawn, one of which cuts a perpendicular to the diameter, and the other is drawn to the point where the perpendicular meets the circumference; the latter of these lines is a mean proportional between the cutting line, and that part of it which is intercepted between the perpendicular and the extremity of the diameter.

45. In the diameter of a circle produced, to determine a point, from which a tangent drawn to the circumference, shall be equal to the diameter.

46. To determine a point in the perpendicular at the extremity of the diameter of a semicircle, from which if a line be drawn to the

other extremity of the diameter, the part without the circle may be equal to a given straight line.

47. Through a given point without a given circle, to draw a straight line to cut the circle, so that the two perpendiculars drawn from the points of intersection to that diameter which passes through the given point, may together be equal to a given line, not greater than the diameter of the circle.

48. If from each extremity of any number of equal adjacent arcs in the circumference of a circle, lines be drawn through two given points in the opposite circumference, and produced till they meet; the angles formed by these lines will be equal.

49. To determine a point in the circumference of a circle, from which lines drawn to two other given points, shall have a given ratio.

50. If any point be taken in the diameter of a circle, which is not the centre; of all the chords which can be drawn through that point, that is the least which is at right angles to the diameter.

51. If from any point without a circle lines be drawn touching it; the angle contained by the tangents is double the angle contained by the line joining the points of contact and the diameter drawn through one of them.

52. If from the extremities of the diameter of a circle tangents be drawn, and produced to intersect a tangent to any point of the circumference; the straight lines joining the points of intersection and the centre of the circle form a right angle.

53. If from the extremities of the diameter of a circle tangents be drawn; any other tangent to the circle, terminated by them, is so divided at the point of contact, that the radius of the circle is a mean proportional between its segments.

54. Two circles being given in magnitude and position; to find a point in the circumference of one of them, to which if a tangent be drawn cutting the circumference of the other, the part of it intercepted between the two circumferences may be equal to a given line.

55. To draw a straight line cutting two concentric circles, so that the part of it which is intercepted by the circumference of the greater may be double the part intercepted by the circumference of the less.

56. If two circles intersect each other, the centre of the one being in the circumference of the other, and any line be drawn from that centre; the parts of it which are cut off by the common chord and the two circumferences will be in continued proportion.

57. If a semicircle be described on the side of a quadrant, and from any point in the quadrantal arc a radius be drawn, the part of the radius intercepted between the quadrant and semicircle, is equal to the perpendicular let fall from the same point on their common tangent.

COR. Any chord of the semicircle drawn from the centre of the quadrant is equal to the perpendicular drawn to the other side, from the point in which the chord produced meets the quadrantal arc.

58. If a semicircle be described on the side of a quadrant, and a line be drawn from the centre of the quadrant to a common tangent; this line, the parts of it cut off by the circumferences of the quadrant, and of the semicircle, and the segment of the diameter of the semicircle made by a perpendicular from the point where the line meets its circumference, are in continued proportion.

59. If the chord of a quadrant be made the diameter of, a semicircle, and from its extremities two straight lines be drawn to any point in the circumference of the semicircle; the segment of the greater line intercepted between the two circumferences shall be equal to the less of the two lines.

60. If two circles cut each other, so that the circumference of one passes through the centre of the other, and from either point of intersection a straight line be drawn cutting both circumferences; the part intercepted between the two circumferences will be equal to the chord drawn from the other point of intersection to the point where it meets the inner circumference.

61. If from each extremity of the diameter of a circle lines be drawn to any two points in the circumference; the sums of the lines so drawn to each point will have to one another the same ratio that the lines have, which join those points and the opposite extremity of a diameter perpendicular to the former.

62. If from any two points in the circumference of a circle there be drawn two straight lines to a point in a tangent to that circle; they will make the greatest angle when drawn to the point of contact:

63. From a given point within a given circle to draw a straight line which shall make with the circumference an angle less than the angle made by any other line drawn from that point.

64. To determine a point in the arc of a quadrant, from which if lines be drawn to the centre and the point of bisection of the radius, they shall contain the greatest possible angle.

65. If the radius of a circle be a mean proportional to two distances from the centre in the same straight line; the lines drawn from their extremities to any point in the circumference will have the same ratio that the distances of these points from the circumference have.

66. Two circles being given in position and magnitude; to draw a straight line cutting them so that the chords in each circle may be equal to a given line, not greater than the diameter of the smaller circle.

67. To determine a point in the arc of a quadrant, through which if a tangent be drawn meeting the sides of the quadrant produced, the intercepted parts may have a given ratio.

68. If a tangent be drawn to a circle at the extremity of a chord which cuts the diameter at right angles, and from any point in it a perpendicular be let fall; the segment of the diameter intercepted between that perpendicular and chord is to the intercepted part of the tangent as the chord is to the diameter.

69. If a straight line be placed in a circle, and from its extremities perpendiculars be let fall upon any diameter; these perpendiculars together will have to the part of the diameter intercepted between them, the same ratio that a line placed in the circle perpendicular to the former line, has to the former line itself.

70. In a circle to place a straight line of given length, so that perpendiculars drawn to it from two given points in the circumference may have a given ratio.

71. If from any point in the arc of a segment of a circle a line be drawn perpendicular to the base; and from the greater segment of the base and arc, parts be cut off respectively equal to the less; the remaining part of the base shall be equal to the chord of the remaining arc.

72. If from the point of bisection of any arc of a circle a perpendicular be drawn to the diameter which passes through one extremity; it will bisect the segment of the chord cut off by the line joining the point of bisection of the arc and the other extremity of the diameter.

73. In a given circle to draw a chord parallel to a straight line given in position; so that the chord and perpendicular drawn to it from the centre may together be equal to a given line.

74. Through a given point within a given circle, to draw a straight line such that the parts of it intercepted between that point and the circumference may have a given ratio.

75. From two given points in the circumference of a given circle, to draw two lines to a point in the circumference, which shall cut a line given in position, so that the part of it intercepted by them may be equal to a given line.

76. If a chord and diameter of a circle intersect each other at any angle, and a perpendicular to the chord be drawn from either extremity of it, meeting the circumference and diameter produced; the whole perpendicular has to the part of it without the circle, the same ratio that the greater segment of the chord has to the less.

77. If from the extremities of any chord of a circle, perpendiculars to it be drawn and produced to cut a diameter; and from the points of intersection with the diameter lines be drawn to a point in the chord so as to make equal angles with it; these lines together will be equal to the diameter of the circle.

78. If from a point without a circle two straight lines be drawn, one of which touches and the other cuts the circle; a line drawn from the same point in any direction, equal to the tangent, will be parallel to the chord of the arc intercepted by two lines drawn from its other extremity to the former intersections of the circle.

79. If from a point without a circle two straight lines be drawn touching it, and from one point of contact a perpendicular be drawn to that diameter which passes through the other; this perpendicular will be bisected by the line joining the point without the circle and the other extremity of the diameter.

80. If any chord in a circle be bisected by another, and produced to meet the tangents drawn from the extremities of the bisecting line; the parts intercepted between the tangents and the circumferences are equal.

81. If one chord in a circle bisect another, and tangents drawn from the extremities of each be produced to meet; the line joining their points of intersection will be parallel to the bisected chord.

82. If from a point without a circle two lines be drawn touching the circle, and from the extremities of any diameter lines be drawn to the points of contact, cutting each other within the circle; the line produced, which joins their intersection and the point without the circle, will be perpendicular to the diameter.

83. If on opposite sides of the same extremity of the diameter of a circle equal arcs be taken, and from the extremities of these arcs lines be drawn to any point in the circumference, one of which cuts the diameter, and the other the diameter produced; the distances of the points of intersection from the extremities of the diameter are proportional to each other.

84. If from the extremities of any chord in a circle, perpendiculars be drawn to a diameter, and from either extremity of that diameter a perpendicular be drawn to the chord; it will divide it into segments, which are respectively mean proportionals between the segments of the diameter cut off by the perpendiculars.

85. If from any point in the diameter of a semicircle, a perpendicular be drawn, meeting the circumference, and on it as a diameter a circle be described, to the centre of which a line is drawn from the farther extremity of the diameter of the semicircle, cutting its circumference; and through the point of intersection another line be drawn from the extremity of the perpendicular, meeting the diameter of the semicircle; this diameter will be divided into three segments which are in continued proportion.

86. If from a point without a given circle, any two lines be drawn cutting the circle; to determine a point in the circumference, such that the sum of the perpendiculars from it upon these lines may be equal to a given line.

87. If two circles cut each other, and any two points be taken

in the circumference of one of them, through which lines are drawn from the points of intersection and produced to the circumference of the other; the straight lines joining the extremities of those which are drawn through the same point, are equal.

88. If two circles cut each other; the greatest line that can be drawn through the point of intersection is that which is parallel to the line joining their centres.

89. Having given the radii of two circles which cut each other, and the distance of their centres; to draw a straight line of given length through their point of intersection, so as to terminate in their circumferences.

90. If two circles cut each other; to draw from one of the points of intersection a straight line meeting the circles, so that the part of it intercepted between the circumferences may be equal to a given line.

91. If two circles cut each other; to draw from the point of intersection two lines, the parts of which intercepted between the circumferences may have a given ratio.

92. If a semicircle be described on the common chord of two intersecting circles, and a line drawn from one extremity of the chord, cutting the two circles; the part intercepted between the two shall be divided by the semicircle into segments proportional to perpendiculars drawn in those circles from the other extremity of the chord.

93. Two circles being given, the circumference of one of which passes through the centre of the other; to draw a chord from that centre, such that a perpendicular let fall upon it from a given point, may bisect that part of it which is intercepted between the circumferences.

94. If any number of circles cut each other in the same points, and from one of these points any number of lines be drawn; the parts of them which are intercepted between the several circumferences have the same ratio.

95. In a given circle to place a straight line cutting two radii which are perpendicular to each other, in such a manner that the line itself may be trisected.

96. If a straight line be divided into any two parts, and upon the whole line and one of the parts, as diameters, semicircles be described; to determine a point in the less diameter, from which if a perpendicular be drawn cutting the circumferences, and the points of intersection and the extremities of the respective diameters be joined, and these lines produced to meet; the parts of them without the semicircles may have a given ratio.

97. If a straight line be divided into any two parts, and from the point of section a perpendicular be erected, which is a mean proportional between one of the parts and the whole line, and a circle described through the extremities of the line and the perpendicular; the whole line, the perpendicular, the aforesaid part, and a perpendicular drawn from its extremity to the circumference will be in continued proportion.

98. If the tangents drawn to every two of three unequal circles be produced till they meet, the points of intersection will be in a straight line.

99. If from the extremities of the diameter of a circle any number of chords be drawn, two and two intersecting each other in a perpendicular to that diameter; the lines joining the extremities of every corresponding two will meet the diameter produced in the same point.

100. If from a given point in the diameter of a semicircle produced, three straight lines be drawn, one of which is inclined at a given angle to the diameter, another touches the semicircle, and the third cuts it, in such a manner, that the distance of the given point from the nearer extremity of the diameter, and the perpendiculars drawn from that extremity on the three aforesaid lines may be proportional; then will the lines, which join the extremities of the diameter and of that part of the cutting line which is within the circle, intersect each other in an angle equal to the given angle.

SECTION III. Page 92.

1. **ANY** side of a triangle is greater than the difference between the other two sides.

2. In any right-angled triangle, the straight line joining the right angle and the bisection of the hypotenuse is equal to half the hypotenuse.

3. If from any point within an equilateral triangle perpendiculars be drawn to the sides; they are together equal to a perpendicular drawn from any of the angles to the opposite side.

4. If the points of bisection of the sides of a given triangle be joined; the triangle so formed will be one fourth of the given triangle.

5. The difference of the angles at the base of any triangle is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.

6. If from one of the equal angles of an isosceles triangle any line be drawn to the opposite side, and from the same point a line be drawn to the opposite side produced, so that the part intercepted between them may be equal to the former; the angle contained by the side of the triangle and the first drawn line is double the angle contained by the base and the latter.

7. If from the extremity of the base of an isosceles triangle, a line equal to one of the sides be drawn to meet the opposite side; the angle formed by this line and the base produced is equal to three times either of the equal angles of the triangle.

8. The sum of the sides of an isosceles triangle is less than the sum of the sides of any other triangle on the same base and between the same parallels.

9. If from one of the equal angles of an isosceles triangle, a perpendicular be drawn to the opposite side; the part of it intercepted by a perpendicular from the vertex will have to one of the equal sides, the same ratio that the segment of the base has to the perpendicular upon the base.

10. If from any point in the base of an isosceles triangle lines be

drawn to the opposite sides, making equal angles with the base; the triangles formed by these lines, the segments of the base, and the lines joining the intersections of the sides and the angles opposite, will be equal.

11. If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides; these together shall be equal to a perpendicular drawn from either extremity of the base to the opposite side.

12. Of all triangles having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in the given point.

13. If from the angles at the base of a triangle, perpendiculars be let fall on a line which bisects the vertical angle; the part of this line intercepted between these perpendiculars will be bisected by a perpendicular from the middle of the base.

14. If from one of the angles at the base of a triangle a line be drawn parallel to the opposite side and from any point in it lines be drawn making any angles with the sides (produced, if necessary); they will have the same ratio that lines have which are drawn parallel to them from the other angles, and terminated by the same sides.

15. To bisect a given triangle by a line drawn from one of its angles.

16. To bisect a given triangle by a line drawn from a given point in one of its sides.

17. To determine a point within a given triangle, from which lines drawn to the several angles, will divide the triangle into three equal parts.

18. Trisect a given triangle from a given point within it.

19. From a given point in the side of a triangle, to draw lines which will divide the triangle into any number of parts which shall have a given ratio.

20. If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines, a line be drawn to the opposite angle of the triangle; it will bisect that angle.

21. If in two triangles the vertical angle of the one be equal to that of the other, and one other angle of the former be equal to the exterior angle at the base of the latter; the sides about the third angle of the former shall be proportional to those about the interior angle at the base of the latter.

22. In a given triangle, to draw a line parallel to one of the sides, so that it may be a mean proportional between the segments of the base.

23. To draw a line parallel to the common base of two triangles which have different altitudes, so that the parts of it intercepted by the sides may have a given ratio.

24. If the base of a triangle be produced so that the whole may be to the part produced in the duplicate ratio of the sides; the line joining the vertex and the extremity of the part produced will be a mean proportional between the whole line produced and the part produced.

25. To determine a point within a given triangle, which will divide a line parallel to the base into two segments, such that the excess of each segment above the perpendicular distance between the parallel lines may be to each other in the duplicate ratio of the respective segments.

26. If perpendiculars be drawn to two sides of a triangle from any two points therein; the distance of their concurrence from that of the two sides will be to the distance between the two points as either side is to the perpendicular drawn from its extremity upon the other.

27. If the three sides of a triangle be bisected, the perpendiculars drawn to the sides at the three points of bisection, will meet in the same point.

28. If from the three angles of a triangle lines be drawn to the points of bisection of the opposite sides, these lines intersect each other in the same point.

29. The three straight lines which bisect the three angles of a triangle, meet in the same point.

30. If the three angles of a triangle be bisected, and one of the bisecting lines be produced to the opposite side; the angle contained

by this line produced, and one of the others is equal to the angle contained by the third and a perpendicular drawn from the common point of intersection of the three lines, to the aforesaid side.

31. In a right-angled triangle, if a straight line be drawn parallel to the hypotenuse, and cutting the perpendicular drawn from the right angle; and through the point of intersection a line be drawn from one of the acute angles to the opposite side, and the extremity of this line and of the perpendicular be joined; the locus of its intersection with the line parallel to the hypotenuse will be a straight line.

32. If from the angles of a triangle, lines, each equal to a given line, be drawn to the opposite sides (produced if necessary); and from any point within, lines be drawn parallel to these, and meeting the sides of the triangle; these lines will together be equal to the given line.

33. If the sides of a triangle be cut proportionally, and lines be drawn from the points of section to the opposite angles; the intersections of these lines will be in the same line; viz. that drawn from the vertex to the middle of the base.

34. If from any point in one side of a triangle, two lines be drawn, one to the opposite angle, and the other parallel to the base, and the former intersect a line drawn from the vertex bisecting the base; this point of intersection, that of the line parallel to the base and the third side, and the third angular point are in the same straight line.

35. If one side of a triangle be divided into any two parts, and from the point of section two straight lines be drawn parallel to, and terminating at the other sides, and the points of termination be joined; and any other line be drawn parallel to either of the two former lines, so as to intersect the other, and to terminate in the sides of the triangle; then the two extreme parts of the three segments into which the line so drawn is divided, will always be in the ratio of the segments of the first divided line.

36. If through the point of bisection of the base of a triangle, any line be drawn intersecting one side of the triangle and the other produced, and meeting a parallel to the base from the vertex; this line will be cut harmonically.

37. If from either angle of a triangle, a line be drawn intersecting that which joins the vertex and the bisection of the base, the opposite side, and the line from the vertex parallel to the base; it will be cut harmonically.

38. To draw a line from one of the angles at the base of a triangle, so that the part of it cut off by a line drawn from the vertex parallel to the base, may have a given ratio to the part cut off by the opposite side.

39. To determine that point in the base produced of a right-angled triangle, from which the line drawn to the angle opposite to the base shall have the same ratio to the base produced, which the perpendicular has to the base itself.

40. If the base of any triangle be divided into two parts by a line which is a mean proportional between them, and which being drawn parallel to the second side is terminated in the third; any line parallel to the base will be divided by the mean proportional (produced if necessary) into segments which will be to each other inversely as the whole mean proportional to that segment which is terminated in the third side of the triangle.

41. If from the extremities of the base of any triangle, two straight lines be drawn intersecting each other in the perpendicular, and terminating in the opposite sides; straight lines drawn from thence to the intersection of the perpendicular with the base, will make equal angles with the base.

42. In any triangle, the intersection of the perpendiculars drawn from the angles to the opposite sides, the intersection of the lines from the angles to the middle of the opposite sides, and the intersection of the perpendiculars from the middle of the sides, are all in the same straight line. And the distances of those points from one another are in a given ratio.

43. If straight lines be drawn from the angles of a triangle through any point, either within or without the triangle, to meet the sides, and the lines joining these points of intersection and the sides of the triangle be produced to meet; the three points of concurrence will be in the same straight line.

SECTION IV. Page 120.

1. **THE** diameters of a rhombus bisect each other at right angles.
2. If the opposite sides or opposite angles of a quadrilateral figure be equal; the figure will be a parallelogram.
3. To bisect a parallelogram by a line drawn from a point in one of its sides.
4. If from any point in the diameter (or diameter produced) of a parallelogram straight lines be drawn to the opposite angles; they will cut off equal triangles.
5. From one of the angles of a parallelogram to draw a line to the opposite side, which shall be equal to that side together with the segment of it which is intercepted between the line and the opposite angle.
6. If from one of the angles of a parallelogram a straight line be drawn, cutting the diameter, a side and a side produced; the segment intercepted between the angle and the diameter is a mean proportional between the segments intercepted between the diameter and the sides.
7. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of two opposite sides, are together half of the parallelogram.
8. If a straight line be drawn parallel to one of the sides of a parallelogram, and one extremity of this line be joined to the opposite one of the parallel side, by a line which also cuts the diameter; the segments of the diameter made by this line will be reciprocally proportional to the segments of that part of it which is intercepted between the side and the parallel line.
9. If two lines be drawn parallel and equal to the adjacent sides of a parallelogram; the lines joining their extremities, if produced, will meet the diameter in the same point.
10. If in the sides of a square, at equal distances from the four angles, four other points be taken, one in each side; the figure contained by the straight lines which join them shall also be a square.

11. The sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the four angles from any point within the figure, except from the intersection of the diagonals.

12. Every trapezium is divided by its diagonals into four tri-angles proportional to each other.

13. If two opposite angles of a trapezium be right angles; the angles subtended by either side at the two opposite angular points shall be equal.

14. To determine the figure formed by joining the points of bisection of the sides of a trapezium; and its ratio to the trapezium.

15. To determine the figure formed by joining the points where the diagonals of the trapezium cut the parallelogram (in the last problem); and its ratio to the trapezium.

16. If two sides of a trapezium be parallel; its area is equal to half that of a parallelogram whose base is the sum of those two sides, and altitude the perpendicular distance between them.

17. If from any angle of a rectangular parallelogram a line be drawn to the opposite side, and from the adjacent angle of the trapezium thus formed another be drawn perpendicular to the former; the rectangle contained by these two lines is equal to the given parallelogram.

18. To divide a parallelogram into two parts which shall have a given ratio, by a line drawn parallel to a given line.

19. To bisect a trapezium by a line drawn from one of its angles.

20. To bisect a trapezium by a line drawn from a given point in one of its sides.

21. If two sides of a trapezium be parallel; the triangle contained by either of the other sides, and the two straight lines drawn from its extremities to the bisection of the opposite side, is half the trapezium.

22. To divide a given trapezium, whose opposite sides are parallel, in a given ratio, by a line drawn through a given point, and terminated by the two parallel sides.

23. If a trapezium, which has two of its adjacent angles right angles, be bisected by a line drawn from the middle of one of those

sides which are not parallel; the sum of the parallel sides will have to one of them the same ratio, that the side which is not bisected has to that segment of it which is adjacent to the other.

24. If the sides of an equilateral and equiangular pentagon be produced to meet; the angles formed by these lines are together equal to two right angles.

25. If the sides of an equilateral and equiangular hexagon be produced to meet; the angles formed by these lines are together equal to four right angles.

26. The area of any two parallelograms described on the two sides of a triangle is equal to that of a parallelogram on the base, whose side is equal and parallel to the line drawn from the vertex of the triangle to the intersection of the two sides of the former parallelograms produced to meet.

27. The perimeter of an isosceles triangle is greater than the perimeter of a rectangular parallelogram, which is of the same altitude with, and equal to the given triangle.

28. If from one of the acute angles of a right-angled triangle, a line be drawn to the opposite side; the squares of that side and the line so drawn are together equal to the squares of the segment adjacent to the right angle and of the hypotenuse.

29. In any triangle, if a line be drawn from the vertex at right angles to the base; the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.

30. In any triangle, if a line be drawn from the vertex bisecting the base; the sum of the squares of the two sides of the triangle is double the sum of the squares of the bisecting line and of half the base.

31. If from the three angles of a triangle lines be drawn to the points of bisection of the opposite sides; the squares of the distances between the angles and the common intersection are together one third of the squares of the sides of the triangle.

32. If from any point within or without any rectilineal figure, perpendiculars be let fall on every side; the sum of the squares of the alternate segments made by them, will be equal.

33. If from any point within a rectangular parallelogram lines be drawn to the angular points; the sums of the squares of those which are drawn to the opposite angles are equal.

34. The squares of the diagonals of a parallelogram are together equal to the squares of the four sides.

35. If two sides of a trapezium be parallel to each other; the squares of its diagonals are together equal to the squares of its two sides which are not parallel, and twice the rectangle contained by its parallel sides.

36. The squares of the diagonals of a trapezium are together double the squares of the two lines joining the bisections of the opposite sides.

37. The squares of the diagonals of a trapezium are together less than the squares of the four sides, by four times the square of the line joining the points of bisection of the diagonals.

38. In any trapezium, if two opposite sides be bisected; the sum of the squares of the two other sides, together with the squares of the diagonals, is equal to the sum of the squares of the bisected sides, together with four times the square of the line joining those points of bisection.

39. If squares be described on the sides of a right-angled triangle; each of the lines joining the acute angles and the opposite angle of the square, will cut off from the triangle an obtuse-angled triangle, which will be equal to that cut off from the square by a line drawn from the intersection with the side to that angle of the square which is opposite to it.

40. If squares be described on the two sides of a right-angled triangle; the lines joining each of the acute angles of the triangle and the opposite angle of the square will meet the perpendicular drawn from the right angle upon the hypotenuse, in the same point.

41. If squares be described on the three sides of a right-angled triangle, and the extremities of the adjacent sides be joined; the triangles so formed are equal to the given triangle and to each other.

42. If the sides of the square described on the hypotenuse of a right-angled triangle be produced to meet the sides (produced if

necessary) of the squares described upon the legs; they will cut off triangles equiangular and equal to the given triangle.

43. If from the angular points of the squares described upon the sides of a right-angled triangle perpendiculars be let fall upon the hypotenuse produced; they will cut off equal segments; and the perpendiculars will together be equal to the hypotenuse.

44. If on the two sides of a right-angled triangle squares be described; the lines joining the acute angles of the triangle and the opposite angles of the squares will cut off equal segments from the sides; and each of these equal segments will be a mean proportional between the remaining segments.

45. If squares be described on the hypotenuse and sides of a right-angled triangle, and the extremities of the sides of the former and the adjacent sides of the others be joined; the sum of the squares of the lines joining them will be equal to five times the square of the hypotenuse.

46. If a line be drawn parallel to the base of a triangle, and terminated in the sides; to draw a line cutting it, and terminated also by the sides, so that the rectangle contained by their segments may be equal.

47. If the sides, or sides produced, of a triangle be cut by any line; the solids formed by the segments which have not a common extremity are equal.

48. If through any point within a triangle, three lines be drawn parallel to the sides; the solids formed by the alternate segments of these lines are equal.

49. If through any point within a triangle lines be drawn from the angles to cut the opposite sides; the segments of any one side will be to each other in the ratio compounded of the ratios of the segments of the other sides.

50. If from each of the angles of any triangle, a line be drawn through any point within the triangle to the opposite side; the solid formed by the segments thereof, intercepted between the angles and the point, will have to the solid formed by the three remaining segments, the same ratio that the solid formed by the three sides of the triangle has to either of the (equal) solids formed by the alternate segments of the sides.

SECTION V. Page 153.

1. A STRAIGHT line of given length being drawn from the centre at right angles to the plane of a circle; to determine that point in it which is equally distant from the upper end of the line, and the circumference of the circle.

2. To determine a point in a line given in position, to which lines drawn from two given points may have the greatest difference possible.

3. A straight line being divided in two given points; to determine a third such that its distances from the extremities may be proportional to its distances from the given points.

4. In a straight line given in position, to determine a point, at which two straight lines drawn from given points on the same side, will contain the greatest angle.

5. To determine the position of a point, at which lines drawn from three given points shall make with each other angles equal to given angles.

6. To divide a straight line into two parts such that the rectangle contained by them may be equal to the square of their difference.

7. If a straight line be divided into any two parts; to produce it so that the rectangle contained by the whole line so produced and the part produced, may be equal to the rectangle contained by the given line and one segment.

COR. 1. To produce the line, so that the rectangle contained by the whole line and the part produced may be equal to the rectangle contained by two given lines.

COR. 2. To produce the line, so that the rectangle contained by the whole line produced and the part produced may be equal to a given square.

8. To determine two lines such that the sum of their squares may be equal to a given square, and their rectangle equal to a given rectangle.

9. To divide a straight line into two parts, so that the rectangle

contained by the whole and one of the parts may be equal to the square of a given line, which is less than the line to be divided.

10. To divide a given line into two such parts, that the rectangle contained by the whole line, and one of the parts may be (m) times the square of the other part; (m) being whole or fractional.

11. To divide a given line into two such parts, that the square of the one shall be equal to the rectangle contained by the other and a given line.

12. A straight line being given in magnitude and position; to draw to it, from a given point, two lines, whose rectangle shall be equal to a given rectangle, and which shall cut off equal segments from the given line.

13. To draw a straight line which shall touch a given circle, and make with a given line, an angle equal to a given angle.

14. Through a given point to draw a line terminating in two lines given in position, so that the rectangle contained by the two parts may be equal to a given rectangle.

15. From a given point to draw a line cutting two given parallel lines, so that the difference of its segments may be equal to a given line.

16. From a given point without a circle, to draw a straight line cutting the circle, so that the rectangle contained by the part of it without, and the part within the circle shall be equal to a given square.

17. From a given point in the circumference of a semicircle, to draw a straight line meeting the diameter, so that the difference between the squares of this line and a perpendicular to the diameter from the point of intersection may be equal to a given rectangle.

18. From a given point to draw two lines to a third given in position, so that the rectangle contained by those lines may be equal to a given rectangle, and the difference of the angles which they make with that part of the third which is intercepted between them may be equal to a given angle.

19. Two points being given without a given circle; to determine a point in the circumference, from which lines drawn to the two given points shall contain the greatest possible angle.

20. From the bisection of a given arc of a circle, to draw a straight line such that the part of it intercepted between the chord of that and the opposite circumference shall be equal to a given straight line.

21. To draw a straight line through a given point, so that the sum of the perpendiculars to it from two other given points may be equal to a given line.

22. To draw a straight line through one of three points given in position, so that the rectangle contained by the perpendiculars let fall upon it from the other two, may be equal to a given square.

23. A given straight line being divided into two parts; to cut off a part which shall be a mean proportional between the two remaining segments.

24. To draw a straight line making a given angle with one of the sides of a given triangle, so that the triangle cut off may be to the whole in a given ratio.

25. Between two given straight lines containing a given angle, to place a straight line of given length, and subtending that angle, so that the segment of the one of them adjacent to the angle may be to the segment of the other which is not adjacent, in the ratio of two given lines.

26. From two given points to draw two lines to a point in a third, such that the difference of their squares may be equal to a given square.

27. To divide a given straight line into two such parts, that the square of one may be to the excess of a given rectangle above the square of the other, in a given ratio.

28. From any angle of a triangle, not isosceles about the angle, to draw a line without the triangle to the opposite side produced, which shall be a mean proportional between the segments of the side.

29. From the obtuse angle of any triangle, to draw a line within the triangle to the opposite side, which shall be a mean proportional between the segments of the side.

30. From the common extremity of the diameters of two semi-circles, given in magnitude and position; to draw a line meeting the

circumferences so that the rectangle contained by the two chords may be equal to a given square.

31. To draw a line parallel to a given line, which shall be terminated by two others given in position, so as to form with them a triangle equal to a given rectilineal figure.

32. To bisect a triangle by a line drawn parallel to one of its sides.

33. To divide a given triangle into any number of parts, having a given ratio to each other, by lines drawn parallel to one of the sides of the triangle.

34. To divide a given triangle into any number of equal parts, by lines drawn parallel to a given line.

35. To divide a trapezium, which has two sides parallel, into any number of equal parts, by lines drawn parallel to those sides.

36. From one of the angular points of a given square, to draw a line meeting one of the opposite sides and the other produced, in such a manner, that the exterior triangle formed thereby may have a given ratio to the square.

37. From a given point in the side produced of a given rectangular parallelogram, to draw a line which shall cut the perpendicular sides and the other side produced, so that the trapezium cut off, which stands on the aforesaid side, may be to the triangle which stands upon the produced part of the opposite side, in a given ratio.

38. Through a given point between two straight lines containing a given angle, to draw a line which shall cut off a triangle equal to a given figure.

39. Between two lines given in position, to draw a line equal to a given line, so that the triangle thus formed may be equal to a given rectilineal figure.

40. From two given lines to cut off two others, so that the remainder of one may have to the part cut off from the other, a given ratio; and the difference of the squares of the other remainder and part cut off from the first may be equal to a given square.

41. From two given lines to cut off two others which shall have a given ratio, so that the difference of the squares of the remainders may be equal to a given square.

42. From two given lines to cut off two others, so that the remainders may have a given ratio, and the sum of the squares of the parts cut off may be equal to the square of a given line.

43. Two points being given in a given straight line; to determine a third, such that the rectangles contained by its distances from each extremity and the given point adjacent to that extremity may be equal.

44. Through the point of intersection of two given circles, to draw a line in such a manner, that the sum of the respective rectangles contained by the parts thereof, which are intercepted between the said point and their circumferences, and given lines *A* and *B*, may be equal to a given square.

45. Through a given point, to draw an indefinite line such, that if lines be drawn from two other given points and forming given angles with it, the rectangle contained by the segments intercepted between the given point and the two lines so drawn, shall be equal to the square of a given line.

46. Through a given point between two straight lines containing a given angle, to draw a line such that a perpendicular upon it from the given angle may have a given ratio to a line drawn from one extremity of it, parallel to a line given in position.

47. Through a given point between two indefinite straight lines, not parallel to one another, to draw a line, which shall be terminated by them, so that the rectangle contained by its segments shall be less than the rectangle contained by the segments of any other line drawn through the same point.



SECTION VI. Page 184.

1. To describe an isosceles triangle on a given finite straight line.

2. To describe a square which shall be equal to the difference of two squares, whose sides are given.

COR. Hence a mean proportional between the sum and difference of two given lines may be determined.

3. To describe a rectangular parallelogram, which shall be equal to a given square, and have its adjacent sides together equal to a given line.

4. To describe a rectangular parallelogram, which shall be equal to a given square, and have the difference of its adjacent sides equal to a given line.

5. To describe a triangle, which shall be equal to a given equilateral and equiangular pentagon, and of the same altitude.

6. To describe an equilateral triangle, equal to a given isosceles triangle.

7. To describe a parallelogram, the area and perimeter of which shall be respectively equal to the area and perimeter of a given triangle.

8. To describe a parallelogram, which shall be of given altitude, and equiangular and equal to a given parallelogram.

9. To describe a square which shall be equal to the sum of any number of given squares.

10. Having given the difference between the diameter and side of a square; to describe the square.

11. To divide a circle into any number of concentric equal annuli.

COR. To divide it into annuli which shall have a given ratio.

12. In any quadrilateral figure circumscribing a circle, the opposite sides are equal to half the perimeter.

13. If the opposite angles of a quadrilateral figure be equal to two right angles, a circle may be described about it.

14. A quadrilateral figure may have a circle described about it, if the rectangles contained by the segments of the diagonals be equal.

15. If from any point within a regular figure circumscribed about a circle, perpendiculars be drawn to the sides; they will together be equal to that multiple of the semi-diameter which is expressed by the number of the sides of the figure.

16. If the radius of a circle be cut in extreme and mean ratio; the greater segment will be equal to the side of an equilateral and equiangular decagon inscribed in that circle.

17. Any segment of a circle being described on the base of a triangle; to describe on the other sides segments similar to that on the base.

18. If an equilateral triangle be inscribed in a circle; the square described on a side thereof is equal to three times the square described upon the radius.

19. To inscribe a square in a given right-angled isosceles triangle.

20. To inscribe a square in a given quadrant of a circle.

21. To inscribe a square in a given semicircle.

22. To inscribe a square in a given segment of a circle.

23. Having given the distance of the centres of two equal circles which cut each other; to inscribe a square in the space included between the two circumferences.

24. In a given segment of a circle to inscribe a rectangular parallelogram, whose sides shall have a given ratio.

25. In a given circle to inscribe a rectangular parallelogram equal to a given rectilineal figure.

26. In a given segment of a circle to inscribe an isosceles triangle, such that its vertex may be in the middle of the chord, and the base and perpendicular together equal to a given line.

27. In a given triangle, to inscribe a parallelogram similar to a given parallelogram.

28. In a given triangle, to inscribe a triangle similar to a given triangle.

29. In a given equilateral and equiangular pentagon, to inscribe a square.

30. In a given triangle, to inscribe a rhombus, one of whose angles shall be in a given point in the side of the triangle.

31. To inscribe a circle in a given quadrant.

32. To describe a circle, the circumference of which shall pass through a given point, and touch a given straight line in a given point.

33. To describe a circle, which shall pass through a given point, have a given radius, and touch a given straight line.

34. To describe a circle, which shall pass through two given points, and touch a given straight line.

35. To describe a circle, the circumference of which shall pass through a given point, and touch a circle in a given point; the two points not being in a tangent to the given circle.

36. To describe a circle, the centre of which may be in the perpendicular of a given right-angled triangle, and the circumference pass through the right angle and touch the hypotenuse.

37. To describe a circle, which shall pass through the extremities of a given line, so that if from any point in its circumference a line be drawn making a given angle with the given line; the rectangle contained by the segment it cuts off and the given line, may be equal to the square of the line drawn from the same point to the farther extremity of the given line.

38. To determine a point in the perpendicular let fall from the vertical angle of any triangle on the base; about which as a centre a circle may be described touching the longer side, and passing through the opposite angular point.

39. To describe a circle which shall have a given radius, its centre in a given straight line, and shall also touch another given straight line inclined at a given angle to the former.

40. To describe a circle which shall touch a straight line in a given point, and also touch a given circle.

41. To describe two circles, each having a given radius, which shall touch each other, and the same given straight line on the same side of it.

42. To describe a circle passing through two given points, and touching a given circle.

43. To describe a circle which shall pass through a given point, and touch a given circle and a given straight line.

44. To describe a circle which shall touch a straight line and two circles given in magnitude and position.

45. To describe a circle which shall touch two given straight lines, and pass through a given point between them.

46. To describe a circle which shall touch two given straight lines, and also touch a given circle.

47. To describe a circle which shall touch a circle and straight line, both given in position, and have its centre also in a given straight line.

48. Through two given points within a given circle, to describe a circle, which shall bisect the circumference of the other.

49. Through two given points without a given circle, to describe a circle which shall cut off from the one, an arc equal to a given arc.

50. To describe three circles of equal diameters, which shall touch each other.

✓ 51. Every thing remaining as in the last proposition ; to describe a circle which shall touch the three circles.

52. To determine how many equal circles may be described round another circle of the same diameter, touching each other and the interior circle.

53. To draw two lines parallel to the adjacent sides of a given rectangular parallelogram, which shall cut off a portion, whose breadth shall be every where the same, and whose area shall be to that of the parallelogram in any given ratio.

54. To describe a triangle equal to a given rectilinear figure, having its vertex in a given point in a side of the figure, and its base in the base (produced if necessary) of the figure.

55. On the base of a given triangle to describe a quadrilateral figure equal to the triangle, and having two of its sides parallel, one of them being the base of the triangle ; and one of its angles being an angle at the base, and the other equal to a given angle.

56. A trapezium being given, two of whose sides are parallel ; to describe on one of those sides another trapezium, having its opposite side also parallel to this, one of the angles at the base the same as the former, and the other equal to a given angle.

57. If with any point in the circumference of a circle as centre, and distance from its centre as radius, a circular arc be described ; and any two chords be drawn, one from the centre of the circular arc, and the other through the point where this cuts the arc, and parallel to the line joining the centres ; the segments of each chord intercepted between the circumferences which are concave to each other, will be equal respectively to those of the other between the other circumference.

58. If the diagonals of an equilateral and equiangular pentagon be drawn to cut one another, the greater segments will be equal to the side of the pentagon; and the diagonals cut each other in extreme and mean ratio.

59. If the sides of a triangle inscribed in the segment of a circle be produced to meet lines drawn from the extremities of the base, forming with it angles equal to the angle in the segment; the rectangle contained by these lines will be equal to the square described on the base.

60. If two triangles (one of them right-angled) have the same base and altitude, and the hypotenuse intersect a line which is drawn bisecting the right angle; a line passing through this point of intersection parallel to the base, and terminated by the sides of the other triangle, shall be a side of the square inscribed within it.

61. If on the side of a rectangular parallelogram, as a diameter, a semicircle be described, and from any point in the circumference lines be drawn through its extremities to meet the opposite side produced; the altitude of the parallelogram will be a mean proportional between the segments cut off.

62. If on the diameter of a semicircle a rectangular parallelogram be described, whose altitude is equal to the chord of half the semicircle, and lines drawn from any point in the circumference to the extremities of the base intersect the diameter, the squares of the distances of each point of section from the farthest extremity of the diameter will be together equal to the square of the diameter.

COR. The square of the part of the diameter intercepted between the two lines drawn from the point in the semicircle is double of the rectangle contained by the two extreme segments.

63. If on the radius drawn from the point of contact of a circle and its circumscribed square, another circle be described; and from any point in the outer circumference a line be drawn through its centre to the inner circumference, and through the same point another line be drawn parallel to the common tangent to the circles, and terminated by the side of the square and its diagonal; these two lines are equal.

64. If two sides of a trapezium inscribed in a circle be produced, and from the same point in one side produced, a line be

drawn parallel to the other, intersecting the adjacent side of the trapezium, and a second line to the extremity of that other intersecting the circumference: the line joining the two points of intersection will pass through the same point.

65. If the diagonals of a quadrilateral figure inscribed in a circle, cut each other at right angles; the rectangles contained by the opposite sides are together double of the quadrilateral figure.

66. If a rectangular parallelogram be inscribed in a right-angled triangle, and they have the right angle common; the rectangle contained by the segments of the hypotenuse is equal to the sum of the rectangles contained by the segments of the sides about the right angle.

67. If on the diameter of a semicircle two equal circles be described, and in the curvilinear space included by the three circumferences a circle be inscribed; its diameter will be to that of the equal circles in the proportion of two to three.

68. If through the middle point of any chord of a circle two chords be drawn; the lines joining their extremities will intersect the first chord at equal distances from the middle point.

69. The longest side of a trapezium being given, and made the diameter of the circumscribed circle; also the distance between its extremity and the intersection of the opposite side produced to meet it, and the angle formed by the intersection of the diagonals: to construct the trapezium.

70. The diagonals of a quadrilateral figure inscribed in a circle are to one another as the sums of the rectangles of the sides which meet their extremities.

71. The square described on the side of an equilateral and equiangular pentagon inscribed in a circle, is equal to the sum of the squares of the side of a regular hexagon and decagon inscribed in the same circle.

72. If the opposite sides of an irregular hexagon inscribed in a circle be produced till they meet; the three points of intersection will be in the same straight line.

SECTION VII. Page 232.

1. **T**HE vertical angle of an oblique-angled triangle inscribed in a circle, is greater or less than a right angle, by the angle contained by the base and the diameter drawn from the extremity of the base.

2. If from the vertex of an isosceles triangle a circle be described with a radius less than one of the equal sides, but greater than the perpendicular; the parts of the base cut off by it will be equal.

3. If a circle be inscribed in a right-angled triangle; the difference between the two sides containing the right angle and the hypotenuse, is equal to the diameter of the circle.

4. If a semicircle be inscribed in a right-angled triangle so as to touch the hypotenuse and perpendicular, and from the extremity of its diameter a line be drawn through the point of contact, to meet the perpendicular produced; the part produced will be equal to the perpendicular.

5. If the base of any triangle be bisected by the diameter of its circumscribing circle, and from the extremity of that diameter a perpendicular be let fall upon the longer side; it will divide that side into segments, one of which will be equal to half the sum, and the other to half the difference of the sides.

6. The same supposition being made as in the last proposition; if from the point where the perpendicular meets the longer side, another perpendicular be let fall on the line bisecting the vertical angle; it will pass through the middle of the base.

7. If a point be taken without a circle, and from it tangents be drawn to the circle, and another point be taken in the circumference between the two tangents, and a tangent be drawn to it; the sum of the sides of the triangle thus formed is equal to the sum of the two tangents.

8. Of all triangles on the same base and between the same parallels, the isosceles has the greatest vertical angle.

COR. Of all triangles on the same base, and having the same vertical angle, the isosceles is the greatest.

9. If through the vertex of an equilateral triangle a perpendicular be drawn to the side, meeting a perpendicular to the base drawn

from its extremity; the line intercepted between the vertex and the latter perpendicular is equal to the radius of the circumscribing circle.

10. If a triangle be inscribed in a semicircle, and a perpendicular drawn from any point in the diameter meeting one side, the circumference, and the other side produced; the segments cut off will be in continued proportion.

11. If a triangle be inscribed in a semicircle, and one side be equal to the semidiameter; the other side will be a mean proportional between that side and a line equal to that side and the diameter together.

12. If a circle be inscribed in a right-angled triangle; to determine the least angle that can be formed by two lines drawn from the extremity of the hypotenuse to the circumference of the circle.

13. If an equilateral triangle be inscribed in a circle, and through the angular points another be circumscribed; to determine the ratio which they bear to each other.

14. A straight line drawn from the vertex of an equilateral triangle inscribed in a circle to any point in the opposite circumference, is equal to the two lines together, which are drawn from the extremities of the base to the same point.

15. If the base of a triangle be produced both ways, so that each part produced may be equal to the adjacent side, and through the extremities of the parts produced and the vertex a circle be described; the line joining its centre and the vertex of the triangle will bisect the angle at the vertex.

16. If an isosceles triangle be inscribed in a circle, and from the vertical angle a line be drawn meeting the circumference and the base; either equal side is a mean proportional between the segments of the line thus drawn.

17. If from the extremities of one of the equal sides of an isosceles triangle inscribed in a circle, tangents be drawn to the circle and produced to meet; two lines drawn to any point in the circumference from the point of concurrence and one point of contact, will divide the base (produced if necessary) in geometrical proportion.

18. If on the sides of a triangle segments of circles be described

similar to a segment on the base, and from the extremities of the base tangents be drawn intersecting their circumferences; the points of intersection and the vertex of the triangle will be in the same straight line.

19. The centre of the circle which will touch two semicircles described on the sides of a right-angled triangle is in the middle point of the hypotenuse.

COR. Its diameter will be equal to the sides together.

20. If on the three sides of a right-angled triangle semicircles be described, and with the centres of those described on the sides, circles be described touching that described on the base; they will also touch the other semicircles.

21. If from any point in the circumference of a circle perpendiculars be drawn to the sides of the inscribed triangle; the three points of intersection will be in the same straight line.

22. The base of a right-angled triangle not being greater than the perpendicular; if on any line drawn from the vertex to the base a semicircle be described, and a chord equal to the perpendicular placed in it, and bisected; the point of bisection will always fall within the triangle.

23. The straight line bisecting any angle of a triangle inscribed in a given circle, cuts the circumference in a point, which is equidistant from the extremities of the side opposite to the bisected angle, and from the centre of a circle inscribed in the triangle.

24. The perpendicular from the vertex on the base of an equilateral triangle is equal to the side of an equilateral triangle inscribed in a circle whose diameter is the base.

25. If an equilateral triangle be inscribed in a circle, and the adjacent arcs cut off by two of its sides be bisected; the line joining the points of bisection will be trisected by the sides.

26. If any triangle be inscribed in a circle, and from the vertex a line be drawn parallel to a tangent at either extremity of the base; this line will be a fourth proportional to the base and two sides.

27. If a triangle be inscribed in a circle, and from its vertex lines be drawn parallel to tangents at the extremities of its base; they will cut off similar triangles.

COR. 1. The rectangle contained by the segments of the base adjacent to the angles is equal to the square of either line drawn from the vertex.

COR. 2. Those segments are also in the duplicate ratio of the adjacent sides.

28. If one circle be circumscribed and another inscribed in a given triangle, and a line be drawn from the vertical angle to the centre of the inner, and produced to the circumference of the outer circle; the whole line thus produced has to the part produced the same ratio that the sum of the sides of the triangle has to the base.

29. If in a right-angled triangle, a perpendicular be drawn from the right angle to the hypotenuse, and circles inscribed within the triangles on each side of it; their diameters will be to each other as the subtending sides of the right-angled triangle.

30. To find the locus of the vertex of a triangle, whose base and ratio of the other two sides are given.

31. A given straight line being divided into any three parts; to determine a point such, that lines drawn to the points of section and to the extremities of the line shall contain three equal angles.

32. If two equal lines touch two unequal circles, and from the extremities of them lines containing equal angles be drawn cutting the circles, and the points of section joined; the triangles so formed will be reciprocally proportional.

33. If from an angle of a triangle a line be drawn to cut the opposite side, so that the rectangle contained by the sides including the angle be equal to the rectangle contained by the segments of the side together with the square of the line so drawn; that line bisects the angle.

34. In any triangle, if perpendiculars be drawn from the angles to the opposite sides, they will all meet in a point.

35. If from the extremities of the base of any triangle, two perpendiculars be let fall on the line bisecting the vertical angle; and through the points where they meet that line, and the point in the base, whereon the perpendicular from the vertical angle falls, a circle be described; that circle will bisect the base of the triangle.

36. If from one of the angles of a triangle a straight line be

drawn through the centre of its inscribed circle, and a perpendicular be drawn to this line from one of the other angles; the point of intersection of the perpendicular, and the two points of contact of the inscribed circle which are adjacent to the remaining angle, are in the same straight line.

37. If from the three angles of any triangle three straight lines be drawn to the points where the inscribed circle touches the sides; these lines shall intersect each other in the same point.

38. If three circles touch each other, two of which are equal; the vertical angle of the triangle formed by joining the points of contact, is equal to either of the angles at the base of the triangle, which is formed by joining their centres.

39. If three equal circles touch each other; to compare the area of the triangle formed by joining their centres with the area of the triangle formed by joining the points of contact.

40. If four straight lines intersect each other, and form four triangles; the circles which circumscribe them will pass through one and the same point.

41. Having given the base and vertical angle of a triangle; to determine the locus of the extremity of the line, which always bisects the vertical angle, and is equal to half the sum of the sides containing that angle.

42. If from the extremities of the base of a triangle inscribed in a circle, perpendiculars be drawn to the opposite sides, intersecting a diameter which is perpendicular to the base; the segments of the diameter intercepted between these points and a point in it, whose distance from the base is equal to the lesser segment of the diameter made by the base, will be to one another in the ratio of the sides of the triangle.

43. If the exterior angle of a triangle be bisected by a straight line which cuts the base produced; the square of the bisecting line is equal to the difference of the rectangles of the segments of the base and of the sides of the triangle.

SECTION VIII. Page 263.

1. If from the centre of a circle a line be drawn to any point in the chord of an arc; the square of that line together with the rectangle contained by the segments of the chord will be equal to the square described on the radius.

2. If two straight lines in a circle cut each other at right angles; the sums of the squares of the two lines joining their extremities will be equal.

3. If two points be taken in the diameter of a circle, equidistant from the centre; the sum of the squares of the two lines drawn from these points to any point in the circumference will be always the same.

4. If from any point in the diameter of a semicircle there be drawn two straight lines to the circumference, one to its point of bisection, and the other at right angles to the diameter; the squares of these two lines are together double of the square of the semidiameter.

5. If a straight line be drawn at right angles to the diameter of a circle, and be cut by any other line; the rectangle contained by the segments of this cutting line, together with the square of that part of the perpendicular line which is intercepted between it and the diameter, is always of the same magnitude.

6. A straight line being drawn from the centre of a quadrant, bisecting the arc and meeting a tangent drawn from one extremity: if from any point in the bounding radius a line be drawn parallel to the tangent; the sum of the squares of the segments of it, cut off by the aforesaid line and by the circumference will be equal to the square of the radius.

7. If from a point without a circle there be drawn two straight lines, one of which is perpendicular to a diameter, and the other cuts the circle; the square of the perpendicular is equal to the rectangle contained by the whole cutting line and the part without the circle, together with the rectangle contained by the segments of the diameter.

8. If any straight line be drawn perpendicular to the diameter of a given circle, and produced to cut any chord; the rectangle con-

tained by the segments of the diameter will be less or greater than the rectangle contained by the segments of the chord, by the square of the line intercepted between them, according as it is drawn without or within the circle.

9. If a diameter of a circle be produced to bisect a line at right angles, the length of which is the double of a mean proportional between the whole line through the centre and the part without the circle; and from any point in the double of the mean proportional a line be drawn cutting the circle; the sum of the squares of the segments of the double mean proportional will be equal to twice the rectangle contained by this cutting line and the part without the circle.

10. If from a point without a circle two straight lines be drawn, one through the centre to the circumference, and the other perpendicular to it, and on the former a mean proportional be taken between the whole line and the part without the circle; any other line passing through that extremity of the mean proportional which is within the circle, and terminated by the circumference and perpendicular, will be similarly divided.

11. If a chord be drawn parallel to the diameter of a circle, and from any point in the diameter lines be drawn to its extremities; the sum of their squares will be equal to the sum of the squares of the segments of the diameter.

12. If through a point within or without a circle, two straight lines be drawn at right angles to each other, and meeting the circumference; the squares of the segments of them are together equal to the square of the diameter.

13. If from a point without a circle there be drawn two straight lines, one of which touches the circle and the other cuts it, and from the point of contact a perpendicular be drawn to the diameter; the square of the line which touches the circle is equal to the square of that part of the cutting line which is intercepted by the perpendicular, together with the rectangle contained by the segments of that part of it which is within the circle.

14. A straight line drawn from the concourse of two tangents to the concave circumference of a circle is divided harmonically by the convex circumference and the chord which joins the points of contact.

15. If from the extremities of any chord in a circle straight lines be drawn to any point in the circumference meeting a diameter perpendicular to the chord; the rectangle contained by the distances of their points of intersection from the centre is equal to the square described upon the radius.

16. If from any point in the base, or base produced, of the segment of a circle, a line be drawn making therewith an angle equal to the angle in the segment, and from the extremity of the base any line be drawn to the former, and cutting the circumference; the rectangle contained by this line and the part of it within the segment is always of the same magnitude.

17. To determine the locus of the extremities of any number of straight lines drawn from a given point, so that the rectangle contained by each and a segment cut off from each by a line given in position may be equal to a given rectangle.

18. If from a given point two straight lines be drawn containing a given angle, and such that their rectangle may be equal to a given rectilineal figure, and one of them be terminated by a straight line given in position; to determine the locus of the extremity of the other.

19. If from the vertical angle of a triangle two lines be drawn to the base making equal angles with the adjacent sides; the squares of those sides will be proportional to the rectangles contained by the adjacent segments of the base.

20. If a line placed in one circle be made the diameter of a second, the circumference of the latter passing through the centre of the former, and any chord in the former circle be drawn through this diameter perpendicularly; the rectangle contained by the segments made by the circumference of the latter circle will be equal to that contained by the whole diameter and a mean proportional between its segments.

21. If semicircles be described on the segments of the base made by a perpendicular drawn from the right angle of a triangle; they will cut off from the sides, segments which will be in the triplicate ratio of the sides.

22. If from any point in the diameter of a semicircle a perpendicular be drawn, and from the extremities of the diameter lines be

drawn to any point in the circumference, and meeting the perpendicular; the rectangle contained by the segments which they cut off from the perpendicular, will be equal to the rectangle contained by the segments of the diameter.

23. If from the point of bisection and any other point in a given arc of a circle, two parallel lines be drawn, the former terminated by the circumference, the latter by the chord of the arc; the rectangle contained by these two lines will be equal to that contained by the lines which join the latter point with each extremity of the given arc.

24. If two circles cut each other, and from either point of intersection lines be drawn meeting both circumferences; the rectangles contained by the segments of these lines are to one another in the ratio of the perpendiculars drawn from their intersection with the inner circumferences upon the line joining the intersections of the circles.

25. If on opposite sides of any point in the chord of a circle, two lines be taken, one terminating in the chord the other in the chord produced, whose rectangle is equal to that contained by the segments of the chord; and the extremities of the lines so taken be joined to those of any other chord passing through the same point; the line joining their intersections of the circle will be parallel to the first chord.

26. If from two points without a circle two tangents be drawn, the sum of the squares of which is equal to the square of the line joining those points; and from one of them a line be drawn cutting the circle, and two lines from the other point to the intersections with the circumference; the points in which these two lines cut the circle, are in the same straight line with the former point.

27. If from the vertex of a triangle there be drawn a line to any point in the base, from which point lines are drawn parallel to the sides; the sum of the rectangles of each side and its segment adjacent to the vertex will be equal to the square of the line drawn from the vertex, together with the rectangle contained by the segments of the base.

28. If on the chord of a quadrantal arc a semicircle be described; the area of the lune so formed will be equal to the area of the triangle formed by the chord and terminating radii of the quadrant.

29. If from the extremities of the side of a square circles be described with radii equal, the former to the side and the latter to the diagonal of the square; the area of the lune so formed will be equal to the area of the square.

30. If on the sides of a triangle inscribed in a circle, semicircles be described; the two lunes formed thereby will together be equal to the area of the triangle.

31. If on the two longer sides of a rectangular parallelogram as diameters, two semicircles be described towards the same parts; the figure contained by the two remaining sides of the parallelogram and the two circumferences shall be equal to the parallelogram.

32. If two points be taken at equal distances from the extremities of a quadrant, and perpendiculars be drawn from these points to the radius; the mixtilinear space cut off, shall be equal to the sector which stands on the arc between them.

33. If the arc of a semicircle be trisected, and from the points of section lines be drawn to either extremity of the diameter; the difference of the two segments thus made will be equal to the sector which stands on either of the arcs.

34. If a straight line be placed in a circle, and on the radius passing through one extremity, as a diameter, another circle be described; the segments of the two circles cut off by the above straight line will be similar, and in the ratio of four to one.

35. If on any two segments of the diameter of a semicircle semicircles be described; the area included between the three circumferences will be equal to the area of a circle whose diameter is a mean proportional between the segments.

36. If the diameter of a semicircle be divided into any number of parts, and on them semicircles be described; their circumferences will together be equal to the circumference of the given semicircle.

37. If two equal circles cut each other, and from either point of section a line be drawn meeting the two circumferences; the area cut off by the part of this line between the two circumferences will be equal to the area of the triangle contained by that part and lines drawn to its extremities from the other point of section.

38. If two equal circles touch each other externally, and through

the point of contact another be described with the same radius; the area contained by the convex circumferences cut off from the touching circles, and the part of the third without them, is equal to the area of the quadrilateral figure formed by lines drawn from the points of intersection to the point of contact, and to the point where the third circle is cut by a tangent drawn to the point of contact of the two circles.

39. If a straight line be divided into any two parts, and upon the whole and the two parts semicircles be described; and from the point of section a perpendicular be drawn, on each side of which circles are described touching it and the semicircles; these circles will be equal.

SECTION IX. Page 291.

1. **GIVEN** one angle, a side adjacent to it, and the difference of the other two sides; to construct the triangle.

2. **Given** one angle, a side opposite to it, and the difference of the other two sides; to construct the triangle.

3. **Given** the base, and one of the angles at the base; to construct the triangle when the side opposite to the given angle is equal to half the sum of the other side and a given line.

4. **Given** the base of a right-angled triangle, and the sum of the hypotenuse and a straight line, to which the perpendicular has a given ratio; to construct the triangle.

5. **Given** the perpendicular drawn from the vertical angle to the base, and the difference between each side and the adjacent segment of the base made by the perpendicular; to construct the triangle.

6. **Given** the vertical angle, and the base; to construct the triangle when the line drawn from the vertex cutting the base in any given ratio, bisects the vertical angle.

7. **Given** the vertical angle, and one of the sides containing it; to construct the triangle, when the line drawn from the vertex making a given angle with the base, bisects the triangle.

8. Given one angle, a side opposite to it, and the sum of the other two sides; to construct the triangle.

9. Given the vertical angle, the line bisecting the base, and the angle which the bisecting line makes with the base; to construct the triangle.

10. Given the vertical angle, the perpendicular drawn from it to the base, and the ratio of the segments of the base made by it; to construct the triangle.

11. Given the vertical angle, the base, and a line drawn from either of the angles at the base to cut the opposite side in a given ratio; to construct the triangle.

12. Given the perpendicular, the line bisecting the vertical angle, and the line bisecting the base; to construct the triangle.

13. Given the line bisecting the vertical angle, the line bisecting the base, and the difference of the angles at the base; to construct the triangle.

14. Given the vertical angle, and the line drawn to the base bisecting the angle, and the difference between the base and the sum of the sides; to construct the triangle.

15. Given the line bisecting the vertical angle, the perpendicular drawn to it from one of the angles at the base, and the other angle at the base; to construct the triangle.

16. Given the line bisecting the vertical angle, and the perpendiculars drawn to that line from the extremities of the base; to construct the triangle.

17. Given the vertical angle, the difference of the two sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex; to construct the triangle.

18. Given the base, and vertical angle; to construct the triangle, when the square of one side is equal to the square of the base, and three times the square of the other side.

19. Given the base and perpendicular; to construct the triangle, when the rectangle contained by the sides is equal to twice the rectangle contained by the segments of the base made by the line bisecting the vertical angle.

20. In a right-angled triangle, having given the sum of the base and hypotenuse, and the sum of the base and perpendicular; to construct the triangle.

21. Given the perimeter of a right-angled triangle whose sides are in geometrical progression; to construct the triangle.

22. Given the difference of the angles at the base, the ratio of the segments of the base made by the perpendicular, and the sum of the sides; to construct the triangle.

23. Given the difference of the angles at the base, the ratio of the sides, and the length of a third proportional to the difference of the segments of the base made by a perpendicular from the vertex and the shorter side; to construct the triangle.

24. Given the base of a right-angled triangle; to construct it, when parts, equal to given lines, being cut off from the hypotenuse and perpendicular, the remainders have a given ratio.

25. Given one angle of a triangle, and the sums of each of the sides containing it and the third side; to construct the triangle.

26. Given the vertical angle, and the ratio of the sides containing it, as also the diameter of the circumscribing circle; to construct the triangle.

27. Given the vertical angle, and the radii of the inscribed and circumscribing circles; to construct the triangle.

28. Given the vertical angle, the radius of the inscribed circle, and the rectangle contained by the straight lines drawn from the centre of that circle to the angles at the base; to construct the triangle.

29. Given the base, one of the angles at the base, and the point in which the diameter of the circumscribing circle drawn from the vertex meets the base; to construct the triangle.

30. Given the vertical angle, the base, and the difference between two lines drawn from the centre of the inscribed circle to the angles at the base; to construct the triangle.

31. Given that segment of the line bisecting the vertical angle which is intercepted by perpendiculars let fall upon it from the angles at the base; the ratio of the sides; and the ratio of the radius of the

inscribed circle to the segment of the base which is intercepted between the line bisecting the vertical angle and the point of contact of the inscribed circle ; to construct the triangle.

32. Given the line bisecting the vertical angle, and the differences between each side and the adjacent segment of the base made by the bisecting line ; to construct the triangle.

33. Given one of the angles at the base, the side opposite to it, and the rectangle contained by the base and that segment of it made by the perpendicular which is adjacent to the given angle ; to construct the triangle.

34. Given the vertical angle, and the lengths of two lines drawn from the extremities of the base to the points of bisection of the sides ; to construct the triangle.

35. Given the lengths of three lines drawn from the angles to the points of bisection of the opposite sides ; to construct the triangle.

36. Given the segments of the base made by the perpendicular, and one of the angles at the base triple the other ; to construct the triangle.

37. The area and hypotenuse of a right-angled triangle being given ; to construct the triangle.

38. Given one angle, and a line drawn from one of the others bisecting the side opposite to it ; to construct the triangle, when the area is also given.

39. In two similar right-angled triangles, the sum of the base of one and perpendicular of the other is given ; to determine the triangles such that their hypotenuses may contain the right angle of another triangle similar to them, and the sum of the three areas may be equal to a given area.

40. Given the vertical angle, the area, and the distance between the centres of the inscribed circle and the circle which touches the base and the two sides produced ; to construct the triangle.

41. Given the area, the line from the vertex dividing the base into segments which have a given ratio, and either of the angles at the base ; to construct the triangle.

42. Given the difference between the segments of the base made by the perpendicular, the sum of the squares of the sides, and the area; to construct the triangle.

43. Given the base, one of the angles at the base, and the difference between the side opposite to it and the perpendicular; to construct the triangle.

44. Given the vertical angle, the difference of the base and one side, and the sum of the perpendicular drawn from the angle at the base contiguous to that side upon the opposite side and the segment cut off by it from that opposite side contiguous to the other angle at the base; to construct the triangle.

45. Given the base, the difference of the sides, and the segment intercepted between the vertex and a perpendicular from one of the angles at the base upon the opposite side; to construct the triangle.

46. Given the vertical angle, the side of the inscribed square, and the rectangle contained by one side and its segment adjacent to the base made by the angular point of the inscribed square; to construct the triangle.

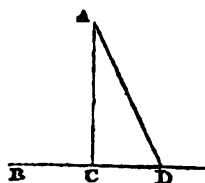


GEOMETRICAL PROBLEMS.

SECT. I.

(1.) *From a given point, to draw the shortest line possible to a given straight line.*

Let A be the given point, and BD the given line. From A let fall the perpendicular AC ; this will be less than any other line AD drawn from A to BD .

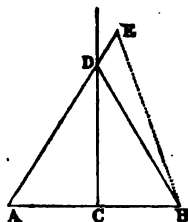


For since AC is perpendicular to BD , the angle ACD is a right angle, therefore the angle ADC is less than a right angle (Eucl. i. 32.) and consequently less than ACD . But the greater angle is subtended by the greater side (Eucl. i. 19.); therefore AD is greater than AC . In the same manner every other line drawn from A to BD may be shewn to be greater than AC ; therefore AC is the least.

(2.) *If a perpendicular be drawn bisecting a given straight line; any point in this perpendicular is at equal*

distances, and any point without the perpendicular is at unequal distances from the extremities of the line.

From C the point of bisection let CD be drawn at right angles to AB ; any point D is at equal distances from A and B .

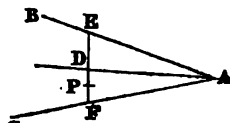


Join AD , DB . Since $AC = CB$ and CD is common, and the angle $ACD = BCD$ being right angles, $AD = DB$. And the same may be proved of lines drawn from any other point in CD to A and B .

But if a point E be taken which is not in CD , join EA cutting the perpendicular in D ; join EB , DB . Then $AD = DB$ from the first part, and AE is equal to AD , DE , that is, to BD , DE , and is therefore greater than BE , (Eucl. i. 20.); therefore, &c.

(3.) *Through a given point to draw a straight line which shall make equal angles with two straight lines given in position.*

Let P be the given point, and BE , CF the lines given in position. Produce BE , CF to meet in A , and bisect the angle BAC by the line AD . From P let fall the perpendicular PD , and produce it both ways to E and F . It will be the line required.

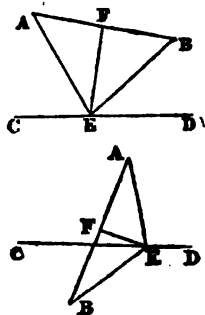


For the angle EAD is equal to the angle FAD , the angles at D right angles, and AD common, therefore (Eucl. i. 26.) the angle AED is equal to the angle AFD ; therefore, &c.

(4.) *From two given points to draw two equal straight lines which shall meet in the same point of a line given in position.*

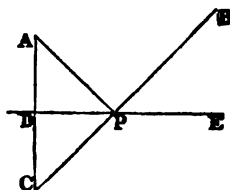
Let A and B be the given points, and CD the given straight line. Join AB , and bisect it in F , and from F draw FE at right angles to AB meeting CD in E ; E is the point required.

Join AE , EB . Since $AF = FB$, and FE is common, and the angles at F are right angles, therefore $AE = EB$.



(5.) *From two given points on the same side of a line given in position, to draw two lines which shall meet in that line, and make equal angles with it.*

Let A and B be the given points, and DE the line given in position. From A let fall the perpendicular AD , and produce it to C making $DC = AD$. Join CB , AP . AP , PB will be the lines required.

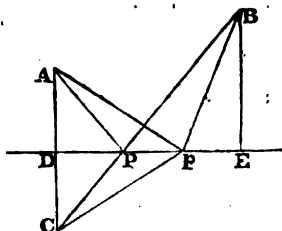


Since $AD = DC$, and DP is common, and the angles at D are right angles, therefore the triangles APD , CPD are equal, and the angle $APD = CPD =$ the vertically opposite angle BPE .

(6.) *From two given points on the same side of a line given in position, to draw two lines which shall meet in a point in this line, so that their sum shall be less*

than the sum of any two lines drawn from the same points and terminated at any other point in the same line.

Let A and B be the given points, and DE the line given in position; from A and B let fall the perpendiculars AD , BE , and produce AD to C making $CD = DA$. Join BC cutting DE in P . Join AP ; AP and PB shall be

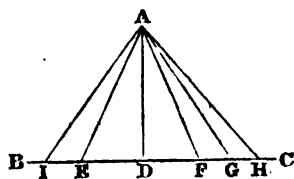


less than any other two lines Ap , pB drawn from A and B to any other point p in the line DE .

For $AD = DC$ and DP is common and the angles at D are right angles, $\therefore AP = PC$. In the same manner, if pC be joined, it may be shewn that $Ap = pC$. Hence AP and BP together are equal to BC , and Ap , pB are equal to Cp , pB . Now (Eucl. i. 20.) BC is less than Bp , pC , and therefore AP , PB are less than Ap , pB ; therefore, &c.

(7.) Of all straight lines which can be drawn from a given point to an indefinite straight line, that which is nearer to the perpendicular is less than the more remote. And from the same point there cannot be drawn more than two straight lines equal to each other, viz. one on each side of the perpendicular.

Let A be the given point, and BC the given indefinite straight line. From A let fall the perpendicular AD , and draw any other lines AF , AG , AH , &c. of which AF is nearer to AD than AG is, and



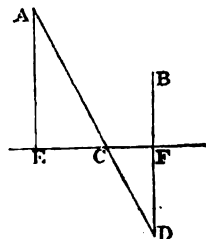
AG than AH ; then AF will be less than AG and AG than AH .

For since the angle at D is a right angle, the angle AFG is greater than a right angle (Eucl. i. 16.), and therefore greater than AGF , hence (Eucl. i. 19.) AG is greater than AF . In the same manner it may be shewn that AH is greater than AG .

And from A there can only be drawn to BC two straight lines equal to each other, viz. one on each side of AD . Make $DE = DF$, and join AE . Then $AE = AF$ (i. 2.). And besides AE no other line can be drawn equal to AF . For, if possible, let $AI = AF$. Then because $AI = AF$ and $AF = AE$, therefore $AI = AE$, i. e. a line more remote is equal to one nearer the perpendicular, which is impossible; therefore AI is not equal to AF . In the same manner it may be shewn that no other but AE can be equal to AF , therefore, &c.

(8.) *Through a given point, to draw a straight line, so that the parts of it intercepted between that point and perpendiculars drawn from two other given points may have a given ratio.*

Let A and B be the points from which the perpendiculars are to be drawn, and C the point through which the line is to be drawn. Join AC , and produce it to D , making $AC : CD$ in the given ratio; join BD , and through C draw ECF perpendicular to BD . ECF is the line required.

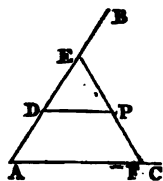


Draw AE parallel to BD , and $\therefore$ perpendicular to

EF. The triangles ACE , DFC , having each a right angle, and the angles at C equal, are equiangular, whence $CE : CF :: AC : CD$, i. e. in the given ratio.

(9.) *From a given point between two indefinite right lines given in position, to draw a line which shall be terminated by the given lines, and bisected in the given point.*

Let AB , AC be the given lines, meeting in A . From P the given point draw PD parallel to AC one of the lines, and make $DE = DA$. Join EP , and produce it to F ; then will EF be bisected in P .



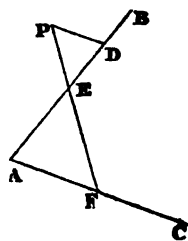
For since DP is parallel to AF , (Eucl. vi. 2.)

$EP : PF :: ED : DA$, i. e. in a ratio of equality.

COR. If it be required to draw a line through P which shall be terminated by the given lines, and divided in any given ratio in P , draw PD parallel to AC , and take $AD : DE$ in the given ratio, and draw EPF , it will be the line required.

(10.) *From a given point without two indefinite right lines given in position; to draw a line such that the parts intercepted by the point and the lines may have a given ratio.*

Let AB , AC be the given lines, and P the given point. Draw PD parallel to AC , and take $AD : DE$ in the given ratio. Join PE , and produce it to F . Then $PF : PE$ will be in the given ratio.



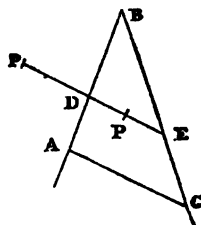
For the triangles PDE and AEF are similar, having the angles at E equal, as also the angles PDE , EAF , (Eucl. i. 39.)

$$\therefore FE : EP :: AE : ED$$

and comp. $PF : PE :: AD : DE$, i.e. in the given ratio.

(11.) *From a given point to draw a straight line, which shall cut off from lines containing a given angle, segments that shall have a given ratio.*

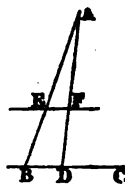
Let ABC be the given angle, and P the given point, either without or within. In BA take any point A , and take $AB : BC$ in the given ratio. Join AC , and from P draw PDE parallel to AC . PDE is the line required.



For since DE is parallel to AC , (Eucl. vi. 2.) $DB : BE :: AB : BC$, i.e. in the given ratio.

(12.) *If from a given point any number of straight lines be drawn to a straight line given in position; to determine the locus of the points of section which divide them in a given ratio.*

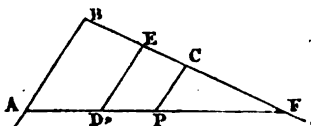
Let A be the given point, and BC the line given in position. From A draw any line AB , and divide it at E in the given ratio; through E draw EF parallel to BD ; it is the locus required.



From A draw any other line AD meeting EF in F ; then (Eucl. vi. 2.) $AF : FD :: AE : EB$, i.e. in the given ratio. In the same manner any other line drawn from A to BD will be divided in the given ratio by EF , which therefore is the locus required.

(13.) *A straight line being drawn parallel to one of the lines containing a given angle and produced to meet the other; through a given point within the angle, to draw a line cutting the other three, so that the part intercepted between the two parallel lines may have a given ratio to the part intercepted between the given point and the other line.*

Let $\angle ABC$ be the given angle, DE parallel to AB , and P the given point.



From P draw PC parallel to DE or AB , and take $BE : CF$ in the given ratio. Join FP and produce it to A ; APF is the line required.

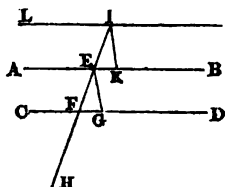
For since DE and CP are parallel to AB ,

$$AD : EB :: DF : EF :: PF : CF$$

$\therefore$ alt. $AD : PF :: EB : CF$ i. e. in the given ratio.

(14.) *Two parallel lines being given in position; to draw a third, such that, if from any point in it lines be drawn at given angles to the parallel lines, the intercepted parts may have a given ratio.*

Let AB, CD be the given parallel lines; in AB take any point E ; and draw EF, EG making angles equal to the given angles; produce EF , and take $EH : EG$ in the given ratio; produce FE to I so that $FI : IE :: HE : EF$; through I draw LI parallel to AB ; it is the line required.



Draw IK parallel to EG ; then the triangles IEK, EFG are equiangular,

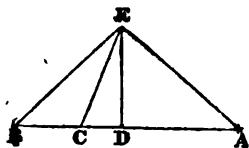
$$\therefore IE : IK :: EF : EG$$

$$\text{but } FI : IE :: HE : EF;$$

$\therefore$ *ex æqu.* $FI : IK :: HE : EG$, i. e. in the given ratio; and $IKE = EGF$ which is one of the given angles, and by construction IFG is equal to the other. Also lines drawn from any point in LI , making with AB and CD angles equal to the given angles will be parallel and equal to FI , IK , and $\therefore$ in the given ratio.

(15.) *If three straight lines drawn from the same point and in the same direction be in continued proportion, and from that point also a line equal to the mean proportional be inclined at any angle; the lines joining the extremity of this line and of the proportionals will contain equal angles.*

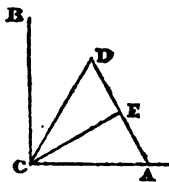
Let $AB : AC :: AC : AD$, and from A let AE be drawn equal to AC , inclined at any angle to AB ; join EB, EC, ED ; the angle BEC is equal to the angle CED .



For since $AB : AC :: AC : AD$, and $AE = AC$;
 $\therefore AB : AE :: AE : AD$, i. e. the sides about the angle A are proportional, and $\therefore$ the triangles AEB, AED are similar, and the angle AED is equal to EBA . Also since $CA = AE$, the angle $AEC = ECA$; but ECA is equal to the two BEC, EBC , (Eucl. i. 32.) $\therefore$ also AEC is equal to the two BEC, EBC ; of which DEA is equal to EBC ; $\therefore$ the remainder DEC is equal to the remainder BEC .

(16.) *To trisect a right angle.*

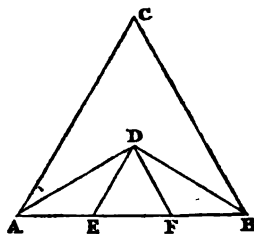
Let ACB be a right angle. In CA take any point A , and on CA describe an equilateral triangle ACD , and bisect the angle DCA by the straight line CE ; the angles BCD , DCE , ECA are equal to one another.



For the angle DCA being one of the angles of an equilateral triangle is one third of two right angles, and therefore equal to two thirds of a right angle BCA ; consequently BCD is one third of BCA ; and since the angle DCA is bisected by CE , the angles DCE , ECA are each of them equal to one third of a right angle, and are therefore equal to BCD and to each other.

(17.) *To trisect a given finite straight line.*

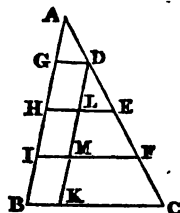
Let AB be the given straight line. On it describe an equilateral triangle ABC ; bisect the angles CAB , CBA by the lines AD , BD meeting in D , and draw DE , DF parallel to CA and CB respectively. AB will be trisected in E and F .



Because ED is parallel to AC , the angle $EDA = DAC = DAE$ and therefore $AE = ED$. For the same reason $DF = FB$. But DE being parallel to CA and DF to CB , the angle DEF is equal to the angle CAB , and DFE to CBA , and therefore $EDF = ACB$; and hence the triangle EDF is equiangular, and consequently equilateral; therefore $DE = EF = FD$, and hence $AE = EF = FB$, and AB is trisected.

(18.) *To divide a given finite straight line into any number of equal parts.*

Let AB be the given straight line. Let AC be any other indefinite straight line making any angle with AB , and in it take any point D , and take as many lines DE, EF, FC &c. each equal to AD as the number of parts into which AB is to be divided. Join CB , and draw DG, EH, FI &c. parallel to BC ; and therefore parallel to each other; and draw DK parallel to AB .

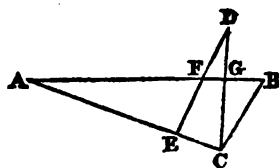


Then because GD is parallel to HE one of the sides of the triangle AHE , $AG : GH :: AD : DE$; hence $AG = GH$. For the same reason $DL = LM$. But DM being parallel to GI , and DG, LH to MI , the figures DH, HM are parallelograms; therefore $DL = GH$ and $LM = HI$, consequently $GH = HI$. In the same manner it may be shewn that $HI = IB$; and so on, if there be any other parts; therefore AG, GH, HI, IB , &c. are all equal, and AB is divided as was required.

COR. If it be required to divide the line into parts which shall have a given ratio; take AD, DE, EF , &c. in the given ratio, and proceed as in the proposition.

(19.) *To divide a given finite straight line harmonically.*

Let AB be the given straight line. From B draw any straight line BC , and join AC ; and from any point E in AC draw ED parallel to CB , and make $FD =$



FE, join *DC* cutting *AB* in *G*. *AB* is harmonically divided in *G* and *F*.

Since *BC* is parallel to *FD*, the angle *BCG* is equal to *GDF* and the vertically opposite angles at *G* are equal; therefore the triangles *DGF*, *BGC* are similar,

$$\text{and } BC : BG :: FD : FG.$$

But *FE* being parallel to *BC*,

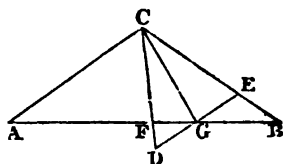
$$(\text{Eucl. vi. 2.}) \quad AB : BC :: AF : FE = FD.$$

$$\therefore \text{ex æquali, } AB : BG :: AF : FG$$

$$\text{or } AB : AF :: BG : FG.$$

(20.) *If a given finite straight line be harmonically divided, and from its extremities and the points of division lines be drawn to meet in any point, so that these from the extremities of the second proportional may be perpendicular to each other, the line drawn from the extremity of this proportional will bisect the angle formed by the lines drawn from the extremities of the other two.*

Let the straight line *AB* be divided harmonically in the points *G* and *F*, and let the lines *AC*, *BC*, *GC*, *FC* be drawn to any point *C*, so that *GC* may be perpendicular to *CA*; the angle *BCF* will be bisected by *CG*.



Through *G* draw *EGD* parallel to *CA*, meeting *CF* in *D*; then *EG* being parallel to *AC*, the triangles *EGB*, *ACB* are similar; as also the triangles *ACF*, *DFG*; hence

$$AF : AC :: FG : DG$$

$$\text{but } AB : AF :: GB : GF,$$

$$\therefore \text{ex æquo } AB : AC :: BG : GD.$$

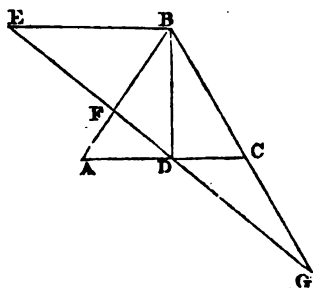
But $AB : AC :: GB : GE$

$\therefore$ (Eucl. v. 15.) $BG : GD :: BG : GE$,

and therefore $GD = GE$, and GC is common, and the angles at G are right angles, therefore the angle $DCG = GCE$, and FCB is bisected by CG .

(21.) *If a straight line be drawn through any point in the line bisecting a given angle, and produced to cut the sides containing that angle, as also a line drawn from the angle perpendicular to the bisecting line; it will be harmonically divided.*

Let the angle ABC be bisected by the line BD , and through any point D in this line draw $GDFE$ meeting the sides in G and F , and BE a perpendicular to BD in E ; then will $EG : EF :: GD : FD$.



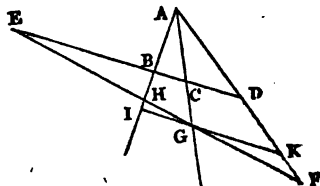
For through D draw AC parallel to BE and therefore perpendicular to BD ; then the angles ADB , CDB being right angles are equal, and $ABD = CBD$, and BD is common to the triangles ADB , CDB , $\therefore AD = DC$. But DC being parallel to EB ,

$EG : GD :: EB : DC :: EB : AD :: EF : FD$,
since the triangles EFB and AFD are similar,
 $\therefore EG : EF :: GD : FD$.

(22.) *If from a given point there be drawn three straight lines forming angles less than right angles, and*

from another given point without them a line be drawn intersecting the others so as to be harmonically divided; then will all lines drawn from that point meeting the three lines be harmonically divided.

From A let AB, AC, AD be drawn making each of the angles BAC, CAD less than a right angle, and from a given point E let EBD be drawn so as to be harmoni-



cally divided in C and B ; then will any other line EF be harmonically divided in G and H .

Through G draw IK parallel to BD ,

then $DC : CB :: KG : GI$,

But $DC : CB :: DE : EB$

$\therefore$ (Eucl. v. 15.) $DE : EB :: KG : GI$

and alt. $DE : KG :: EB : GI$

and since DE is parallel to GK , (Eucl. vi. 2.)

$DE : KG :: EF : FG$

and EB being parallel to GI ,

$\therefore EB : GI :: EH : HG$,

whence (Eucl. v. 15.)

$EF : FG :: EH : HG$

and alt. $EF : EH :: FG : HG$.

(23.) *If a straight line be divided into two equal, and also into two unequal parts, and be produced, so that the part produced may have to the whole line so produced the same ratio that the unequal segments of the line have to each other; then shall the distances of the point of unequal section from one extremity of the given line,*

from its middle point, from the extremity of the part produced, and from the other extremity of the given line, be proportionals.

Let AB be divided into two equal parts in C and into two unequal parts in D , and produced to E , so that $BE : EA :: BD : DA$; then will $AD : DC :: ED : DB$.

For since $BE : EA :: BD : DA$

inv. $AE : EB :: AD : DB$

div. $AB : BE :: 2CD : DB$

and $AC : BE :: CD : DB$

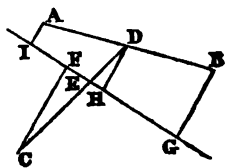
alt. $AC : CD :: BE : BD$

$\therefore$ comp. $AD : DC :: ED : DB$.

COR. The converse may easily be proved to be true.

(24.) *Three points being given; to determine another, through which if any straight line be drawn, perpendiculars upon it from two of the former, shall together be equal to the perpendicular from the third.*

Let A, B, C be the three given points. Join AB , and bisect it in D . Join CD , from which cut off DE equal to a third part of it. E is the point required.

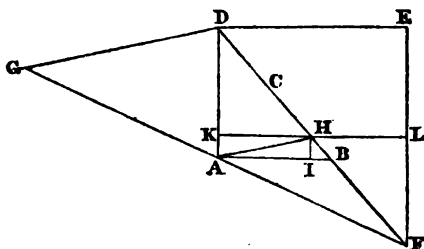


Through E , let any line FG be drawn, and let fall on it the perpendiculars AI, BG, DH, CF ; then the angles at F and H being right angles, and the vertical angles at E equal, the triangles CFE, DHE are equiangular,

$\therefore FC : DH :: CE : ED :: 2 : 1$,
 $\therefore FC = 2 DH$; but since AI, BG, DH are parallel, and
 $AD = DB$, $\therefore AI + BG = 2 DH = FC$.

(25.) *From a given point in one of two straight lines given in position, to draw a line to cut the other, so that if from the point of intersection a perpendicular be let fall upon the former, the segment intercepted between it and the given point, together with the first drawn line may be equal to a given line.*

Let AB, BC be the lines given in position, and A the



given point. Draw AD perpendicular to AB , and meeting BC in D ; draw DE parallel to AB , and equal to the given line. And draw EF parallel to AD , meeting CB in F . Join FA , and produce it, and from D draw $DG = DE$, meeting FG in G , and draw AH parallel to DG , and let fall the perpendicular HI ; AH and AI together are equal to the given line.

Through H draw KL parallel to DE ; then, since GD is parallel to AH and HL to DE ,

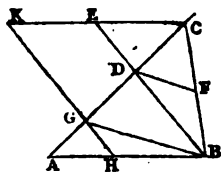
$$\therefore DG : AH :: FD : FH :: DE : HL,$$

$$\text{but } DG = DE, \therefore AH = HL,$$

$$\therefore AH + AI = KL = DE = \text{the given line.}$$

(26.) One of the lines which contain a given angle, is also given. To determine a point in it such, that if from thence to the indefinite line there be drawn a line having a given ratio to that segment of it which is adjacent to the given angle; the line so drawn, and the other segment of the given line, may together be equal to another given line.

Let AB be the given line, and BAC the given angle. From B draw BD to AC , such that it may be to AB in the given ratio*; produce it till $BE =$ the other given line. Through E draw EC parallel to AB , meeting AC in C . Join BC , and draw DF so that it may $= DE$, and draw BG , GH respectively parallel to FD , EB ; H is the point required.



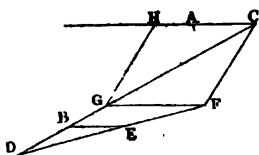
For produce HG to meet CE in K ;
 then (Eucl. vi. 2.) $ED : KG :: CD : CG :: DF : BG$,
 but $ED = DF$, $\therefore KG = BG$,
 and $HG + GB = HG + GK = BE =$ the given line,
 and $HG : HA :: BD : AB$ i. e. in the given ratio.

(27.) Two straight lines and a point in each being given in position; to determine the position of another point in each, so that the straight line joining these latter points may be equal to a given line, and their respective distances from the former points in a given ratio.

Let A and B be the given points in the lines AC , BD which are given in position, and produced to meet

* That is, the given ratio must be less than that of AB to the perpendicular on AD .

in C . Take $BD : AC$ in the given ratio, and from B draw BE parallel and equal to AC . Join DE , and produce it to meet CF drawn at any angle from C , equal to the given line; draw FG parallel to EB , and from G draw GH parallel to FC ; G and H are the points required,



For BE being parallel to GF ,

$$DG : GF :: DB : BE,$$

$$\text{or } DG : HC :: DB : AC,$$

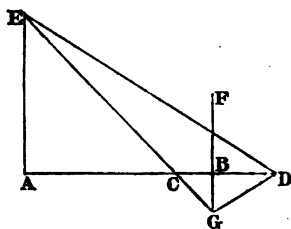
$\therefore$ (Eucl. v. 19. Cor.)

$$BG : AH :: DB : AC \text{ in the given ratio,}$$

$$\text{and } HG = CF = \text{the given line.}$$

(28.) *If a straight line be divided into any two parts, and produced so that the segments may have the same ratio that the whole line produced has to the part produced, and from the extremities of the given line perpendiculars be erected; then any line drawn through the point of section, meeting these perpendiculars, will be divided at that point into parts, which have the same ratio that those lines have, which are drawn from the extremity of the produced line to the points of intersection with the perpendiculars.*

Let AB be divided into any two parts in C and produced to D so that $AC : CB :: AD : DB$, and from A and B let AE , BF be drawn perpendiculars to AB , and through C let any line ECG be drawn meeting them in E and G , and join DE , DG ; then $DE : DG :: CE : CG$.



For because $AC : CB :: AD : DB$

and $EA : BG :: AC : CB$,

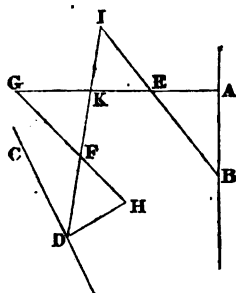
(by sim. tri. ACE, BCG)

$\therefore$ (Eucl. v. 11.) $EA : BG :: DA : DB$,

$\therefore$ (Eucl. vi. 6.) the triangles EAD, GDB are equiangular,
and $ED : DG :: AE : BG :: CE : CG$.

(29.) *From two given points, to draw two straight lines which shall contain a given angle, and meet two lines given in position, so that the parts intercepted between those points and the lines may have a given ratio.*

Let AB, CD be the lines given in position, and E, F the given points. From E draw EA perpendicular to AB , and make the angle AGF equal to the given angle. In GF produced take FH such, that the ratio of $EA : FH$ may be the same as the given ratio. Draw HD perpendicular to GH meeting CD in D . Draw DFI and BEI to include the given angle. These are the lines required.



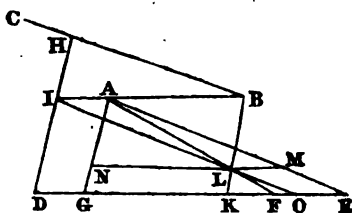
For, since the angles FGE, FIE are equal, as also FKG, EKI , $\therefore GFK, IEK$ or their vertically opposite angles DFH, AEB are equal, and the angles at H and A are right angles, $\therefore$ the triangles FDH, AEB are equiangular, and

$EB : FD :: EA : FH$, i. e. in the given ratio.

(30.) *The length of one of two lines which contain*

a given angle being given; to draw from a given point without them a straight line which shall cut the given line produced, so that the part produced may be in a given ratio to the part cut off from the indefinite line.

Let AB be the given line, and ABC the given angle; and D the given point. Draw AE , DE parallel to BC , BA respectively; and take $EF : EA$ in the given ratio. Divide



DF so that $FE : DG :: FG : AB$. Join AG ; and draw DH parallel to AG , and it will be the line cutting BC in H , and BA produced in I , as was required.

Join AF ; and draw BK parallel to AG cutting AF in L ; and draw LM parallel to KE cutting AE in M and AG in N .

Then $FE : LM :: GF : (NL =) AB$

and $FE : DG :: FG : AB$ by construction;

$\therefore LM = DG = IA$; if therefore ILO be drawn, IL must be equal and parallel to AM , and IO to AE (Eucl. i. 33.). In the same manner it is evident that $HB = IL = AM$; and by similar triangles AFE , ALM ,

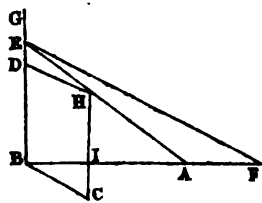
$$FE : EA :: LM : MA$$

$$:: IA : HB,$$

$$\therefore IA : HB \text{ in the given ratio.}$$

(31.) From two given straight lines to cut off two parts, which may have a given ratio; so that the ratio of the remaining parts may also be equal to the ratio of two other given lines.

Let AB be one of the given lines; draw BG to make any angle with AB , and let BD be equal to the other given line. Take $AB : BE$ in the given ratio of the remaining parts, and $BF : BE$ in the given ratio of the parts to be cut off. Join AE, FE ; and draw DH and BC parallel to EF , and HC parallel to DB meeting BC in C , and AB in I .



Then (Eucl. vi. 2.) $AI : IH :: AB : BE$ in the given ratio of the remainders; and the triangles BCI, BFE having the angle $CBI =$ the alternate angle BFE , and $CIB = FBE$, are equiangular,

$$\therefore BI : IC :: BF : BE,$$

in the ratio of the parts to be cut off; and $AB, HC (= DB)$ are the given lines.

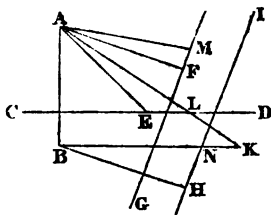
(92.) *Three lines being given in position; to determine a point in one of them, from which if two lines be drawn at given angles to the other two, the two lines so drawn may together be equal to a given line.*

Let AB, AC, BC be the three lines given in position, take $AD =$ the given line, and making with AB an angle equal to one of the given angles. Through D draw Dba parallel to AB , and meeting AC and BC in a and b . Draw AE to meet CB in E making the angle $AEC =$ the given angle to be made by the line to be drawn, with BC . In AE take $Ad = AD$, and join ad cutting BC in F . Draw FG parallel to EA meeting AC in G , which is the point required.

Draw any other line AB , and make the angle $BAF = DAE$. Then the angle $BAD = FAE$ and $ADB = AEF$, $\therefore$ the triangles ABD , AEF are equiangular, whence $AB : AF :: AD : AE$, in the given ratio. The same may be proved of any other lines drawn from A and containing an angle equal to the given angle, and one of them terminated in BC .

(34.) *If from two given points, straight lines be drawn, containing a given angle, and from each of them segments be cut off, having a given ratio; and the extremities of the segments of the lines drawn from one of the points be in a straight line given in position; to determine the locus of the extremities of the segments of lines drawn from the other.*

Let A and B be the given points, and CD the line given in position. From A to CD draw any line AE . Make the angle $EAF =$ the given angle, and $AE : AF$ in the given ratio, and let FG be the locus of the points F (i. 33.). Draw BH equal and parallel to AF , and through H draw HI parallel to GF . It is the locus required.



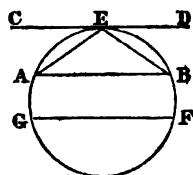
Draw any lines AK , BK containing the angle at $K =$ the given angle. Make the angle $LAM =$ the given angle; $AL : AM$ in the given ratio, and M is in the line GF . And since AF is parallel to BH , and FM to HN , and BK to AM (since the angles BKA , LAM are equal) and $AF = BH$, $\therefore$ the triangles BHN , AFM are similar and equal, $\therefore AM = BN$;

but $AL : AM$ is equal to the given ratio,
 $\therefore$ also $AL : BN$ is equal to the given ratio.
 And the same may be proved of any other lines drawn in
 the same manner.

SECT. II.

(1.) *If a straight line be drawn to touch a circle, and be parallel to a chord; the point of contact will be the middle point of the arc cut off by that chord.*

Let CD be drawn touching the circle ABE in the point E , and parallel to the chord AB ; E is the middle point of the arc AEB .



Join AE, EB . The angle BAE is equal to the alternate angle CEA , and therefore to the angle EBA in the alternate segment, whence $AE = EB$, and (Eucl. iii. 28.) the arc AE is equal to the arc EB .

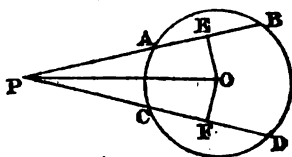
COR. 1. Parallel lines placed in a circle cut off equal parts of the circumference.

If FG be parallel to AB ; the arc $EF = EG$, whence $AG = BF$.

COR. 2. The two straight lines in a circle, which join the extremities of two parallel chords are equal to each other. For if AB, FG be parallel, the arcs AG, BF are equal, therefore (Eucl. iii. 29.) the straight lines AG, BF are also equal.

(2.) *If from a point without a circle, two straight lines be drawn to the concave part of the circumference, making equal angles with the line joining the same point and the centre, the parts of the lines which are intercepted within the circle are equal.*

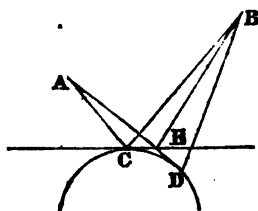
From the point P without the circle ABC let two lines PB , PD be drawn making equal angles with PO , the line joining P and the centre; AB shall be equal to CD .



Let fall the perpendiculars OE , OF ; then since the angle at E is equal to the angle at F , and $EPO = FPO$, and the side PO , opposite to one of the equal angles in each is common, $\therefore OE = OF$, and consequently (Eucl. iii. 14.) $AB = CD$.

(3.) *Of all straight lines which can be drawn from two given points to meet on the convex circumference of a given circle; the sum of those two will be the least, which make equal angles with the tangent at the point of concurrence.*

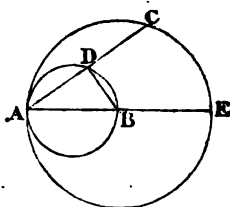
Let A and B be two given points, CE a tangent to the circle at C , where the lines AC , BC make equal angles with it; and let lines AD , BD be drawn from A and B to any other point D on the convex circumference; AC and CB together are less than AD , DB together.



Let AD meet the tangent in E . Join EB ; then (i. 6.) AC and CB together are less than AE and EB ; but AE , EB are less than AD , DB (Eucl. i. 19.), $\therefore$ *a fortiori* AC , CB are less than AD , DB . And the same may be proved of lines drawn to every other point in the convex circumference.

(4.) *If a circle be described on the radius of another circle; any straight line drawn from the point where they meet, to the outer circumference, is bisected by the interior one.*

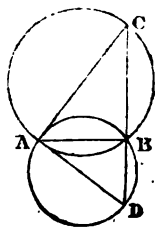
Let ADB be a circle described on the radius AB of the circle ACE . Draw any line AC meeting the circle ADB in D ; AD is equal to DC .



Join DB . Then the angle ADB being in a semicircle is a right angle; and therefore BD being drawn from the centre B of the circle ACE bisects AC (Eucl. iii. 3.).

(5.) *If two circles cut each other, and from either point of intersection diameters be drawn; the extremities of these diameters and the other point of intersection shall be in the same straight line.*

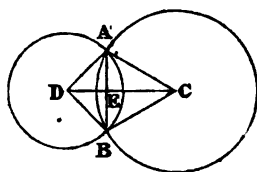
Let the two circles ABC , ABD cut each other in A and B ; draw the diameters AC , AD , and join BC , BD ; CB and BD are in the same straight line.



Join AB ; the angles ABC , ABD being angles in semicircles are right angles, and therefore (Eucl. i. 13.) CB and BD are in the same straight line.

(6.) *If two circles cut each other, the straight line joining their points of intersection, is bisected at right angles by the straight line joining their centres.*

Let the two circles whose centres are C and D cut each other in A and B ; join AB , DC . DC bisects AB at right angles.



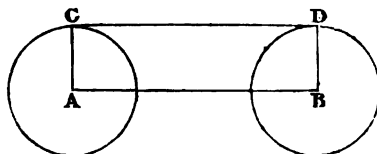
Join BD , DA , AC , CB .

Since $AD = DB$, and DC is common to the triangles ADC , BDC , and the base $AC = CB$, $\therefore$ (Eucl. i. 8.) the angle $ADE = BDE$. Hence the two sides AD , DE are equal to the two BD , DE , and the included angles are equal, $\therefore$ (Eucl. i. 4.) $AE = EB$, and the angle $DEA = DEB$, and being adjacent, they are right angles, i. e. DC bisects AB at right angles.

(7.) *To draw a straight line which shall touch two given circles.*

1. If the circles be equal.

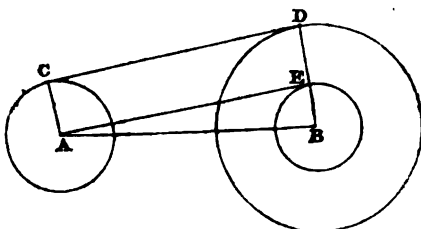
Let A and B be the centres, join AB ; and from A



and B draw AC , BD at right angles to it; join CD . Then AC being parallel and equal to DB ; CD is parallel to AB , $\therefore$ $CABD$ is a rectangular parallelogram; and the angles at C and D being right angles, CD is a tangent to both circles (Eucl. iii. 16. Cor.).

2. If the circles be unequal, and the line be required to touch them on the same side of the line joining the centres.

Let A and B be the centres ; join AB ; and with the

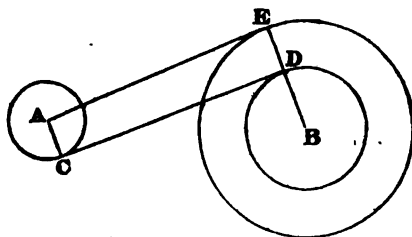


centre B , and distance equal to the difference of the given radii, describe a circle; and from A draw AE touching it. Join BE , and produce it to D ; draw AC parallel to BD , and join CD .

Then AC being parallel and equal to DE , CD is equal and parallel to AE , $\therefore ACDE$ is a parallelogram ; and the angle AEB being a right angle, AED is also a right angle ; hence the angles at C and D are right angles, and therefore CD touches both circles.

3. If the line be required to touch them on opposite sides of the line joining the centres.

With the centre B and radius equal to the sum of

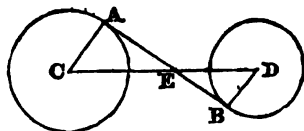


the given radii describe a circle, to which from A draw a tangent AE . Join BE , and let it cut the given circle in D . Draw AC parallel to BE ; join CD .

Then AC being equal and parallel to ED , $ACDE$ is a parallelogram; and the angle AED being a right angle, the angles at C and D are right angles, and therefore CD touches both circles.

(8.) *If a line touching two circles cut another line joining their centres, the segments of the latter will be to each other as the diameters of the circles.*

Let the line AB touch the circles, whose centres are C and D , in A and B , and cut CD in the point E ; CE will be to ED in the ratio of the diameters of the circles.

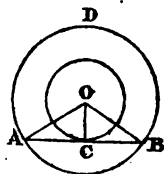


Join CA , BD . Then the angles at A and B are right angles, and the angles at E are vertically opposite, therefore the triangles AEC , BED are equiangular, and consequently

$$CE : ED :: CA : BD \\ :: 2CA : 2BD.$$

(9.) *If a straight line touch the interior of two concentric circles, and be placed in the outer; it will be bisected at the point of contact.*

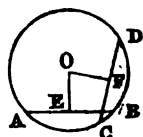
Let AB touch the interior of two circles, whose common centre is O , in the point C ; AB is bisected in C .



Join OC ; then (Eucl. iii. 18.) the angles at C are right angles; and OC drawn from the centre of the circle ADB at right angles to AB , bisects it (Eucl. iii. 3.).

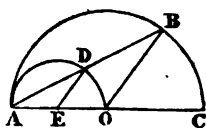
(10.) *If any number of equal straight lines be placed in a circle; to determine the locus of their points of bisection.*

Let there be any number of lines AB , CD , placed in the circle whose centre is O , and let them be bisected in E , F ; join OE , OF ; then (Eucl. iii. 14.) these lines are equal, and therefore the locus will be a circle whose centre is O , and radius equal to the distance of the points of bisection from O .



(11.) *If from a point in the circumference of a circle any number of chords be drawn; the locus of their points of bisection will be a circle.*

From the given point A let any chord AB be drawn in the circle, whose centre is O ; bisect it in D . Join AO , BO , and draw DE parallel to BO .

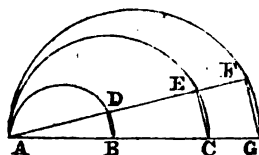


Then DE being parallel to BO , the triangles ADE , ABO are similar, and BO is equal to AO , $\therefore DE = EA$; but $AE : AO :: AD : AB$ (Eucl. vi. 2.), whence $AE = \frac{1}{2}AO$, $\therefore ED = EA = \frac{1}{2}AO$, and the locus will be a circle described on AO as a diameter.

(12.) *If on the radius of a given semicircle, another semicircle be described, and from the extremity of the diameters any lines be drawn cutting the circumferences, and produced so that the part produced may always have*

a given ratio to the part intercepted between the two circumferences; to determine the locus of the extremities of these lines.

On AB the radius of the semicircle AEC let a semicircle ADB be described; and from A draw any line ADE , which produce till $EF : ED$ in the given ratio.

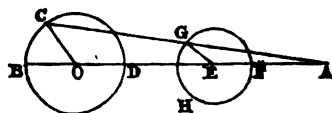


Produce AC to G , making $CG : CB$, in the given ratio, and join DB, EC, FG ;

then since $EF : ED :: GC : CB$,
 $\therefore FE : GC :: ED : CB :: DA : AB :: EA : CA$,
 whence (Eucl. vi. 2.) FG is parallel to CE and DB , and the angle AFG is a right angle, and is in a semicircle whose diameter is AG ; hence the locus required is a semicircle.

(13.) If from a given point without a given circle, straight lines be drawn, and terminated by the circumference; to determine the locus of the points which divide them in a given ratio.

Let A be the given point and BCD the given circle. Find O its centre and join AO , and divide it in E , so



that $AO : AE$ in the given ratio; and find a point F , so that EF may be to OD in the given ratio; and with the centre E and radius EF describe a circle; it will be the locus required.

Draw any line AGC ; join OC, EG . Since $AO : AE$ in a given ratio, as also $OD : EF$;

$$\therefore OC : EG :: AO : AE,$$

hence OC is parallel to EG ,
and $AC : AG :: OC : EG$, i. e. in the given ratio.

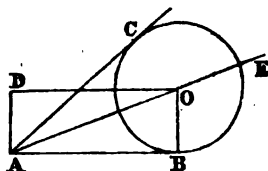
In the same manner it may be shewn that every line drawn from A to BCD will be divided by the circumference of the circle GFH in the same ratio, i. e. GFH will be the locus required.

(14.) *Having given the radius of a circle; to determine its centre, when the circle touches two given lines which are not parallel.*

Let BA, AC be the two lines which touch the circle, whose radius is given.

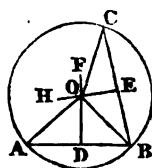
Bisect the angle BAC by the line AE , the centre of the circle will be in this line (Eucl. iv. 4.)

From A draw AD at right angles to AB , and make it equal to the given radius; through D draw DO parallel to AB meeting AE in O ; then the centre of the circle being in this line also, must be at the point of intersection O .



(15.) *Through three given points which are not in the same straight line, a circle may be described; but no other circle can pass through the same points.*

Let A, B, C be the three given points. Join AB, BC , and bisect them in D and E ; from which points draw DO, EO at right angles to them; these lines will meet in some point O ; for if not, they are



parallel, and therefore AB , BC must be parallel, which is contrary to the supposition. Join AO , BO , CO .

Since $AD = DB$, and DO is common, and the angles at D equal, $\therefore AO = BO$. In the same manner it may be shewn that $BO = CO$; and the three lines OA , OB , OC being equal, a circle described from the centre O at the distance of any one of them will pass through the extremities of the other two.

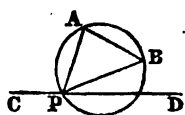
And besides this, no other circle can pass through A , B , C : for if it could, its centre would be in DF and EH , and $\therefore$ in their intersection; but two right lines cut each other only in one point, $\therefore$ only one circle can be described.

(16.) *From two given points on the same side of a line given in position, to draw two straight lines which shall contain a given angle, and be terminated in that line.*

Let A and B be the given points, and CD the given line.

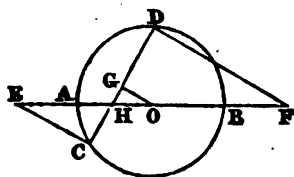
Join AB , and on it describe a segment of a circle containing an angle equal to the given angle, and (if the problem be possible) meeting CD in P ; P is the point required.

For join PA , PB ; the angle APB being in the segment is equal to the given angle.



(17.) *If from the extremities of any chord in a circle perpendiculars be drawn, meeting a diameter; the points of intersection are equally distant from the centre.*

At C and D the extremities of the chord CD , let perpendiculars to it be drawn meeting a diameter AB in E and F ; E and F are equally distant from the centre O .



Draw OG perpendicular to CD , and therefore bisecting it; then OG is parallel to DF ;

whence $GD : OF :: HG : HO :: HC : HE$

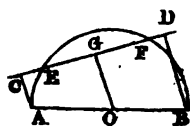
since the triangles HGO , HEC are equiangular;

$\therefore$ (Eucl. v. 18, 15.) $DG : OF :: GC : OE$

but $GD = GC$, $\therefore OF = OE$.

(18.) *If from the extremities of the diameter of a semicircle perpendiculars be let fall on any line cutting the semicircle; the parts intercepted between those perpendiculars and the circumference are equal.*

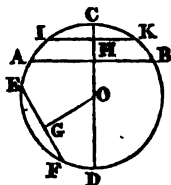
From A and B , the extremities of the diameter AB , let AC , BD be drawn perpendicular to any line CD cutting the semicircle in E and F ; CE is equal to FD .



From O the centre draw OG perpendicular to CD , it will be parallel to AC and BD ,
whence $CG : GD :: AO : OB$, i. e. in a ratio of equality.
But (Eucl. iii. 3.) $EG = GF$, and $\therefore CE = FD$.

(19.) *In a given circle to place a straight line parallel to a given straight line, and having a given ratio to it.*

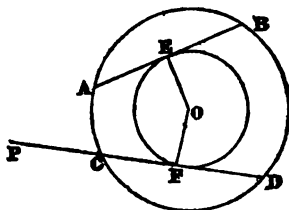
Let AB be the given line in the circle ABC whose centre is O . Draw the diameter CD at right angles to AB : and taking a line EF which has to AB the given ratio (Eucl. vi. 12.), place it in the circle ABC ; bisect it in G and join OG ; make $OH = OG$, and through H , draw IK parallel to AB ; IK is the line required.



For since $OG = OH$, $\therefore$ (Eucl. iii. 14.) $IK = EF$, and $EF : AB$ in the given ratio; $\therefore IK : AB$ in the given ratio.

(20.) *Through a given point, either without or within a given circle, to draw a straight line, the part of which intercepted by the circle, shall be equal to a given line, not greater than the diameter of the circle.*

Let P be the given point without the circle ABC , whose centre is O . In the circle place a straight line AB equal to the given straight line; which bisect in E ; and join OE . With the centre O and radius OE describe a circle; this will touch AB in E , since the angles at E are right angles (Eucl. iii. 3.); from P draw PCD touching the circle in F . PCD is the line required.

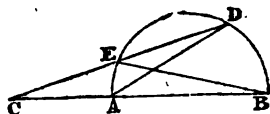


Join OF . Then OF being equal to OE , CD will be equal to AB (Eucl. iii. 14.), i. e. to the given line.

(21.) *From a given point in the diameter of a semi-circle produced, to draw a line cutting the semicircle, so*

that lines drawn from the points of intersection to the extremities of the diameter, cutting each other, may have a given ratio.

Let C be the given point in the diameter BA produced. Make $BC : CD$ in the given ratio; and from the points E and D , in which CD cuts the semicircle, draw EB , AD to the extremities of the diameter. CD is the line required.

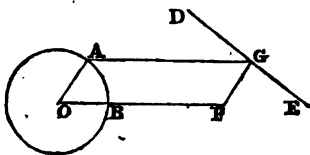


Since the angles EDA , EBA in the same segment are equal, and the angle at C common to the two triangles ACD , CEB , the triangles are equiangular, whence

$BE : AD :: BC : CD$, i.e. in the given ratio.

(22.) From the circumference of a given circle to draw to a straight line given in position, a line which shall be equal and parallel to a given straight line.

Let AB be the given circle whose centre is O , and DE the line given in position. From O draw OF parallel and equal to the given line; and



with the centre F , and radius equal to OB , the radius of the given circle, describe a circle cutting DE in G : join FG , and draw OA parallel to it; join AG ; AG is the line required.

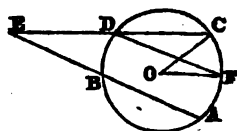
Since $FG = OB = OA$, and is parallel to it, AG is equal and parallel to OF , and $\therefore$ equal and parallel to the given line.

(23.) The bases of two given circular segments being in the same straight line; to determine a point in it

Through D draw DF parallel to BA . Find O the centre of the circle; join CO , FO . Then AB being parallel to FD , (ii. 1.) AF is equal to BD ; and the angle CEA is equal to CDF , i.e. to half the angle COF , which stands on the arc CF equal to CA and BD together.

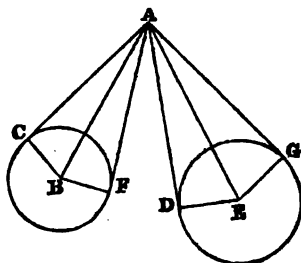
2. Next, let AB , CD intersect in E , without the circle.

The same construction being made, the angle CEA is equal to the angle CDF , i.e. to half COF , i.e. to half the angle standing on CF which is the difference between CA and AF , or CA and BD .



(25.) *If from a point without two circles which do not meet each other, two lines be drawn to their centres, which have the same ratio that their radii have; the angle contained by tangents drawn from that point towards the same parts will be equal to the angle contained by lines drawn to the centres.*

From the point A let the lines AB , AE be drawn to the centres of two circles, and let them have the same ratio that the radii BC , DE , have; from A draw the tangents AC , AD ; as also AF , AG ; each of the angles CAD , FAG will be equal to BAE .



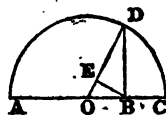
Since $AB : BC :: AE : ED$, and the angles at C and D are right angles, $\therefore$ the triangles ABC , ADE are

equiangular (Eucl. vi. 7.), and the angle $CAB = DAE$; to each of these add the angle BAD ; and $CAD = BAE$.

In the same manner FAG may be shewn to be equal to BAD .

(26.) *To determine the Arithmetic, Geometric and Harmonic means between two given straight lines.*

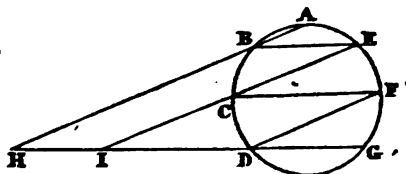
Let AB, BC be the two given lines. Let them be placed in the same straight line, and on AC describe a semicircle ADC . Through B draw BD at right angles to AC , join OD , and upon it let fall the perpendicular BE . Then AO being half of the sum of AB, BC is the arithmetic mean; and since (Eucl. vi. 8.) $AB : BD :: BD : BC$, $\therefore BD$ is the geometric mean. And DE is the harmonic mean, for (Eucl. vi. 8.) $(DO =) AO : DB :: DB : DE$, i. e. it is a third proportional to the arithmetic and geometric means, and $\therefore$ is the harmonic mean.



(27.) *If on each side of any point in a circle any number of equal arcs be taken, and the extremities of each pair joined; the sum of the chords so drawn will be equal to the last chord produced to meet a line drawn from the given point through the extremity of the first arc.*

Let AB, BC, CD , &c., AE, EF, FG , &c. be equal arcs and let their extremities BE, CF, DG be joined;

from A through B draw ABH meeting the last chord GD in H ; GH is equal to EB, CF, DG , &c. together.

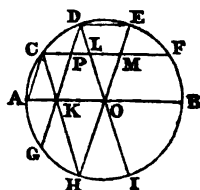


Join DF, EC , and produce EC to I . Since $AE = BC$, ABH is parallel to ECI . And since the arcs are equal, the lines BE, CF, DG are parallel, whence BI is a parallelogram, and $BE = HI$. In the same manner it may be shewn, that $CF = ID$; and so on, whatever be the number of equal arcs; hence GH is equal to the sum of BE, CF, DG .

(28.) *If the circumference of a semicircle be divided into an odd number of equal parts, and through the points which are equally distant from the diameter lines be drawn; the segments of these lines intercepted between radii drawn to the extremities of the most remote, will together be equal to a radius of the circle.*

Let the circumference of the semicircle ADB be divided into any odd number of equal parts, e. g. five, (the demonstration being the same for any odd number) in the points C, D, E, F .

Join DE, CF , which are parallel, since they intercept equal arcs. Join OD, OE ; DE and LM together are equal to the radius of the circle.

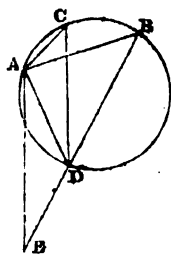


For complete the circle, and divide the opposite semi-circle in the same manner; join AC, DG, EH , which will be parallel to one another; CH will also be parallel to DI . Hence $DE = OK$, and OK is also equal to each of the two PM, CL , $\therefore PM = CL$, whence $LM = CP$, which is equal to AK , since CF is parallel to AB , and $DE + LM = KO + AK = AO$ the radius of the circle.

(29.) *If from the extremities and the point of bisection of any arc of a circle, lines be drawn to any point in the opposite circumference; the sum of those drawn from the extremities will have to that from the point of bisection, the same ratio that the line joining the extremities, has to that joining one of them and the point of bisection.*

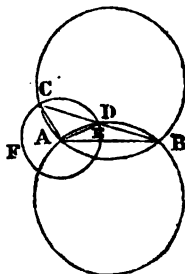
Let the arc AB be bisected in C , and AB, AC joined; to any point D in the circumference draw AD, BD, CD ; then $AD + DB : DC :: AB : AC$.

Draw AE parallel to CD , and let it meet BD produced in E . The angle $EAD = ADC = CAB$ (Eucl. iii. 27.); to each of these add DAB , and $EAB = CAD$; also $ABD = ACD$, $\therefore$ the triangles EAB, ACD are equiangular, whence $BE : CD :: BA : AC$. But since CDB is equal to each of the angles AED, DAE , they are equal to one another, $\therefore DA = DE$, and $AD + DB : DC :: AB : AC$.



(30.) *If two equal circles cut each other, and from either point of intersection a circle be described cutting them; the points where this circle cuts them and the other point of intersection of the equal circles are in the same straight line.*

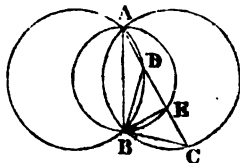
Let the two equal circles cut each other in A and B ; and with the centre A , and any distance AC , describe a circle FCD cutting their circumferences in C and D ; C, D, B will be in a straight line.



Join CB , and let it meet the circumference ADB in E . Join AE, AC . Since the angle ABC is an angle in each of the two equal circles, the circumference AC is equal to the circumference AE (Eucl. iii. 26.), $\therefore$ the line AC is equal to the line AE ; and $\therefore E$ is a point in the circle FDC , and being by construction in the circumference ADB , it must coincide with D ; $\therefore CB$ passes through D , or C, D, B are in a straight line.

(31.) *If two equal circles cut each other, and from either point of intersection a line be drawn meeting the circumferences; the part of it intercepted between the circumferences will be bisected by the circle whose diameter is the common chord of the equal circles.*

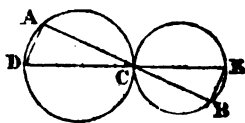
Let the two equal circles ADB, ACB cut each other in A and B ; join AB , and on it as a diameter let a circle AEB be described, and from A draw any line ADC meeting the circumferences in D and C ; DC is bisected in E .



Join BD , BE , BC . Since the angle CAB is in each of the two equal circles, the circumferences BD , BC on which it stands are equal, and $\therefore$ the straight lines BD , BC are equal, and consequently the angle BDE is equal to the angle BCE ; and the angle BED in a semi-circle is a right angle, and $\therefore$ equal to BEC , and BE is common to the two triangles BED , BEC , $\therefore DE = EC$.

(32.) *If two circles touch each other externally or internally; any straight line drawn through the point of contact, will cut off similar segments.*

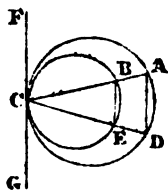
Let the circles ADC , BCE touch each other in the point C , and let any line ACB be drawn through the point of contact; it will cut off similar segments.



For draw the diameters CD , CE ; and join AD , BE . Then DCE being a straight line (Eucl. iii. 12.), the angle ACD is equal to BCE , and $DAC = CBE$ each being in a semicircle, and $\therefore$ a right angle; whence the angles ADC , CEB are equal, and the segments ADC , CEB similar; and $\therefore$ the segments AC and CB are also similar.

(33.) *If two circles touch each other externally or internally; two straight lines drawn through the point of contact will intercept arcs, the chords of which are parallel.*

Let the two circles ACD , ECB touch each other in C , and let ABC , DEC be any two lines drawn through the point of contact. Draw the tangent FCG ; and join AD , EB , AD , EB are parallel.



For (Eucl. iii. 32.) the angle $ADC = (FCA =) BEC$, whence (Eucl. i. 28.) AD is parallel to BE .

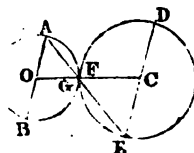
(34.) *If two circles touch each other internally or externally; any two straight lines drawn through the point of contact and terminated both ways by the circumferences will be cut proportionally by the circumference.*

Let the two circles touch each other in C , (see last Fig.) and let ACB , DCE be any two lines drawn through the point of contact; then it may be shewn (as in the last prop.), that AD is parallel to BE , and the triangles ACD , BCE are similar,

$$\therefore AC : CB :: DC : CE.$$

(35.) *If two circles touch each other externally, and parallel diameters be drawn; the straight line joining the extremities of these diameters, will pass through the point of contact.*

Let ABG , DGE be two circles touching each other externally in the point G ; and let AB , DE be parallel diameters; join AE ; AE will pass through G .



Join O, C the centres of the circles; OC will pass through G : let it meet AE in F . The vertically opposite angles at F being equal, and also the alternate angles OAF, FEC , the triangles AOF, FCE are equiangular,

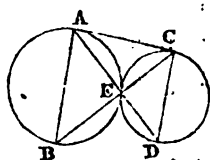
$$\therefore AO : CE :: OF : FC,$$

$$\text{comp. } AO + CE : CE :: OF + FC : FC.$$

But $OC = AO + CE$, and $\therefore FC = CE = CG$, and consequently F and G coincide, or AE intersects OC in the point G , i. e. it passes through the point of contact.

(36.) *If two circles touch each other, and also touch a straight line; the part of the line between the points of contact, is a mean proportional between the diameters of the circles.*

Let AEB, CED be two circles touching each other in E , and a straight line AC in A and C ; draw the diameters AB, CD ; AC is a mean proportional between AB and CD .



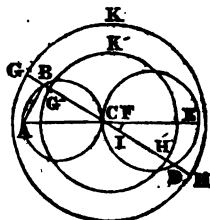
Join AD, BC ; these lines (ii. 25.), pass through the point of contact E . And since CA touches the circle in A , from which AE is drawn, the angle CAD is equal to the angle in the alternate segment ABE ; also the angle ACD being a right angle is equal to the angle CAB , $\therefore$ the triangles ACD, ABC are equiangular, and

$$BA : AC :: AC : CD.$$

(37.) *If two circles touch each other externally, and the line joining their centres be produced to their circumferences; and from its middle point as a centre with*

any radius whatever a circle be described, and any line placed in it passing through the point of contact; the parts of the line intercepted between the circumference of this circle and each of the others will be equal.

Let ABC , DCE be two circles which touch each other externally in C ; and let AFE be the line joining their centres, and produced to the circumferences in A and E . Bisect AE in F ; and with the centre F and any radius, let a circle GHK be described; and in it any line GCH drawn through C meeting the circumferences of the circles in B and D ; then will $GB = DH$.



Join AB , DE , and draw FI parallel to AB ; it will be perpendicular to GH , since ABC is an angle in a semicircle; and $\therefore GH$ is bisected in I . And since IF is parallel to AB ,

(Eucl. vi. 2.) $AF : BI :: FC : IC$,

also the triangles ICF , ECD being similar,

$FC : CI :: EF : ID$,

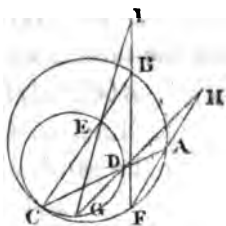
$\therefore$ (Eucl. v. 15.) $AF : BI :: EF : ID$.

But $AF = FE$, $\therefore BI = ID$,

and it has been shewn that $GI = IH$, whence $GB = DH$.

(38.) *If from the point of contact of two circles which touch each other internally, any number of lines be drawn; and through the points, where these intersect the circumferences, lines be drawn from any other point in each circumference, and produced to meet; the angles formed by these lines will be equal.*

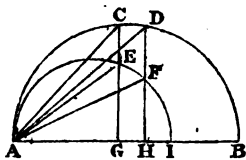
Let the two circles ABC , DEC touch each other internally in C , from which let any lines CA , CB be drawn; and taking any two points G and F , through E and B draw GEI , FBI , and through D and A draw GDH , FAH ; if those lines meet, the angle at I will be equal to the angle at H .



For the angles CBF , CAF standing on the same circumference CF , are equal, $\therefore$ the angle IBE is equal to HAD . Also the angles CEG , CDG , standing on the same circumference CG , are equal, and $\therefore$ the angle IEB is equal to the angle HDA ; $\therefore$ the triangles IEB , HDA have two angles in each equal, and consequently the remaining angles are equal, i. e. $EIB = DHA$.

(39.) *If two circles touch each other internally, and any two perpendiculars to their common diameter be produced to cut the circumferences; the lines joining the points of intersection and the point of contact are proportionals.*

Let the two circles ACB , AEI touch each other internally in the point A , from which let the common diameter AIB be drawn, and from any two points G , H let perpendiculars GC , HD meet the circumferences in C , D , E , F ; join AC , AD , AE , AF ; these lines are proportional.



For since $AB : AD :: AD : AH$,

$AB : AH$ in the duplicate ratio of $AB : AD$.

For the same reason,

$AG : AB$ in the duplicate ratio of $AC : AB$,

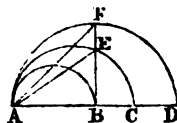
$\therefore AG : AH$ in the duplicate ratio of $AC : AD$.

In the same manner it may be shewn that

$AG : AH$ in the duplicate ratio of $AE : AF$,
 $\therefore$ (Eucl. v. 15.) the duplicate ratio of $AC : AD$,
 is the same with the duplicate ratio of $AE : AF$,
 and $\therefore AC : AD :: AE : AF$.

(40.) *If three circles, whose diameters are in continued proportion touch each other internally, and from the extremity of the least diameter passing through the point of contact, a perpendicular be drawn, meeting the circumferences of the other two circles; this diameter and the lines joining the points of intersection and contact are in continued proportion.*

Let AB, AC, AD the diameters of three circles touching each other in A , be in continued proportion, viz. $AB : AC :: AC : AD$, and from B the perpendicular BF meet the circumferences in E and F ; join AE, AF ; then $AB : AE :: AE : AF$.



For (Eucl. vi. 8.) $AB : AF :: AF : AD$.

But by the hypothesis $AC : AB :: AD : AC$,

$$\therefore AC : AF :: AF : AC,$$

whence $AF = AC$.

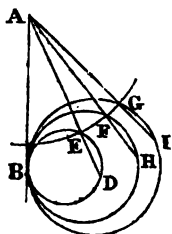
And (Eucl. vi. 8.) $AB : AE :: AE : AC$,

$$\therefore AB : AE :: AE : AF.$$

(41.) *If a common tangent be drawn to any number of circles which touch each other internally; and from*

any point in this tangent as a centre, a circle be described cutting the others, and from this centre lines be drawn through the intersections of the circles respectively; the segments of them within each circle will be equal.

Let the circles touch each other in the point B , to which let a tangent BA be drawn, and from any point A in it as a centre with any radius, let a circle EFG be described. Draw the lines AED , AFH , AGI ; then will the parts DE , HF , IG be equal.



For since AB touches the circle, (Eucl. iii. 36.)

$$DA : AB :: AB : AE,$$

For the same reason, $AB : AH :: AF : AB$,

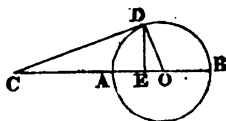
$$\therefore \text{ex æquo } DA : AH :: AF : AE,$$

but $AF = AE$, $\therefore DA = AH$ and consequently $DE = HF$.

In the same manner it may be proved, that $IG = HF$ or DE .

(42.) If from any point in the diameter of a circle produced, a tangent be drawn; a perpendicular from the point of contact to the diameter will divide it into segments which have the same ratio that the distances of the point without the circle from each extremity of the diameter, have to each other.

From any point C in the diameter BA produced, let a tangent CD be drawn, and from D , draw DE perpendicular to AB ; $AE : EB :: AC : CB$.



Take O the centre of the circle, join DO ; then (Eucl. iii. 18.) the angle CDO is a right angle, and $\therefore$ (Eucl. vi. 8.)

$$CO : OD :: OD : OE,$$

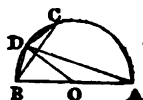
$$\text{or } CO : OA :: OA : OE,$$

$$\therefore \text{div. and comp. } AC : CB :: AE : EB.$$

COR. The converse may easily be proved.

(43.) *If from the extremity of the diameter of a given semicircle a straight line be drawn in it, equal to the radius, and from the centre a perpendicular let fall upon it and produced to the circumference; it will be a mean proportional between the lines drawn from the point of intersection with the circumference to the extremities of the diameter.*

From B the extremity of the diameter AB let a line BC be drawn, equal to the radius BO ; and on it let fall a perpendicular OD meeting the circumference in D ; join DB, DA ; DO is a mean proportional between DA and DB .



Join DC . Then the angles BAD, BCD on the same base are equal. Also since OD bisects BC , it bisects the arc BDC , $\therefore$ also the straight line $BD = DC$ and the angle $DBC = DCB$, but $ODA = OAD$, $\therefore$ the triangles ODA, DBC are similar, $\therefore AD : DO :: (BC =) DO : DB$.

(44.) *If from the extremity of the diameter of a circle, two lines be drawn, one of which cuts a perpen-*

dicular to the diameter, and the other is drawn to the point where the perpendicular meets the circumference; the latter of these lines is a mean proportional between the cutting line, and that part of it which is intercepted between the perpendicular and the extremity of the diameter.

Let CE be at right angles to the diameter AB of the circle ABC , and from A let AD , AC be drawn, of which AD cuts CE in F , then will

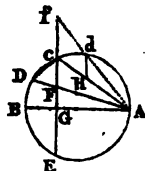
$$AD : AC :: AC : AF.$$

For since the circumference AE is equal to the circumference AC , (Eucl. iii. 27.) the angle ECA is equal to the angle ADC , and the angle at A is common to the two triangles ADC , ACF , $\therefore$ the triangles are similar, and

$$AD : AC :: AC : AF.$$

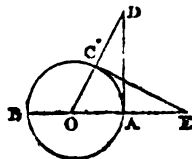
But if the point of intersection f be without the circle, draw dH parallel to CG , then, as before, the angle HdA is equal to ACd , and the angle at A common to the triangles AHd , ACd ,

$$\therefore Ad : AC :: AH : Ad :: AC : Af.$$



(45.) *In the diameter of a circle produced, to determine a point, from which a tangent drawn to the circumference shall be equal to the diameter.*

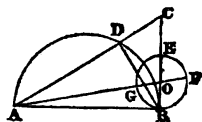
From A the extremity of the diameter AB , draw AD at right angles and equal to AB . Find the centre O , join OD cutting the circle in C ; and through C draw CE at right angles to OD meeting BA produced in E .



Then because the angle OAD is equal to OCE , each being a right angle, and the angle at O is common to the two triangles OAD , OCE , and $OA = OC$, $\therefore AD = CE$. But AD was made equal to AB , $\therefore CE = AB$, and E is the point required.

(46.) *To determine a point in the perpendicular at the extremity of the diameter of a semicircle, from which if a line be drawn to the other extremity of the diameter, the part without the circle may be equal to a given straight line.*

From B the extremity of the diameter of the semicircle ADB , let a perpendicular BC be drawn; in which take BE equal to the given line; and on it as a diameter describe a circle; through the centre of which draw AGF , and with A as centre and radius AF describe a circle cutting BC in C . Join AC ; CD is equal to the given line.

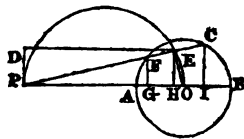


Join BD . Then BD being perpendicular to AC ,
 (Eucl. vi. 8. Cor.) $AC : AB :: AB : AD$,
 and (Eucl. iii. 36.) $AB : AF :: AG : AB$,
 $\therefore$ *ex æquo*, $AC : AF :: AG : AD$,
 whence $AG = AD$, and $\therefore DC = GF = BE$.

(47.) *Through a given point without a given circle, to draw a straight line to cut the circle, so that the two perpendiculars drawn from the points of intersection to that diameter which passes through the given point, may*

together be equal to a given line, not greater than the diameter of the circle.

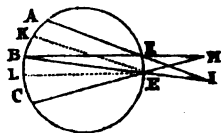
Let P be the given point without the circle ABC , whose centre is O ; AB the diameter which passes through P . On PO describe a semicircle. From P draw PD at right angles to PB and equal to half the given line; through D draw DE parallel to PB meeting the semicircle in E ; join PE ; and produce it to C ; PC is the line required.



For, draw FG, EH, CI perpendiculars to AB . Join OE ; then the angle PEO is a right angle, and $\therefore$ (Eucl. iii. 3.) $EF = EC$; whence FG and CI together are equal to $2EH = 2PD =$ the given line.

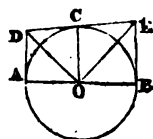
(48.) If from each extremity of any number of equal adjacent arcs in the circumference of a circle, lines be drawn through two given points in the opposite circumference, and produced till they meet; the angles formed by these lines will be equal.

Let AB, BC , be equal arcs, and F, E two points in the opposite circumference, through which let the lines $AFI, BEI; BFH, CEH$ be drawn, so as to meet; the angles at I and H , will be equal.



From E draw EK, EL , respectively parallel to FA ,

From A and B the extremities of the diameter AB let tangents AD , BE be drawn, meeting a tangent to any other point C of the circumference, in D and E ; and let O be the centre; join DO , EO ; the angle DOE is a right angle.



Join CO . Then since $CE = EB$, $CO = OB$, and the angles at C and B , being right angles, are equal, $\therefore$ the angle $CEO = OEB$, and CEB is bisected by EO . In the same manner it may be shewn that the angle ADC is bisected by DO . And since the angles CEB , CDA are equal to two right angles, $\therefore CDO$ and CEO are equal to one right angle, and $\therefore$ (Eucl. i. 32.) DOE is a right angle.

(53.) *If from the extremities of the diameter of a circle tangents be drawn; any other tangent to the circle, terminated by them, is so divided at the point of contact, that the radius of the circle is a mean proportional between its segments.*

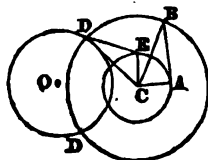
Let AD , BE be two lines touching the circle ABC , (see the last Fig.) at A and B the extremities of its diameter, and meeting DCE any other tangent to the circle; take O the centre, and join CO ; then will $DC : CO :: CO : CE$.

Join DO , EO ; then as in the last proposition, it may be shewn that DOE is a right angle; and since from the right angle OC is drawn perpendicular to the base, $\therefore$ (Eucl. vi. 8.) it is a mean proportional between the segments of the base, or

$$DC : CO :: CO : CE.$$

(54.) *Two circles being given in magnitude and position; to find a point in the circumference of one of them, to which if a tangent be drawn cutting the circumference of the other, the part of it intercepted between the two circumferences may be equal to a given line.*

Let O and C be the centres of the two given circles. To any point A in the circumference of one of them let a tangent AB be drawn, and make AB equal to the given line. With the centre C and distance CB describe a circle DBD cutting the other in the point D , and from D draw DE touching the former given circle; E will be the point required.

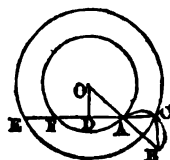


Join CA , CB , CD , CE . Since $CA = CE$ and $CB = CD$, and the angles at A and E are right angles, $\therefore DE$ is equal to BA , i. e. to the given line.

If the circle DBD neither cuts nor touches DD , it is evident the problem will be impossible.

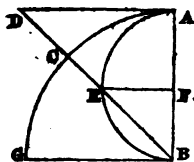
(55.) *To draw a straight line cutting two concentric circles so that the part of it which is intercepted by the circumference of the greater may be double the part intercepted by the circumference of the less.*

Let O be the centre of the two circles. Draw any radius OA of the lesser circle and produce it to B , making $AB = AO$. On AB describe a semicircle ACB cutting the greater circumference in C ; join AC , and produce it to E ; CE is the line required.



Join CB ; and let fall the perpendicular OD . Then

On the radius AB of the quadrant AGB let the semicircle AEB be described, and at A draw the tangent AD . From B draw any line $BECD$ meeting the tangent in D , and the circumferences in C , E ; from E let fall the perpendicular EF ; then BD , BC , BE , BF are in continued proportion.



Since FE is perpendicular to BA , it is parallel to AD ,

$$\therefore BF : BE :: (BA =) BC : BD,$$

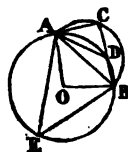
But (Eucl. vi. 8.) $BF : BE :: BE : (BA =) BC$,

$\therefore$ (Eucl. v. 15.) also $BE : BC :: BC : BD$,

and $BF : BE :: BE : BC :: BC : BD$.

(59.) *If the chord of a quadrant be made the diameter of a semicircle, and from its extremities two straight lines be drawn to any point in the circumference of the semicircle; the segment of the greater line intercepted between the two circumferences shall be equal to the less of the two lines.*

Let O be the centre of the quadrant ADB ; join AB , and on it let a semicircle ACB be described; from any point C in which let lines CA , CB be drawn to A and B , of which CB is the greater; $CD = CA$.

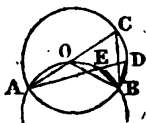


Join AD , and complete the circle ABE ; take any point E , and join EA , EB . Since $ADBE$ is a quadrilateral figure inscribed in a circle, the angles AEB , ADB are equal to two right angles, and $\therefore$ equal to ADB , ADC ; whence $AEB = ADC$; but AEB is half of AOB which is a right angle, $\therefore ADC$ is half a right

angle, and ACD being a right angle (Eucl. ii. 31.), CAD is half a right angle, and $\therefore$ equal to CDA , consequently $CA=CD$.

(60.) *If two circles cut each other so that the circumference of one passes through the centre of the other, and from either point of intersection a straight line be drawn cutting both circumferences; the part intercepted between the two circumferences will be equal to the chord drawn from the other point of intersection to the point where it meets the inner circumference.*

Through O the centre of the circle ABC , let the circle AOB be described, cutting ABC in A and B . If any line AED be drawn from A , and BE joined; DE will be equal to EB ,



Draw the diameter AOC ; join BC , BD . Then since the angle AOB is equal to AEB , $\therefore$ the angle COB is equal to DEB . Also the angles OCB , EDB , being in the same segment, are equal to one another, $\therefore$ the triangles OCB , EDB are equiangular, and $\therefore$ since $OB=OC$, the angle OCB is equal to the angle OBC , whence $EDB=EBD$, and $\therefore ED=EB$.

(61.) *If from each extremity of the diameter of a circle lines be drawn to any two points in the circumference; the sums of the lines so drawn to each point will have to one another the same ratio that the lines have, which join those points and the opposite extremity of a diameter perpendicular to the former.*

From A and C the extremities of AC the diameter of the circle ABC , let lines AE , EC , AF , FC be drawn to any points E and F in the circumference, and draw the diameter BD perpendicular to AC ; join ED , FD ; then

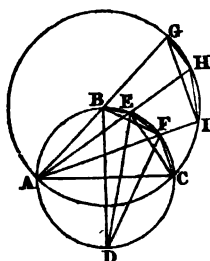
$$AE + EC : AF + FC :: ED : FD.$$

Join AB ; and with the centre B and distance BA describe a circle AGC ; produce AB , AE , AF to the circumference. Join GH , HI , BE , EF , GI , BF . Then since AG and BD are diameters of the circles, the angles AHG , AIG are equal to DEB , DFB ; but BAE , EAF are equal to BDE , EDF , and the angle HIG being $= HAG = BDE = BFE$, $\therefore$ the angle $HIA = EFD$, and the triangles GAH , HAI are similar to BDE , EDF , and

$$\therefore AH : AI :: ED : FD,$$

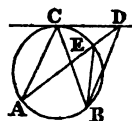
$$\text{But (ii. 60.) } EH = EC, \text{ and } FI = FC,$$

$$\therefore AE + EC : AF + FC :: ED : FD.$$



(62.) *If from any two points in the circumference of a circle there be drawn two straight lines to a point in a tangent to that circle; they will make the greatest angle when drawn to the point of contact.*

Let A and B be the two points, and CD the tangent at C ; join AC , CB ; the angle ACB is greater than any other angle ADB formed by lines drawn to any other point D .



Join BE . Then the angles ACB , AEB in the same

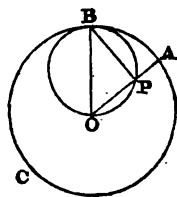
segment are equal; but ADB is less than the exterior angle AEB , and $\therefore$ is less than ACB .

COR. If two circles touch each other in C , it might be shewn in a similar manner, that the angle formed by two straight lines drawn from A and B to C the point of contact will be greater than the angle formed by lines drawn from the same points to any point in the exterior circle.

(63.) *From a given point within a given circle to draw a straight line which shall make with the circumference an angle less than the angle made by any other line drawn from that point.*

Let P be the given point within the circle ABC .

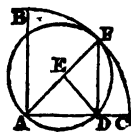
Find O the centre, join OP , and produce it to the circumference. From P draw PB at right angles to OA ; it is the line required.



Join OB , and on it as a diameter describe a circle OPB , which will touch the circle ABC in B . Then OBP is the greatest angle that can be included between lines drawn from O and P to the circumference ABC (ii. 62. Cor.), $\therefore$ the angle contained by PB and the circumference AB will be the least.

(64.) *To determine a point in the arc of a quadrant, from which if lines be drawn to the centre and the point of bisection of the radius, they shall contain the greatest possible angle.*

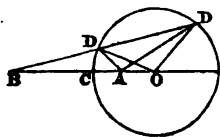
Let BC be the arch of a quadrant whose centre is A , and let the radius AC be bisected in D . On AD describe an equilateral triangle ADE , and produce AE to F ; F is the point required.



Join FD . Then $AF=AC$, and $AD=AE$, but AD is half of AC , and $\therefore AE$ is half of AF , and $\therefore$ equal to EF ; and EA, ED, EF are equal; whence a circle described from the centre E at the distance of any one of them will pass through the extremities of the other two, and touch the arc BC in F , because their centres are in the same straight line; and AFD (ii. 62. Cor.) is greater than any other angle formed by lines drawn from any point in BC to A and D .

(65.) *If the radius of a circle be a mean proportional to two distances from the centre in the same straight line; the lines drawn from their extremities to any point in the circumference will have the same ratio that the distances of these points from the circumference have.*

Let A and B be the two points, such that $AO : CO :: CO : BO$, and from A and B let lines AD, BD be drawn to any point D in the circumference; these have always the same ratio, viz. $AD : BD :: AC : BC$.



Join OD . Then since $OD=OC$, $OA : OD :: OD : OB$, i. e. the sides about the common angle AOD of the triangles AOD, BOD are proportional, and $\therefore$ the triangles are similar, consequently

$$DA : DB :: OA : (OD=) OC.$$

But since $AO : CO :: CO : BO$,

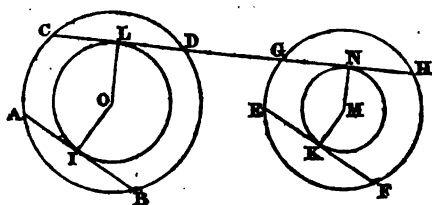
div. $AO : AC :: CO : CB$,

alt. $AO : CO :: AC : CB$,

 $\therefore$ (Eucl. v. 15.) $DA : DB :: AC : CB$.

(66.) *Two circles being given in position and magnitude; to draw a straight line cutting them so that the chords in each circle may be equal to a given line, not greater than the diameter of the smaller circle.*

Let ABC , EFG be the given circles whose centres are O and M . In each place a line AB , EF , equal to



the given line; and from the centres draw the perpendiculars OI , MK ; and with these distances, and centres O and I , describe circles which will touch AB , EF in I and K ; draw $CDGH$ (ii. 7.) which shall touch these circles in L and N ; each of the chords CD and GH will be equal to the given line.

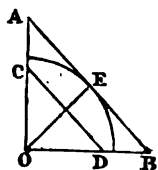
Join OL , MN ; these lines are perpendicular to CD and GH , and being respectively equal to OI and MK , $CD = AB$ (Eucl. iii. 14.), and $GH = EF$; but AB and EF are each equal to the given line, $\therefore$ CD and GH are also each equal to the given line.

COR. If the intercepted parts are required to have

a given ratio, take AB and EF in that ratio, and make the same construction as in the proposition.

(67.) *To determine a point in the arc of a quadrant, through which if a tangent be drawn meeting the sides of the quadrant produced, the intercepted parts may have a given ratio.*

Let OA , OB be the sides of a quadrant produced; and take M and N two right lines which are in the given ratio, and let OC be a mean proportional between the radius of the quadrant and M , and OD a mean proportional between the radius and N , Join CD , and draw the radius OE cutting it at right angles; E is the point required.



Through E draw the tangent AEB , which being perpendicular to OE (Eucl. iii. 18.), will be parallel to CD ,

$$\therefore AO : OB :: CO : OD,$$

and since OC and OD are mean proportionals between M and the radius, and N and the radius respectively,

$$M : N \text{ in the duplicate ratio of } OC : OD,$$

$$\text{i. e. in the duplicate ratio of } AO : OB.$$

But (Eucl. vi. 8. Cor.)

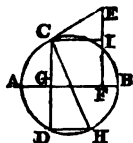
$$AE : EB \text{ in the duplicate ratio of } AO : OB,$$

$$\therefore AE : EB :: M : N, \text{ i. e. in the given ratio.}$$

(68.) *If a tangent be drawn to a circle at the extremity of a chord which cuts the diameter at right angles, and from any point in it a perpendicular be let fall; the*

segment of the diameter intercepted between that perpendicular and chord is to the intercepted part of the tangent, as the chord is to the diameter.

Let the chord CD be perpendicular to the diameter AB , and let CE touch the circle at C ; from any point E in which let EF be drawn perpendicular to AB ;



$$FG : CE :: CD : AB.$$

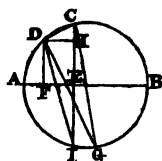
Draw the diameter CH ; join HD , and draw CI perpendicular to EF . Since EC touches the circle, the angle ECH (Eucl. iii. 18.) is a right angle, and $\therefore$ equal to ICD ; whence, taking away from each ICH , the angle $ECI = HCD$, and EIC , HDC are right angles, $\therefore$ the triangles ECI , HDC are equiangular.

$$\text{whence } IC : CE :: DC : CH,$$

$$\text{or } GF : CE :: CD : AB.$$

(69.) *If a straight line be placed in a circle, and from its extremities perpendiculars be let fall upon any diameter; these perpendiculars together will have to the part of the diameter intercepted between them, the same ratio that a line placed in the circle perpendicular to the former line, has to the former line itself.*

Let the line CD be placed in the circle ABC , and from its extremities let CE , DF be drawn perpendicular to a diameter AB . From D let DG be drawn perpendicular to DC ; then will



$$CE + DF : EF :: GD : DC.$$

Join CG , which is therefore a diameter of the circle;

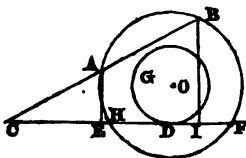
and produce CE to I : join DI , and draw DH perpendicular to CE . Since CI is perpendicular to AB , $CE = EI$, but $HE = DF$, $\therefore HI = CE + DF$. Now (Eucl. iii. 21.) the angle at G is equal to the angle at I , and CDG , DHI are right angles, $\therefore$ the triangles CGD , HID are equiangular,

$$\text{and } HI : HD :: DG : DC,$$

$$\text{or } CE + DF : EF :: DG : DC.$$

(70.) *In a circle to place a straight line of given length, so that perpendiculars drawn to it from two given points in the circumference may have a given ratio.*

Let A and B be the given points in the circumference of the circle whose centre is O . Join BA , and produce it; and take $AC : CB$ in the given ratio. In the circle place a straight line equal to the given straight line, and from the centre O let fall a perpendicular upon it. With O as centre, and distance equal to this perpendicular describe a circle DG , and from C draw $CEDF$ a tangent to it; then HF is the line required.



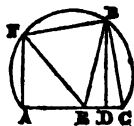
For (Eucl. iii. 14.) it is equal to the given straight line. And if from A and B , AE , BI be drawn perpendicular to CF ; they are parallel to each other, and the triangles CAE , CBI are similar,

$$\therefore AE : BI :: CA : CB, \text{ i. e. in the given ratio.}$$

(71.) *If from any point in the arc of a segment of a circle a line be drawn perpendicular to the base; and*

from the greater segment of the base, and arc, parts be cut off respectively equal to the less; the remaining part of the base shall be equal to the chord of the remaining arc.

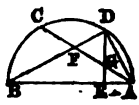
From any point B in the arc ABC , let BD be drawn perpendicular to AC ; make $BF=BC$, and $DE=DC$; join AF ; AF will be equal to AE .



Join FE , EB , FB , BC . Since the arc BC = the arc BF , the straight line $BC = BF$; and DE being equal to DC , and DB common, and at right angles to EC , $\therefore BE = BC = BF$, and the angle BFE is equal to the angle BEF . Now since $AFBC$ is a quadrilateral figure inscribed in a circle, the angles AFB , ACB are equal to two right angles, and $\therefore$ equal to AEB , CEB , of which $ACB = CEB$, $\therefore AFB = AEB$; but $BFE = BEF$, consequently $AFE = AEF$; whence $AF = AE$.

(72.) If from the point of bisection of any arc of a circle a perpendicular be drawn to the diameter, which passes through one extremity; it will bisect the segment of the chord cut off by the line joining the point of bisection of the arc and the other extremity of the diameter.

Let AC be the arc bisected in D . Join AC ; and from D draw DE perpendicular to the diameter AB , and meeting AC in G ; join BD ; $AG = GF$.

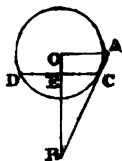


Because AC is bisected in D , the angle CAD is equal to the angle DBA , i. e. to the angle EDA (Eucl.

vi. 8.), $\therefore$ the right-angled triangle ADF is equiangular to the two triangles BED , DEA $\therefore$ the angle $GFD = GDF$, and consequently $GD = GF$; also $GAD = GDA$, $\therefore AG = GD$, whence $AG = GF$.

(73.) *In a given circle to draw a chord parallel to a straight line given in position; so that the chord and perpendicular drawn to it from the centre may together be equal to a given line.*

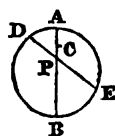
Let O be the centre of the circle, OA the straight line given in position; draw OB perpendicular to it, and equal to the given line. Take OA equal to the half of OB ; and join AB , cutting the circle in C ; through C draw CD parallel to OA ; CD is the chord required.



Because OA is half of OB , and OA , EC are parallel, $\therefore$ (Eucl. vi. 2.) EC is half of EB , and $DC = EB$; therefore DC and OE together are equal to BE and OE together, i. e. to BO , or to the given line.

(74.) *Through a given point within a given circle, to draw a straight line such that the parts of it intercepted between that point and the circumference may have a given ratio.*

Let P be the given point within the circle ABD . Through P draw the diameter APB , and take $AP : PC$ in the given ratio. With P as centre, and radius equal to a mean proportional between BP and PC ,



describe a circle cutting ADB in D ; join DP , and produce it to E ; DE is the chord required.

Since $BP : PD :: PD : PC$,
and (Eucl. iii. 35.) $BP : PD :: PE : PA$,

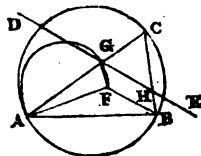
$\therefore PE : PA :: PD : PC$,

and *alt.* $PE : PD :: AP : PC$, i.e. in the given ratio.

COR. Since one circle cuts another in two points, there will be two chords which answer the conditions. If C coincides with A , the ratio is one of equality, and DE will be perpendicular to AB .

(75.) *From two given points in the circumference of a given circle, to draw two lines to a point in the circumference, which shall cut a line given in position, so that the part of it intercepted by them may be equal to a given line.*

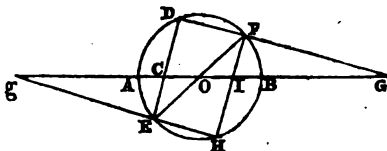
Let A, B be the given points in the circumference of the circle ABC ; DE the line given in position. From B draw BF parallel to DE , and equal to the given line. Join AF ; and on it describe a segment of a circle AGF containing an angle equal to the angle in the segment ACB ; and let it cut DE in G . Join AG , and produce it to C ; and join BC cutting DE in H . AC, BC are the lines required.



Join GF . Since the angle $AGF = ACB$, GF is parallel to CB ; but FB is parallel to GH , whence $FGHB$ is a parallelogram, and $GH = FB$.

(76.) *If a chord and diameter of a circle intersect each other at any angle, and a perpendicular to the chord be drawn from either extremity of it, meeting the circumference and diameter produced; the whole perpendicular has to the part of it without the circle, the same ratio that the greater segment of the chord has to the less.*

Let the diameter AB and chord DE intersect each other at C ; and from D draw DG perpendicular to DE ,



meeting AB produced in G ;

then $GD : GF :: DC :: CE$.

Through F draw FI parallel to DE , and meeting the diameter in I . Join FE , cutting the diameter in O . Since the angle FDE is a right angle, FE is a diameter and O is the centre. And since the angle IFO is equal to the alternate angle OEC , and the angles at O are equal, and $FO = OE$, $\therefore$ the triangles OFI , OEC are equal, and $CE = FI$. And since FI is parallel to DC ,

(Eucl. vi. 2.) $GD : GF :: DC : (FI =) CE$.

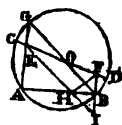
In a similar manner it may be shewn that

$gH : gE :: DC : CE$.

(77.) *If from the extremities of any chord of a circle, perpendiculars to it be drawn and produced to cut a diameter; and from the points of intersection with the diameter lines be drawn to a point in the chord, so as to*

make equal angles with it; these lines together will be equal to the diameter of the circle.

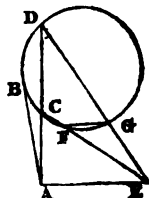
Let AB be any chord of the circle ABC ; draw AE and BF perpendicular to it, meeting the diameter CD in E and F ; from which let the lines EH , FH be drawn making equal angles with AB ; EH and HF together are equal to CD .



Take O the centre, and join BO , and produce it; it will meet AE produced in G . Produce EH , FB to meet in I . Then since the angle $IHB = AHE = FHB$, and HB is perpendicular to FI , the triangles FHB , HBI are equal, and $FH = HI$. And since EG is parallel to FB , the angle $EGO = OBF$, and the vertical angles at O are equal, and $GO = OB$, $\therefore EG = FB = BI$; whence $EI = GB$, and $\therefore EH$, HF together are equal to EI i. e. to GB or CD the diameter of the circle.

(78.) *If from a point without a circle two straight lines be drawn, one of which touches and the other cuts the circle; a line drawn from the same point in any direction, equal to the tangent, will be parallel to the chord of the arc intercepted by two lines drawn from its other extremity to the former intersections of the circle.*

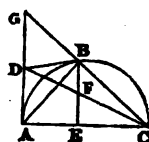
From the point A let AB , AD be drawn, of which AB touches the circle BCD , and AD cuts it; and draw $AE = AB$, in any direction; join CE , DE , cutting the circle in F and G ; the chord FG will be parallel to AE .



Because (Eucl. iii. 36.) $DA : AB :: AB : AC$,
 and $AE = AB$, $\therefore DA : AE :: AE : AC$,
i.e. the sides about the angle A of the triangles ADE ,
 ACE are proportional, $\therefore$ (Eucl. vi. 6.) the triangles are
 equiangular, and the angle AEC is equal to the angle
 ADE . But since $CDGF$ is a quadrilateral figure in the
 circle, the angles CDG , CFG are equal to two right
 angles, *i.e.* to EFG , CFG , $\therefore CDG = EFG$, whence
 $AEF = EFG$, and FG is therefore parallel to AE .

(79.) *If from a point without a circle, two straight lines be drawn touching it, and from one point of contact a perpendicular be drawn to that diameter which passes through the other; this perpendicular will be bisected by the line joining the point without the circle and the other extremity of the diameter.*

Let DA , DB be drawn from a point D without the circle ABC , touching it in A and B ; and from B let BE be drawn perpendicular to AC the diameter passing through A ; join CD ; BE is bisected by CD in the point F .



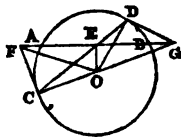
For produce AD and CB to G ; join AB . Then since $DA = DB$, the angle DAB is equal to the angle DBA . Now the angle ABG , being a right angle, is equal to BAG , BGA , of which $ABD = BAG$, $\therefore DBG = DGB$, and $DG = DB = DA$; and since AG is parallel to EB ,

$$BF : GD :: CF : CD :: EF : AD,$$

and $GD = DA$, $\therefore BF = FE$.

(80.) *If any chord in a circle be bisected by another, and produced to meet the tangents drawn from the extremities of the bisecting line; the parts intercepted between the tangents and the circumferences are equal.*

Let AB be bisected in E by CD ; and to C and D let tangents be drawn, meeting AB produced in F and G ; AF is equal to BG .

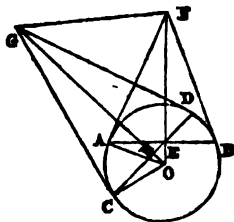


Find O the centre; join OC , OD , OE , OF , OG . Since OE is drawn from the centre to the point of bisection of AB (Eucl. iii. 3.) the angle OEF is a right angle; and the angle OCF is a right angle (Eucl. iii. 18.); $\therefore$ a circle may be described about $OEFC$. Also since ODG and OEG are right angles a circle may be described about $OEDG$; and the angle DOG is equal to the angle DEG in the same segment; but DEG is equal to FEC , i. e. to FOC , $\therefore DOG = FOC$; and ODG , OCF are equal, being right angles; and $OC = OD$, $\therefore OF = OG$, and consequently $FE = EG$. But $AE = EB$, $\therefore FA = BG$.

(81.) *If one chord in a circle bisect another, and tangents drawn from the extremities of each be produced to meet; the line joining their points of intersection will be parallel to the bisected chord.*

Let AB be bisected by the line CD in E , and let the tangents AF , BF meet each other in F , and DG , CG in G . Join GF ; GF is parallel to AB .

Join AO , CO , GO , FO ; then



GO bisects CD in H , and OHE is a right angle; for the same reason FO passes through E , and AEO is a right angle. And since FAO is a right angle (Eucl. iii. 18.), and from A , AE is drawn perpendicular to the base,

(Eucl. vi. 8. Cor.) $FO : OA :: OA : OE$,

for the same reason,

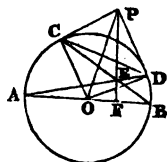
$(OC=) OA : OG :: OH : (OC=) OA$,

$\therefore$ *ex æquo per.* $FO : OG :: OH : OE$,

$\therefore$ the sides of the triangles FOG , OHE about the common angle O are proportional, and consequently the triangles are equiangular, and the angle GFO equal to EHO , and $\therefore$ a right angle, and equal to the alternate angle FEB , $\therefore AB$ is parallel to GF .

(82.) *If from a point without a circle two lines be drawn touching the circle, and from the extremities of any diameter lines be drawn to the points of contact, cutting each other within the circle; the line produced, which joins their intersection and the point without the circle, will be perpendicular to the diameter.*

From the point P without the circle ABC let there be drawn two tangents PC , PD . From A and B the extremities of a diameter, draw AD , BC to the points of contact, intersecting each other in E ; join PE , and produce it to F ; PF is perpendicular to AB .

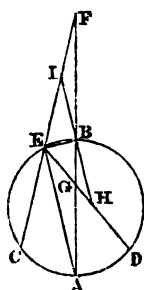


Take O the centre; join CO , DO , CD , DB . Since CPD , COD , are together equal to two right angles,

CPD is equal to AOC, BOD together, i. e. to twice ADC, BCD together, $\therefore CPD$ is equal to the angle at the centre of a circle passing through C, E , and D ; and since $PC=PD$, P is the centre itself; $\therefore PE=PD$, and the angle PED is equal to the angle PDE . But the angle $DBA=PDE=PED=AEF$, and the angle at A is common, $\therefore AFE=ADB$, and (Eucl. iii. 31.) is $\therefore$ a right angle.

(83.) *If on opposite sides of the same extremity of the diameter of a circle equal arcs be taken, and from the extremities of these arcs lines be drawn to any point in the circumference, one of which cuts the diameter, and the other the diameter produced; the distances of the points of intersection from the extremities of the diameter are proportional to each other.*

On opposite sides of the point A in AB the diameter of the circle ABC let equal arcs AC, AD be taken; from C and D let CE, DE be drawn to any point E in the circumference, of which CE cuts AB produced in F , and DE cuts AB in G ; then will $AF : FB :: AG : GB$.



Join AE, BE , and through B draw HBI parallel to AE . Since AEB is a right angle, CEA and BEI are together equal to a right angle, and $\therefore$ equal to AED, DEB ; and since $AC=AD$, $CEA=AED$, $\therefore BEI=DEB$. Again since AE is parallel to IH , the angle EIB is equal to $CEA=AED=$ the alternate angle EHB , $\therefore$ the two triangles EIB, EHB having two angles in each equal, and one side EB

common are equal, and $BI=BH$. And from the similar triangles AGE, BGH ,

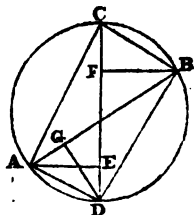
$$AG : AE :: BG : (BH=) BI,$$

$$\text{alt. } AG : GB :: AE : BI :: AF : BF,$$

since the triangles AEF, FIB are similar.

(84.) *If from the extremities of any chord in a circle, perpendiculars be drawn to a diameter, and from either extremity of that diameter a perpendicular be drawn to the chord; it will divide it into segments, which are respectively mean proportionals between the segments of the diameter cut off by the perpendiculars.*

Let AB be any chord, and CD a diameter of the circle ABC ; AE, BF perpendiculars from A and B to the diameter, and DG perpendicular from D to AB ; the segment GB is a mean proportional between DF and CE ; and AG a mean proportional between DE and CF .



Join AC, AD, BC, BD . Then the angle DBA being equal to DCA , and the angles DGB, DAC, AEC being right angles, the triangles DGB, DAC, ACE are similar; also the triangles DBF, DBC are similar; whence

$$DF : DB :: DB : DC :: BG : AC$$

$$\text{but } DB : BG :: AC : CE,$$

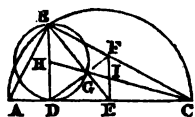
$$\therefore \text{ex æquo } DF : BG :: BG : CE.$$

And in the same manner it may be proved that

$$DE : AG :: AG : CF.$$

(85.) *If from any point in the diameter of a semicircle, a perpendicular be drawn, meeting the circumference, and on it as a diameter a circle be described, to the centre of which a line is drawn from the farther extremity of the diameter of the semicircle, cutting its circumference; and through the point of intersection another line be drawn from the extremity of the perpendicular, meeting the diameter of the semicircle; this diameter will be divided into three segments which are in continued proportion.*

From any point D in the diameter AC of the semicircle ABC , let a perpendicular DB be drawn, on which describe a circle DBG . Find its centre H , join HC cutting the circumference in G ; join BG and produce it to E ; $AD : DE :: DE : EC$.



Join BC , and draw EF parallel to BD ; join DG , GF , AB . Since EF is parallel to DB , and DB is bisected in H , $\therefore EF$ is bisected in I . Also the angle HBG is equal to the alternate angle GEI , and $BHG = GIE$, $\therefore$ the triangles BHG , GIE are equiangular,

$$\text{and } BH : HG :: EI : IG,$$

$$\text{or } HD : HG :: FI : IG,$$

i.e. the sides about the equal angles are proportional, $\therefore$ the triangle FGI is equiangular to HGD , and the angle FGI is equal to HGD ; whence HI being a straight line, FDG is also. Again the angle GDE is equal to the angle in the alternate segment DBG , whence the triangles BDE , FDE are similar,

$$\therefore BD : DE :: DE : EF,$$

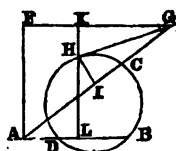
but $AD : DB :: EF : EC$, since the tri-

angles ADB, EFC are similar,

$$\therefore AD : DE :: DE : EC.$$

(86.) *If from a point without a given circle, any two lines be drawn cutting the circle; to determine a point in the circumference, such that the sum of the perpendiculars from it upon these lines may be equal to a given line.*

From the point A without the circle BDC let AB, AC be drawn cutting the circle; draw AF perpendicular to AB , and equal to the given line; FG parallel to AB , and meeting AC produced in G ; from G draw GH bisecting the angle AGF , and (if the problem be possible) meeting the circle in H ; H is the point required.

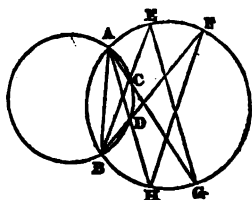


Through H draw KL perpendicular to AB , and HI perpendicular to AC ; then the angle KGH being equal to HGI , and the angle at K to the angle at I , and the side HG , opposite to one of the equal angles in each common, $HK = HI$; whence HI and HL together are equal to HK and HL together, i. e. to AF , i. e. to the given line.

If GH cuts the circle, there are two points which answer the conditions.

(87.) *If two circles cut each other, and any two points be taken in the circumference of one of them, through which lines are drawn from the points of intersection and produced to the circumference of the other; the straight lines joining the extremities of those which are drawn through the same point, are equal.*

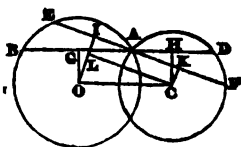
Let the two circles ACB , AEB cut each other in A and B , and in ACB let any two points C and D be taken, through which draw ACG , BCE , ADH , BDF ; and join EG , FH ; $EG = FH$.



For the angles CAD , CBD being on the same circumference CD are equal to one another, $\therefore$ the circumference EF is equal to the circumference GH . Add to each FG , and the circumference EFG is equal to FGH , $\therefore$ (Eucl. iii. 29.), the straight line $EG = FH$.

(88.) *If two circles cut each other; the greatest line that can be drawn through the point of intersection is that which is parallel to the line joining their centres.*

Let the two circles ABE , AFD cut each other in A . Join O , C their centres, and through A let BAD be drawn parallel to OC ; BAD is greater than any other line EAF which can be drawn through A .



Draw OG , CH , perpendicular to BD , and OI , CK perpendicular to EF . Then AG being half of AB , and AH of AD , GH is half of BD . For the same reason IK is half of EF . Draw CL parallel to EF , and therefore at right angles to OI , and equal to IK . Then since the angle CLO is a right angle, it is greater than COL , $\therefore$ the side CO is greater than CL , and GH than IK , consequently BD is greater than EF . In the same way BD may be shewn to be greater than any other line drawn through A .

(89.) *Having given the radii of two circles which cut each other, and the distance of their centres; to draw a straight line of given length through their point of intersection, so as to terminate in their circumferences.*

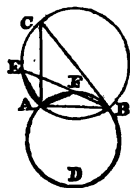
Let the two circles AFD , BGD cut each other in D ; on OC , the line joining their centres O and C , describe a semicircle CEO ; and in it, from C place CE equal to half the given line, and through D draw FDG parallel to it; FG will be the line required.



Through E draw OEH , which (Eucl. iii. 31.) will be perpendicular to FG ; and draw CI parallel to OH , and $\therefore$ perpendicular to DG ; then (Eucl. iii. 3.) FD and DG are bisected in H and I , and $\therefore FG$ is double of HI ; but $HECI$ being a parallelogram, $HI = EC$; $\therefore FG$ is double of EC , and consequently equal to the given line.

(90.) *If two circles cut each other; to draw from one of the points of intersection a straight line meeting the circles, so that the part of it intercepted between the circumferences may be equal to a given line.*

Let the two circles ABC , ADB cut each other in A and B . Join AB , and draw BC touching the circle ABD . Join AC ; and take AF a fourth proportional to BC , BA and the given line; join BF , and produce it to E ; BFE will be the line required.

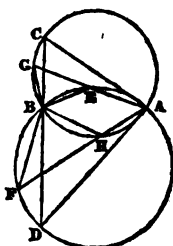


Since the angle AFB together with the angle in the segment ADB or (Eucl. iii. 32.) its equal ABC , are

equal to two right angles, i. e. to the angles AFB , AFE ,
 $\therefore ABC = AFE$; and $ACB = AEF$, being in the same
segment, $\therefore$ the triangles ACB , AEF are equiangular,
and $AB : BC :: AF : FE$,
but $AB : BC :: AF$: the given line,
whence FE is equal to the given line.

(91.) *If two circles cut each other; to draw from the point of intersection two lines, the parts of which intercepted between the circumferences may have a given ratio.*

Let the two circles ABC , ABD cut each other in A and B ; in the circle ABD place BE , BF , which have to each other the given ratio; join AE , AF , and produce AE to G ; EG will have to HF the given ratio.



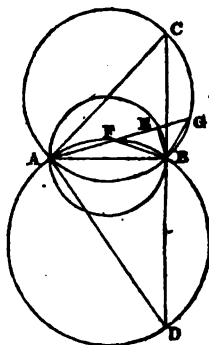
Draw the diameters AC , AD ; join GB , BH , BC , BD ; then $ADBE$ being a quadrilateral figure inscribed in a circle, the angles AEB , ADB are equal to two right angles, and $\therefore$ equal to BEA , BEG , $\therefore BEG = BDA = BFA$. And since $AGBH$ is a quadrilateral figure inscribed in a circle, AHB , AGB are equal to two right angles, i. e. to AHB , BHF , $\therefore AGB = BHF$; hence the triangles GBE , FBH are equiangular,

$\therefore GE : HF :: BE : BF$, i. e. in the given ratio.

(92.) *If a semicircle be described on the common chord of two intersecting circles, and a line be drawn*

from one extremity of this chord cutting the two circles ; the part intercepted between the two shall be divided by the semicircle into segments proportional to perpendiculars drawn in those circles from the other extremity of the chord.

Let the two circles ACB , ADB cut each other in A and B ; and on AB , the line joining the points of intersection, as a diameter, describe the semicircle AEB , and draw any line $AFEG$ cutting the circumferences in F , E , G ; and from B draw BC , BD perpendiculars to AB ; then will $EF : EG :: BD : BC$.



Draw the diameters AC , AD ; and join FB , EB , GB . Then $AFBD$ being a quadrilateral figure inscribed in a circle, the angles AFB , ADB are equal to two right angles, i. e. to AFB , BFE , $\therefore ADB = BFE$, and the angle FEB in a semicircle is equal to ABD , whence the triangles FEB , ABD are equiangular, and $\therefore FE : EB :: DB : BA$.

Again, because the angle $AGB = ACB$, and BEG is a right angle, and $\therefore$ equal to ABC , the triangles EBG , ABC are equiangular, and

$$BE : EG :: AB : BC,$$

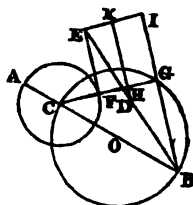
$$\text{but } FE : EB :: DB : BA,$$

$$\therefore \text{ex æquo } FE : EG :: DB : BC.$$

(93.) Two circles being given, the circumference of one of which passes through the centre of the other; to

draw a chord from that centre such, that a perpendicular let fall upon it from a given point, may bisect that part of it which is intercepted between the circumferences.

Let O and C be the centres of the two given circles, the circumference of the former passing through C ; and let D be the given point. Join CO , and produce it both ways to A and B . Join BD , and produce it to E , making $DE = DB$. Draw EF touching the circle AF in F ; join CF , and produce it to G ; and on it let fall the perpendicular DH ; then CG is the chord required, and FG is bisected in H .



Draw EI parallel to CG , meeting BG produced in I ; produce DH to K . Then BG being perpendicular to CG (Eucl. iii. 31.), is parallel to DHK ,

$$\therefore BD : DE :: IK : KE,$$

but $BD = DE$, $\therefore IK = KE$, whence $FH = HG$, and $\therefore FG$ is bisected in H .

COR. If it be required to draw CG such, that the perpendicular DH may divide FG in any given ratio, take $DE : DB$ in that ratio, and proceed as in the proposition.

(94.) *If any number of circles cut each other in the same points, and from one of these points any number of lines be drawn; the parts of these which are intercepted between the several circumferences have the same ratio.*

Let any number of circles ABC , ABE , ABH cut each other in the same points A and B ; and from A draw

AGEC, AHFD, meeting the circumferences; then $HF : GE :: FD : EC$.

Join *BG*, and produce it to *K*;
then (Eucl. iii. 35.) $AL : LB :: LG : LH$,

and $AL : LB :: IL : LF$

and also $:: KL : LD$,

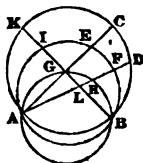
$\therefore$ (Eucl. v. 15.) $IL : LF :: GL : LH$,

and (Eucl. v. 19.) $IG : HF :: IL : LF$.

For the same reason, $IK : FD :: IL : LF$,

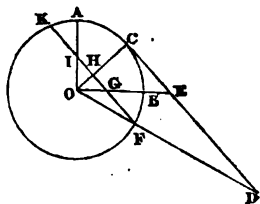
$\therefore IG : HF :: IK : FD$.

In like manner, $GE : GI :: GB : GA :: GC : KG :: EC : IK$,
 $\therefore ex\ aequo\ GE : HF :: EC : FD$.



(95.) *In a given circle to place a straight line cutting two radii which are perpendicular to each other, in such a manner that the line itself may be trisected.*

Let *ABC* be the given circle, *AO* and *OB* being two radii at right angles to each other; bisect the angle *AOB* by *OC*; at *C* draw the tangent *CD*, and make it equal to 3 *CO*; produce *OB* to *E*; join *OD*, and from *F* draw *FGIK* parallel to *DC*; it will be trisected at the points *G* and *I*.



Since the angle at *C* is a right angle, and *COB* is half a right angle, $\therefore$ also *CEO* is half a right angle, and equal to *COE*; whence $CO = CE$. And since *HF* is parallel to *CD*,

$CE : ED :: HG : GF$,

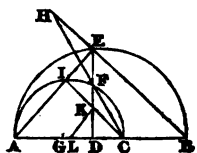
but *ED* is double of *EC*, $\therefore$ *FG* is double of *HG*. But $HG = HI$, since *HO* bisects the angle *IOG*, and is

perpendicular to IG ; $\therefore FG = GI$. Also $HK = HF$,
 $\therefore IK = GF$;

whence $FG = GI = IK$, and FK is trisected.

(96.) *If a straight line be divided into any two parts, and upon the whole line and one of the parts, as diameters, semicircles be described; to determine a point in the less diameter, from which if a perpendicular be drawn cutting the circumferences, and the points of intersection and the extremities of the respective diameters be joined, and these lines produced to meet; the parts of them without the semicircles may have a given ratio.*

Let AB be divided into any two parts in the point C , and on AB , AC let semicircles be described. Take $AG : AC$ the duplicate of the given ratio, and make $CD : CB :: AG : GB$; D will be the point required.



From D draw the perpendicular DFE ; join BE , CF , and produce them to H ; join AE , CI ; and from K draw KL parallel to EA .

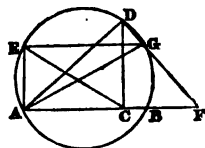
Since $CD : CB :: AG : GB$,
comp. and inv. $DB : CD :: AB : AG$,
 and since CI is parallel to BE , and KL to EA ,
 $BD : CD :: BE : CK :: BA : CL$,
 whence (Eucl. v. 15.) $AB : AG :: AB : CL$,
 $\therefore AG = CL$;

consequently $AG : AC :: CL : CA$,
i.e. in the duplicate ratio of $CK : CF$,
 or (by similar triangles) of $HE : HF$.

But $AG : AC$ is the duplicate of the given ratio,
 $\therefore HE : HF$ is in the given ratio.

(97.) *If a straight line be divided into any two parts, and from the point of section a perpendicular be erected, which is a mean proportional between one of the parts and the whole line, and a circle described through the extremities of the line and the perpendicular; the whole line, the perpendicular, the aforesaid part, and a perpendicular drawn from its extremity to the circumference will be in continued proportion.*

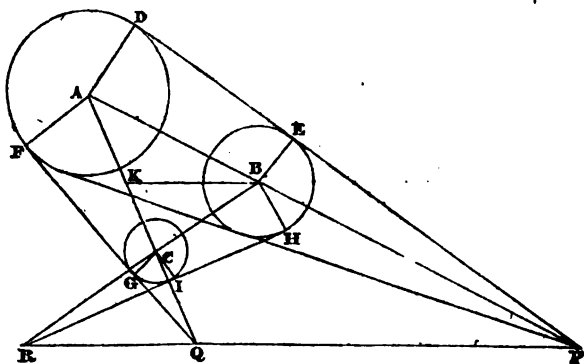
Let AB be divided into any two parts in C , and from C draw the perpendicular CD equal to a mean proportional between AB and AC ; and through A, B, D let a circle be described, and draw AE perpendicular to AB ; AB, CD, AC, AE are in continued proportion.



In AB produced take $BF = AC$. Join FD , meeting the circumference in G ; join AG, AD, GE . Then because $BF = AC$, $\therefore CF = AB$, and CD is a mean proportional between AC and CF , $\therefore \angle ADG$ is a right angle, whence (Eucl. iii. 21.) $\angle AEG$ is also a right angle, and equal to $\angle EAC$; $\therefore EG$ is parallel and equal to AB , i. e. to CF ; whence (Eucl. i. 33.) EC and GF are equal and parallel, and the angle $\angle ACE = \angle CFD = \angle ADC$, and the triangles AEC, ADC, CDF are similar,
 $\therefore (CF =) AB : CD :: CD : CA :: CA : AE$.

(98.) *If the tangents drawn to every two of three unequal circles be produced till they meet; the points of intersection will be in a straight line.*

Let A, B, C be the centres of the three circles; and let DE, FG, HI be respectively tangents to each of two



circles, meeting the lines joining the centres in the points P, Q, R ; P, Q, R are the points in which two tangents to the circles would intersect. Join PQ, QR ; they are in the same straight line.

Join AD, AF, BE, BH, CG, CI , and draw BK parallel to PQ . Then BE and AD being perpendicular to PD are parallel,

$$\therefore AD : BE :: AP : BP,$$

$$\text{or } AF : BH :: AP : BP :: AQ : QK.$$

$$\text{But } (CG =) CI : AF :: CQ : AQ,$$

$$\therefore \text{ex æquo } CI : BH :: CQ : QK.$$

$$\text{But } CI : BH :: CR : BR,$$

$$\therefore (\text{Eucl. v. 15.}) CR : BR :: CQ : QK,$$

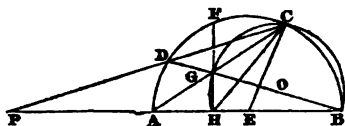
$$\text{and } CR : CB :: CQ : CK;$$

also the vertically opposite angles at C are equal, $\therefore$ the triangles CBK, CQR are similar, and the angle CQR (Eucl. vi. 6.) is equal to BKC , $\therefore CQR$ and CQP are

together equal to BKC , CQP , i. e. (Eucl. i. 29.) to two right angles, whence (Eucl. i. 14.) PQ and QR are in the same straight line.

(99.) *If from the extremities of the diameter of a circle any number of chords be drawn, two and two intersecting each other in a perpendicular to that diameter; the lines joining the extremities of every corresponding two will meet the diameter produced in the same point.*

From A and B , the extremities of the diameter AB of a semicircle, let AC , BD be drawn intersecting each



other in FH , which is perpendicular to AB . Join CD , and produce it to meet BA in P ; P is a fixed point, or the line joining the extremities of every other two chords intersecting each other in FH will pass through P .

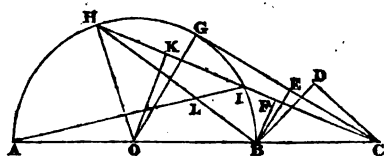
Join BC ; and bisect BG in O ; and with the centre O , and radius OB , describe a circle HGB , which will circumscribe the quadrilateral figure $HGCB$. Take E the centre of the semicircle, and join HC , EC . The angle PCE is equal to PCA , ACE together, i. e. to DBA , CAE together; and the angle CHE is equal to ACH , CAH together, i. e. to DBA , CAH together, $\therefore PCE = CHE$, and the angle at E being common, the triangles CEH , CPE are equiangular;

whence $EH : EC :: EC : EP$,

in which proportion the three first terms being invariable, EP is also, and the point E being fixed, P is also.

(100.) *If from a given point in the diameter of a semicircle produced, three straight lines be drawn, one of which is inclined at a given angle to the diameter, another touches the semicircle, and the third cuts it, in such a manner, that the distance of the given point from the nearer extremity of the diameter, and the perpendiculars drawn from that extremity on the three aforesaid lines may be proportional; then will the lines, which join the extremities of the diameter and of that part of the cutting line which is within the circle, intersect each other in an angle equal to the given angle.*

From a given point C , in the diameter AB produced of the semicircle AGB , draw CD inclined at a given



angle to AC , CG touching, and CIH cutting the circle in such a manner that BD , BE , BF being drawn respectively perpendicular to them, CB may be to BD as BE to BF ; then if AI , BH be joined, the angle ALH or BLI will be equal to BCD .

Join OH , OG ; and draw OK perpendicular to HI . Now the angles at E and F being right angles, as also those at G and K , BE is parallel to OG , and BF to OK ;

$$\therefore (OG =) OH : BE :: CO : CB :: OK : BF,$$

$$\therefore OH : OK :: BE : BF :: BC : BD;$$

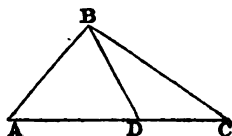
also the angle at D is equal to OKH , $\therefore$ (Eucl. vi. 7.) the triangles OHK , BCD are equiangular, and the angle OHK is equal to BCD . But OHB is equal to OBH ,

i. e. to AIH (Eucl. iii. 21.) $\therefore OHK$ is equal to AIH , LHI together, *i. e.* to ALH (Eucl. i. 32.); wherefore ALH is equal to BCD .

SECT. III.

(1.) *Any side of a triangle is greater than the difference between the other two sides.*

Let ABC be a triangle; any of its sides is greater than the difference of the other two.



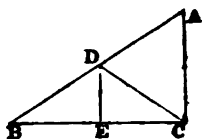
Let AC be greater than AB ; and cut off $AD = AB$; join BD ; then the angle ABD is equal to ADB . But the exterior angle BDC is greater than DBA , *i. e.* than BDA , and $\therefore$ greater than DBC (Eucl. i. 16.); whence BC is greater than DC , *i. e.* than the difference of the sides AC and AB . In the same way it may be shewn that AB is greater than the difference of AC and BC ; and AC greater than the difference of AB and BC .

(2.) *In any right-angled triangle, the straight line joining the right angle and the bisection of the hypotenuse is equal to half the hypotenuse.*

Let ACB be a right-angled triangle, whose hypo-

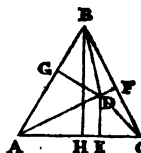
then use AB is bisected in D ; join DC ; DC is equal to the half of AB .

From D draw DE parallel to AC ; $\therefore$ (Eucl. vi. 2.) $BE = EC$, and ED is common and at right angles to BC , $\therefore DC = BD$, i. e. the half of AB .



(3.) *If from any point within an equilateral triangle perpendiculars be drawn to the sides; they are together equal to a perpendicular drawn from any of the angles to the opposite side.*

From any point D within the equilateral triangle ABC , let perpendiculars DE , DF , DG be drawn to the sides; they are together equal to BH a perpendicular drawn from B on the opposite side AC .



Join DA , DB , DC . Since triangles upon the same and equal bases are to one another as their altitudes,

$$ABC : ADC :: BH : DE,$$

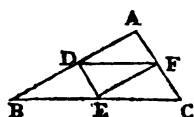
$$\text{also } ABC : BDC :: BH : DF,$$

$$\text{and } ABC : ADB :: BH : DG;$$

whence $ABC : ADC + BDC + ADB :: BH : DE + DF + DG$, in which proportion the first term being equal to the second, $\therefore DE + DF + DG = BH$.

(4.) *If the points of bisection of the sides of a given triangle be joined; the triangle so formed will be one fourth of the given triangle.*

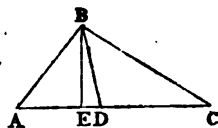
Let the sides of the triangle ABC be bisected in the points D, E, F ; join DE, EF, FD ; the triangle DEF is one fourth of the triangle ABC .



Since AB and AC are bisected in D and F , (Eucl. vi. 2.) DF is parallel to BC ; and for the same reason FE is parallel to AB , and $DFEB$ is a parallelogram, $\therefore$ the triangle DFE is equal to DBE . In the same way it may be shewn to be equal to FEC and ADF ; and $\therefore$ it is one fourth of ABC .

(5.) *The difference of the angles at the base of any triangle is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.*

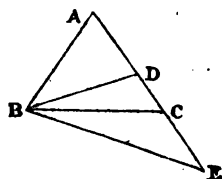
From B the vertex of the triangle ABC let BE be drawn perpendicular to the base, and BD bisecting the angle ABC ; the difference of the angles BAC, BCA is double the angle EBD .



The angle BAC is equal (Eucl. i. 32.) to the difference of the angles BEC and ABE , i. e. of a right angle and ABE . Also the angle BCA is equal to the difference of a right angle and EBC , $\therefore$ the difference of the angles BAC and BCA is equal to the difference of the angles ABE and EBC , i. e. (since $ABD = DBC$) to twice the angle EBD .

(6.) *If from one of the equal angles of an isosceles triangle any line be drawn to the opposite side, and from the same point a line be drawn to the opposite side produced, so that the part intercepted between them may be equal to the former; the angle contained by the side of the triangle and the first drawn line is double of the angle contained by the base and the latter.*

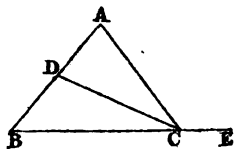
Let ABC be an isosceles triangle, having the side AB equal to AC . From B draw any line BD , and also BE cutting off DE equal to DB ; the angle ABD is double of CBE .



For the angle DCB is equal to the two DEB, CBE , i.e. to the two DBE, CBE , or to DBC and twice CBE ; but DCB is equal to ABC , $\therefore ABC$ is equal to DBC and twice CBE , and taking away the angle DBC , which is common to both, the angle ABD is equal to twice CBE .

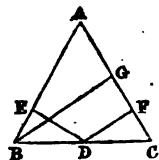
(7.) *If from the extremity of the base of an isosceles triangle, a line equal to one of the sides be drawn to meet the opposite side; the angle formed by this line and the base produced, is equal to three times either of the equal angles of the triangle.*

Let ABC be an isosceles triangle having the side AB equal to AC . From C to AB (produced if necessary) draw CD equal to AC , and let BC be produced; the angle DCE is equal to three times the angle ABC .



(11.) *If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides; these together shall be equal to a perpendicular drawn from either extremity of the base to the opposite side.*

Let ABC be an isosceles triangle, from any point D in the base of which, let DE , DF be drawn perpendicular to the sides; and from B let BG be drawn perpendicular to AC ; BG is equal to DE and DF together.



Since the angle EBD is equal to the angle at C , and the angles at E and F are right angles, the triangles BED , DFC are equiangular, and

$$\therefore BD : DC :: DE : DF,$$

$$\text{comp. } BC : DE + DF :: DC : DF.$$

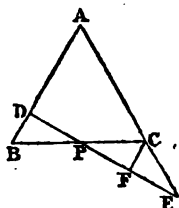
But BG being parallel to DF , $DC : DF :: BC : BG$,

$$\text{whence } BC : BG :: BC : DE + DF,$$

$$\text{and } \therefore BG = DE + DF.$$

(12.) *Of all triangles having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in the given point.*

Let BAC be the vertical angle of any number of triangles, whose bases pass through a given point P ; and let BC be bisected in P : ABC is less than any other triangle ADE .

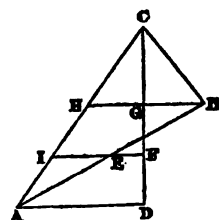


From C draw CF parallel to AB ; then the angle DBP is equal to PCF , and the vertically opposite angles DPB , CPF are equal, and $BP = PC$,

$\therefore$ the triangle DBP is equal to the triangle PCF , and
 $\therefore DPB$ is less than CPE ; add to each the trapezium
 $ADPC$, and ABC is less than ADE . In the same
 manner ABC may be proved to be less than any other
 triangle whose base passes through P .

(13.) *If from the angles at the base of a triangle perpendiculars be let fall on a line which bisects the vertical angle; the part of this line intercepted between these perpendiculars will be bisected by a perpendicular from the middle of the base.*

From A and B let perpendiculars
 AD , BG be drawn to the line CD
 which bisects the angle at C ; the part
 GD will be bisected by a perpendi-
 cular EF from E the middle point of
 the base AB .



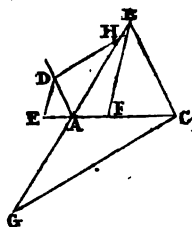
Produce BG , FE to H and I . Then IE being
 parallel to HB , and $AE = EB$, $\therefore$ (Eucl. vi. 2.) $AI = IH$.
 Also since AD is parallel to HG and IF ,

$$DF : FG :: AI : IH,$$

whence $DF = FG$, and DG is bisected in F .

(14.) *If from one of the angles at the base of a triangle a line be drawn parallel to the opposite side, and from any point in it lines be drawn making any angles with the sides (produced, if necessary); they will have the same ratio that lines have, which are drawn parallel to them from the other angles, and terminated by the same sides.*

From A one of the angles of the triangle ABC ; let AD be drawn parallel to BC the opposite side; and from any point D in it, let DE , DH be drawn making any angles with the sides; draw BF , CG parallel to them respectively; $DE : DH :: BF : CG$.



Since DE is parallel to BF , and DA to BC , the triangles DEA , BFC are equiangular,

$$\therefore DE : DA :: BF : BC;$$

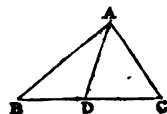
and in a similar manner it may be shewn, that

$$DA : DH :: BC : CG,$$

$$\therefore DE : DH :: BF : CG.$$

(15.) To bisect a given triangle by a line drawn from one of its angles.

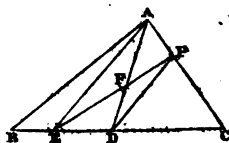
Let ABC be the given triangle; and A the angle, from which the bisecting line is to be drawn. Bisect the opposite side BC in D , and join AD ; AD bisects the triangle.



For the bases BD , DC being equal, (Eucl. i. 38.) the triangles ABD , ADC are also equal.

(16.) To bisect a given triangle by a line drawn from a given point in one of its sides.

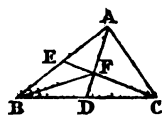
Let ABC be the given triangle, and P the given point. Bisect BC in D ; join AD , PD ; and from A draw AE parallel to PD ; join PE ; PE bisects the triangle ABC .



Since AE is parallel to PD , the triangle APD is equal to the triangle EPD ; from each of them take away the triangle PFD , and $AFP = EFD$. Also since BD is equal to DC , the triangle ABD is equal to the triangle ADC ; parts of which EFD , AFP are equal, $\therefore ABEF$ is equal to $PFDC$; whence $ABEF$ and AFP together, or $ABEP$ will be equal to $PFDC$ and FED together, i.e. to PEC ; and $\therefore$ the triangle ABC is bisected by PE .

(17.) To determine a point within a given triangle, from which lines drawn to the several angles, will divide the triangle into three equal parts.

Let ABC be the given triangle; bisect AB , BC , in E , and D ; join AD , CE , BF ; F is the point required.

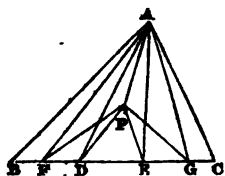


Since $BD = DC$, the triangle BAD is equal to DAC ; and for the same reason the triangle BFD is equal to DFC ; $\therefore$ the triangle BFA is equal to AFC . Again, since $BE = EA$, the triangle BEC is equal to the triangle AEC ; parts of which, the triangles BEF , AEF are equal; $\therefore$ the triangle BFC is equal to AFC ; and $\therefore$ the three BFC , BFA , AFC are equal to one another.

(18.) To trisect a given triangle from a given point within it.

Let ABC be the given triangle, and P the given point within it. Trisect the side BC in D and E ; join

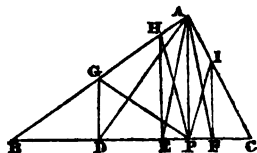
PD , PE ; and from A draw AF , AG respectively parallel to them. Join PF , PG , AP . Those three lines will divide the triangle into three equal parts.



Join AD , AE . Since AF is parallel to PD , the triangle APF is equal to ADF ; to each of these add ABF , $\therefore APFB$ is equal to ADB . In the same manner $APGC$ is equal to AEC ; and $\therefore$ the remainder FPG is equal to DAE . Now the triangles ABD , ADE , AEC , being on equal bases and of the same altitude, are equal, $\therefore APFB$, PFG , $APGC$ are also equal; and the triangle ABC is trisected.

(19.) *From a given point in the side of a triangle, to draw lines, which will divide the triangle into parts which shall have a given ratio.*

Let ABC be the given triangle, and P the given point in the side BC . Divide BC , in the points D , E , F , into parts which shall have the given ratio. Join AD , AE , AF , AP ; and draw DG , EH , FI parallel to AP . Join PG , PH , PI ; they will divide the triangle, as required.

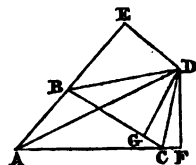


For the triangles ABD , ADE , AEF , AFC being as their bases will be in the given ratio. And since DG is parallel to AP , the triangles DGA , DGP are equal, $\therefore DBA$, GPB are equal. And since the triangle $ADP = AGP$, and $AEP = AHP$, $\therefore ADE = HPG$. Also $APE = AHP$, and $APF = AIP$, $\therefore AEF = AHPI$, and

$AFC = PIC$; $\therefore$ the parts PBG , GPH , $HPIA$, IPC are equal to ABD , ADE , AEF , AFC , and are $\therefore$ in the given ratio. The same may be proved whatever be the number of parts.

(20.) *If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines, a line be drawn to the opposite angle of the triangle; it will bisect that angle.*

Let the exterior angles EBC , BCF , of the triangle ABC , be bisected by the lines BD , CD meeting in D . Join DA ; it will bisect the angle BAC .



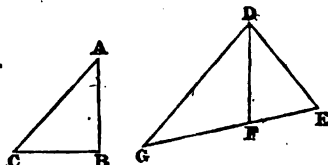
Let fall the perpendiculars DE , DF , DG . Then the angles DBE , DBG being equal, and the angles at E and G being right angles, and DB common to the triangles DBE , DBG , $\therefore DE = DG$. In the same manner $DG = DF$; and $\therefore DE = DF$. Hence in the right-angled triangles DAE , DAF , DE is equal to DF and DA is common, $\therefore$ the triangles are equiangular, and the angles DAE , DAF are equal, i. e. BAC is bisected by AD .

(21.) *If in two triangles the vertical angle of the one be equal to that of the other, and one other angle of the former be equal to the exterior angle at the base of the latter; the sides about the third angle of the former shall be proportional to those about the interior angle at the base of the latter.*

Let ABC , DEF be two triangles having the angle BAC equal to EDF , and ABC equal to the exterior angle DFG , made by producing the side EF ;

then $AC : CB :: DE : EF$.

At the point D in the line FD , make the angle FDG equal to the angle EDF or BAC , and meeting EF pro-



duced in G . Since the angle FDG is equal to the angle BAC , and DFG is equal to ABC , $\therefore$ the triangles ABC , DFG are equiangular, and

$$AC : CB :: DG : GF,$$

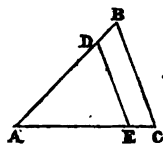
But since the angle GDE is bisected by DF , $\therefore$ (Eucl. vi. 3.)

$$DG : GF :: DE : EF,$$

$$\therefore AC : CB :: DE : EF.$$

(22.) *In a given triangle to draw a line parallel to one of the sides, so that it may be a mean proportional between the segments of the base.*

Let ABC be the given triangle; in the base of which take a point E , such that AE may be to EC in the duplicate ratio of $AC : CB$; draw ED parallel to BC ; ED is the line required.



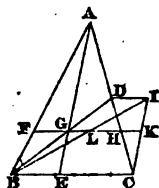
Since ED is parallel to BC , $AE : ED :: AC : CB$. But $AE : EC$ in the duplicate ratio of $AC : CB$, and

therefore also in the duplicate ratio of $AE : ED$;
whence from the definition of the duplicate ratio,

$$AE : ED :: ED : EC.$$

(23.) *To draw a line parallel to the common base of two triangles which have different altitudes, so that the parts of it intercepted by the sides may have a given ratio.*

Let ABC , DBC be two triangles on the same base BC , the vertex D being in the side AC . Divide BC in E , so that $BC : CE$ may be equal to the given ratio. Join AE , cutting BD in G ; and through G draw FH parallel to BC ; FH is the line required.



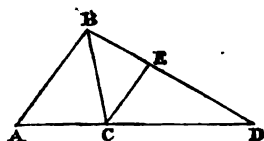
Since FH is parallel to BC , $FH : GH :: BC : CE$, i. e. in the given ratio.

But if the vertex I is not in AC , draw ID parallel to BC ; join BD ; divide the base BC , as before; join AE , and draw FK parallel to BC . Then it is evident that $GH = LK$, and $\therefore FH : LK$ in the given ratio.

COR. If the triangles be upon equal bases, but in the same straight line, the line may be drawn in a similar manner.

(24.) *If the base of a triangle be produced, so that the whole may be to the part produced in the duplicate ratio of the sides; the line joining the vertex and the extremity of the part produced will be a mean proportional between the whole line produced and the part produced.*

Let AC be produced to D , so that AD may be to DC in the duplicate ratio of $AB : BC$; join BD ; it will be a mean proportional between AD and DC .

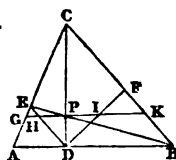


Draw CE parallel to AB ; then $AB : CE :: AD : DC$, i. e. in the duplicate ratio of $AB : BC$, whence $AB : BC :: BC : CE$, i. e. the sides about the equal angles ABC , BCE are proportional; therefore the triangles ABC , BCE are similar, and the angle at A is equal to the angle CBD ; $\therefore$ the triangles ABD , CBD are equiangular, and

$$AB : BD :: BD : DC.$$

(25.) To determine a point within a given triangle, which will divide a line parallel to the base into two segments, such that the excess of each segment above the perpendicular distance between the parallel lines may be to each other in the duplicate ratio of the respective segments.

Let ABC be the given triangle. From C draw CD perpendicular to AB ; and from D draw DE , DF bisecting the angles ADC , BDC . Join BE , cutting CD in P ; P is the point required.



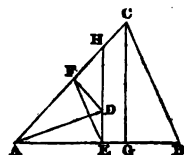
Through P draw $GHIK$ parallel to AB ; then the angle PDH is equal to the angle HDA , i. e. to the alternate angle PHD ; and $\therefore HP$, and in like manner PI will each be equal to PD the perpendicular distance of GK from AB ; and GH , IK will be equal to the

excess of each segment above that distance PD . And since GP is parallel to AB ,

$GP : PK :: AD : DB :: GH : (HP =) PI$,
 hence (Eucl. v. 19. Cor.) $GH : (PI =) PH :: PH : IK$,
 and $\therefore GH : IK$ in the duplicate ratio of $GH : HP$,
 i. e. of $GP : PK$.

(26.) *If perpendiculars be drawn to two sides of a triangle from any two points therein; the distance of their concurrence from that of the two sides will be to the distance between the two points, as either side is to the perpendicular drawn from its extremity upon the other.*

From any two points E, F in the sides AB, AC of the triangle ABC , let perpendiculars ED, FD be drawn, meeting in D . Join AD, EF ; and from C draw CG perpendicular to AB ; $AD : FE :: AC : CG$.

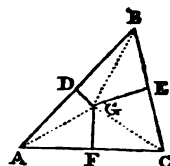


Produce ED to H . And since the angles AED, AFD are right angles, a circle described on AD as a diameter will pass through F and E , and $\therefore$ the angles FAD, FED standing in the same segment are equal; $\therefore$ the triangles AHD, HEF are equiangular;

and $\therefore AD : FE :: AH : HE :: AC : CG$,
 since HE is parallel to CG .

(27.) *If the three sides of a triangle be bisected, the perpendiculars drawn to the sides at the three points of bisection, will meet in the same point.*

Let the sides of the triangle ABC be bisected in the points D, E, F . Draw the perpendiculars EG, FG meeting in G . The perpendicular at D also passes through G .

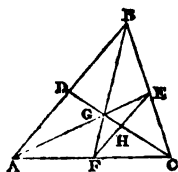


Join GD, GA, GB, GC . Since $AF = FC$, and FG is common to the triangles AFG, CFG , and the angles at F are right angles, $\therefore AG = GC$. In the same way it may be shewn that $GC = GB$; $\therefore AG = GB$; but $AD = DB$, and DG is common to the triangles ADG, BDG , $\therefore$ the angles at D are equal, and $\therefore$ right angles, or the perpendicular at D passes through G .

COR. The point of intersection of the perpendiculars is equally distant from the three angles.

(28.) *If from the three angles of a triangle lines be drawn to the points of bisection of the opposite sides, these lines intersect each other in the same point.*

Let the sides of the triangle ABC be bisected in D, E, F . Join AE, CD , meeting each other in G . Join BG, GF ; BGF is a straight line.



Join EF , meeting CD in H . Then (Eucl. vi. 2.) FE is parallel to AB , and $\therefore$ the triangles DAG, GEH are equiangular,

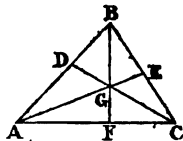
$$\therefore DA : DG :: HE : HG,$$

$$\text{or } DB : DG :: HF : HG,$$

i.e. the sides about the equal angles are proportional; $\therefore$ the triangles BDG, GHF are similar, and the angle $DGB = HGF$; and $\therefore BG$ and GF are in the same straight line.

(29.) *The three straight lines, which bisect the three angles of a triangle, meet in the same point.*

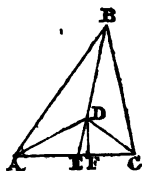
Let the angles BAC , BCA be bisected by the lines AE , CD , and through G their point of intersection draw BGF ; it bisects the angle at B .



For (Eucl. vi. 3.) $BC : CF :: BG : GF :: BA : AF$,
 $\therefore BC : BA :: CF : FA$,
 or BF bisects the angle ABC .

(30.) *If the three angles of a triangle be bisected, and one of the bisecting lines be produced to the opposite side; the angle contained by this line produced, and one of the others is equal to the angle contained by the third, and a perpendicular drawn from the common point of intersection of the three lines to the aforesaid side.*

Let the three angles of the triangle ABC be bisected by the lines AD , BD , CD ; produce BD to E , and from D draw DF perpendicular to AC ; the angle ADE is equal to CDF .

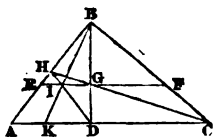


Since the three angles of the triangle ABC are equal to two right angles, $\therefore$ the angles DAB , DBA , DCF are together equal to one right angle, i. e. to DCF , and CDF ; whence the two angles DAB , DBA are together equal to the angle CDF ; but ADE is equal to the same two angles, and $\therefore ADE$ is equal to CDF .

(31.) *In a right-angled triangle, if a straight line be drawn parallel to the hypotenuse, and cutting the*

perpendicular drawn from the right angle ; and through the point of intersection a line be drawn from one of the acute angles to the opposite side, and the extremity of this line and of the perpendicular be joined ; the locus of its intersection with the line parallel to the hypotenuse will be a straight line.

Let EF be drawn parallel to AC the hypotenuse of the right-angled triangle ABC ; and from the right angle B let the perpendicular DB be drawn, meeting EF in G ; through G draw CGH ; join HD ; the locus of I , the intersection of EF and HD is a straight line.



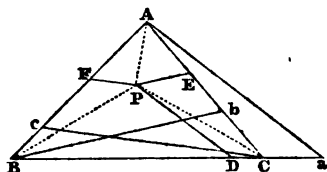
Because EG is parallel to AC the base of the triangles AHC , ABD , $AK : KD :: EI : IG :: AD : DC$. But AD and DC are invariable, $\therefore$ the ratios of $AK : KD$, and $EI : IG$ are also. In the same manner if any other line be drawn parallel to the hypotenuse, and a similar construction be made, the point of intersection will divide the part intercepted between AB and BD in the ratio of $AD : DC$, or $AK : KD$, and will $\therefore$ be in the line BK , which is the locus required.

(32.) *If from the angles of a triangle, lines, each equal to a given line, be drawn to the opposite sides (produced if necessary) ; and from any point within, lines be drawn parallel to these, and meeting the sides of the triangle ; these lines shall together be equal to the given line.*

From the angles of the triangle ABC let the lines Aa , Bb , Cc be drawn to the opposite sides, each equal

to a given line L ; and parallel to them respectively draw, from any point P , the lines PD , PE , PF ; these together will be equal to L .

Join PA , PB , PC . Then since the triangles ABC , APC are on the same base AC , they are to one another



as the perpendiculars from B and P , i.e. by similar triangles, as $Bb : PE$, or as $L : PE$. In the same way,

$$ABC : ABP :: L : PF,$$

$$\text{and } ABC : BPC :: L : PD;$$

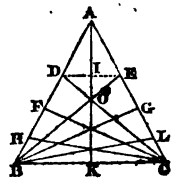
$$\therefore ABC : APC + ABP + BPC :: L : PE + PF + PD;$$

and since the first term is equal to the second, the third will be equal to the fourth, or $L = PD + PE + PF$.

(33.) *If the sides of a triangle be cut proportionally, and lines be drawn from the points of section to the opposite angles; the intersections of these lines will be in the same line, viz. that drawn from the vertex to the middle of the base.*

Let the sides of the triangle ABC be cut proportionally, so that $AD : AE :: DF : EG :: FH : GL :: HB : LC$.

Join BE , BG , BL , CD , CF , CH ; these lines will intersect each other in the line AK drawn from A to K the middle of the base BC .



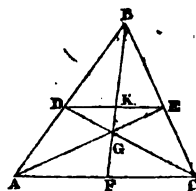
Join DE . Then since when any number of magnitudes are proportional, as one antecedent is to its consequent, so are all the antecedents taken together to all the consequents together, $\therefore AD : AE :: AB : AC$, and DE is parallel to BC . Join KO , and let it meet DE in I . The triangles BOK , IOE are similar, and therefore,

$BK : KO :: EI : IO$, and for the same reason,

$CK : KO :: DI : IO$, whence $EI = DI$, and DE is bisected by KO ; and it is also bisected by AK , $\therefore AK$ passes through O . In the same manner it may be shewn that BG and CF , as also BL , CH intersect each other in points which are in the line AK .

(34.) *If from any point in one side of a triangle, two lines be drawn, one to the opposite angle, and the other parallel to the base, and the former intersect a line drawn from the vertex bisecting the base; this point of intersection, that of the line parallel to the base and the third side, and the third angular point are in the same straight line.*

From any point D in the side AB of the triangle ABC , let DE be drawn parallel to AC , and DC joined; and let DC meet BF drawn from B to the middle of AC in G ; A , G , E are in the same straight line.



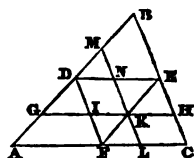
Let DE cut BF in K . The triangles DGK , CGF are equiangular, and

$$\therefore DG : GC :: DK : FC :: DE : AC;$$

hence the triangles DGE , AGC , having one angle in each equal, viz. EDG , GCA , and the sides about them proportional, are therefore similar; whence the angles AGC , DGE are equal; and DGC being a straight line, AGE is also.

(35.) *If one side of a triangle be divided into any two parts, and from the point of section two straight lines be drawn parallel to, and terminating at the other sides, and the points of termination be joined; and any other line be drawn parallel to either of the two former lines, so as to intersect the other, and to terminate in the sides of the triangle; then the two extreme parts of the three segments into which the line so drawn is divided will always be in the ratio of the segments of the first divided line.*

Let AB be divided into any two parts in D , from which draw DE , DF parallel to the other two sides of the triangle; join EF , and draw GH parallel to DE , meeting DF and EF in I and K ;

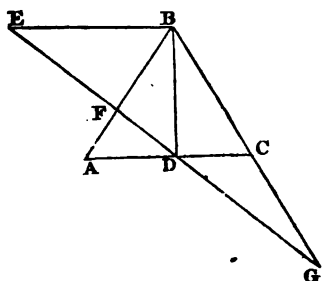


$GI : KH :: AD : DB$; and if LM be parallel to DF ,
 $LK : MN :: AD : DB$.

Since GI is parallel to AF , and NK to DF ,
 $GI : AF :: (ID =) NK : FD :: NE : DE :: KH : FC$,
 $\therefore GI : KH :: AF : FC :: AD : DB$, since DF is parallel to BC . Again since ML is parallel to BC ,
 $MN : BE :: ND : DE :: KF : FE :: KL : EC$,
 $\therefore MN : KL :: BE : EC :: BD : DA$.

(36.) *If through the point of bisection of the base of a triangle any line be drawn, intersecting one side of the triangle, and the other produced, and meeting a parallel to the base from the vertex; this line will be cut harmonically.*

From the vertex B of the triangle ABC , let BE be drawn parallel to the base AC , and through the middle point D let any line EGF be drawn meeting AB , BC , BE , in F , G , E ;



$$EG : DG :: FE : FD.$$

Since AD is parallel to BE ,

$$FE : FD :: BE : AD,$$

$$\text{but } BE : (DC =) DA :: EG : GD,$$

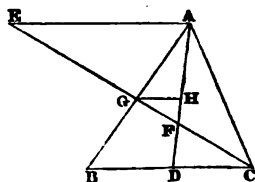
since the triangles BGE , DGC are equiangular,

$$\therefore (\text{Eucl. v. 15.}) EG : GD :: FE : FD,$$

or the line is divided harmonically.

(37.) *If from either angle of a triangle a line be drawn intersecting that which joins the vertex and the bisection of the base, the opposite side, and the line from the vertex parallel to the base; it will be cut harmonically.*

From the vertex A of the triangle ABC , let AE be drawn parallel to the base BC , and AD to its point of bisection D ; and from C draw any line CFG ; then will



$$CE : CF :: EG : FG.$$

Draw GH parallel to BC . Since AE and BC are parallel, (Eucl. vi. 2.)

$$BA : AG :: CE : EG,$$

and since GH is parallel to BD ,

$$BA : AG :: BD : GH :: DC : GH, \\ :: CF : FG,$$

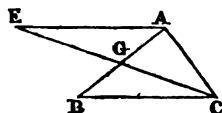
since the triangles DFC , GHF are similar ;

$$\therefore CE : EG :: CF : FG,$$

$$\text{and } CE : CF :: EG : FG.$$

(38.) To draw a line from one of the angles at the base of a triangle, so that the part of it cut off by a line drawn from the vertex parallel to the base, may have a given ratio to the part cut off by the opposite side.

From A let AE be drawn parallel to BC . Divide AB in G , so that $AB : AG$ in the given ratio ; join CG , and produce it to meet AE in E . CGE is the line required.



For the triangles AGE , BGC are equiangular,

$$\therefore CG : EG :: BG : AG,$$

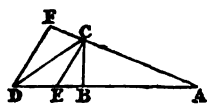
whence (Eucl. v. 18.) $CE : EG :: BA : AG$,

i. e. in the given ratio.

(39.) To determine that point in the base produced of a right-angled triangle, from which the line drawn to the angle opposite to the base shall have the same ratio to the base produced, which the perpendicular has to the base itself.

Let AB be the base, and CB the perpendicular of

a right-angled triangle. Draw CE at right angles to AC , meeting AB produced in E . At the point C make the angle $ECD = CAB$. D is the point required.



From D draw DF perpendicular to AC , and $\therefore$ parallel to CE . Since the angle FDC is equal to the alternate angle DCE , i. e. to CAB , and the angles at F and B are right angles, $\therefore$ the triangles DCF , ACB are equiangular; and DAF is also equiangular to ACB , hence

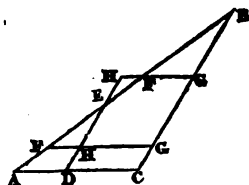
$$FD : DA :: BC : CA,$$

$$\text{and } DC : DF :: AC : AB,$$

$$\therefore \text{ex æquo per. } CD : DA :: CB : BA.$$

(40.) *If the base of any triangle be divided into two parts by a line which is a mean proportional between them, and which being drawn parallel to the second side is terminated in the third; any line parallel to the base will be divided by the mean proportional (produced if necessary) into segments, which will be to each other inversely as the whole mean proportional to that segment which is terminated in the third side of the triangle.*

Let AC the base of the triangle ABC be divided into two parts in D , by a line DE which is parallel to BC , and a mean proportional between AD and DC ; then any line FG parallel to AC , and meeting DE (produced if necessary) in H , will be divided into segments FH , HG , which are to each other inversely as the lines DE , HE .



For since FH is parallel to AD ,

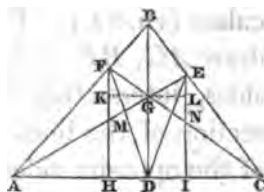
$$FH : AD :: HE : DE,$$

but $AD : DE :: DE : (DC =) HG$,

$$\therefore FH : DE :: HE : HG.$$

(41.) *If from the extremities of the base of any triangle, two straight lines be drawn intersecting each other in the perpendicular, and terminating in the opposite sides; straight lines drawn from thence to the intersection of the perpendicular with the base, will make equal angles with the base.*

From A and C , the extremities of AC , the base of the triangle ABC , let AE , CF be drawn intersecting the perpendicular BD in the same point G . Join FD , ED ; these lines make equal angles FDA , EDC with the base.



Draw EI , FH perpendicular, and KGL parallel to the base; then FH is parallel to BD ,

$$\text{and } \therefore BG : BD :: FM : FH.$$

And in the same manner it may be shewn that

$$BG : BD :: EN : EI;$$

$$\text{whence } FM : FH :: EN : EI;$$

$$\text{and } \therefore FM : EN :: FH : EI.$$

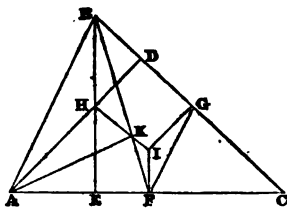
But $FM : EN :: FG : GN :: KG : GL :: HD : DI$,

$$\therefore HD : DI :: HF : EI,$$

whence the two triangles DFH , DEI , having the angle at H equal to the angle at I , and the sides about the equal angles proportional, are equiangular; $\therefore$ the angle HDF is equal to EDI .

(42.) *In every triangle, the intersection of the perpendiculars drawn from the angles to the opposite sides, the intersection of the lines from the angles to the middle of the opposite sides, and the intersection of the perpendiculars from the middle of the sides, are all in the same straight line. And the distances of those points from one another are in a given ratio.*

From the angles A and B of the triangle ABC , let AD , BE be drawn perpendicular to the opposite sides, H will be the intersection of the three perpendiculars (vii. 34.). From A and B draw AG , BF to the points of bisection of the opposite sides, intersecting in K , which $\therefore$ is (iii. 28.) the intersection of the lines drawn from the angles to the middle of the opposite sides; and from F and G draw the perpendiculars FI , GI meeting in I , which $\therefore$ (iii. 27.) is the intersection of the three perpendiculars. Join HK , KI ; HKI is a straight line.



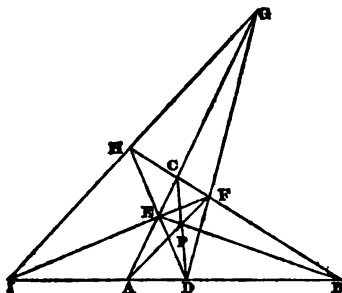
Join GF . (Eucl. vi. 2.) AB is parallel to, and double of GF ; $\therefore$ by similar triangles ABK , KFG , BK is double of KF , and AK double of KG . And the triangles AHB , FIG are equiangular, $\therefore AH$ is double of IG , and BH is double of IF ;

and $\therefore BH : IF :: 2 : 1 :: BK : KF$,
whence the triangles BHK , KIF having the angles at B and F equal, and the sides about them proportional, are similar, $\therefore$ the angle HKB is equal to IKF , $\therefore H$, K , and I are in the same straight line.

And since BK is double of KF , HK is double of KI , and $\therefore$ their distances from each other will be in an invariable ratio.

(43.) *If straight lines be drawn from the angles of a triangle through any point, either within or without the triangle, to meet the sides, and the lines joining these points of intersection and the sides of the triangle be produced to meet; the three points of concurrence will be in the same straight line.*

Let ABC be a triangle from the three angles of which let lines AF , BE , CD be drawn through a point P within the triangle. Join DE , DF , EF , and produce them



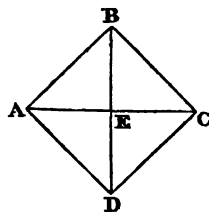
to meet the sides in H , G , I ; these three points will be in the same straight line.

Join GH , HI . Then the three angles of the triangle DHG being equal to two right angles, as also the three EHI , EIH , and $(HEI$ or) DEF , as also the two DFI , GFI ; $\therefore$ the three angles of the triangle DHG together with the angles EHI , EIH , DEF , DFI , GFI are equal to six right angles. Now the angles of the triangles DEF , FGI are together equal to four right angles, whence DHG , DHI are equal to two right angles; or GH , HI are in the same straight line.

SECT. IV.

(1.) *The diameters of a rhombus bisect each other at right angles.*

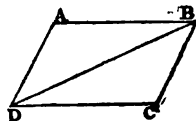
Let $ABCD$ be a rhombus, whose diameters are AC , BD ; they bisect each other at right angles in E .



Since $AB=AD$, and AC is common to the two triangles ABC , ADC , the two BA , AC are equal to the two DA , AC , each to each, and $BC=DC$, $\therefore$ the angle BAC is equal to the angle DAC . Again, since BA , AE are equal to DA , AE , each to each, and the included angles are equal, $\therefore BE=ED$, and the angles AEB , AED are equal, and $\therefore$ are right angles. For the same reason $AE=EC$; also the angles BEC , DEC are right angles.

(2.) *If the opposite sides or opposite angles of a quadrilateral figure be equal, the figure will be a parallelogram.*

Let $ABCD$ be a quadrilateral figure, whose opposite sides are equal. Join BD . Since $AB=DC$, and BD is common, the two AB , BD are equal to

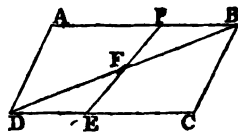


the two CD , DB , each to each, and $AD=BC$, $\therefore$ the angle $ABD=BDC$, whence (Eucl. i. 27.) AB is parallel to DC ; also the angle $ADB=DBC$, whence AD is parallel to BC ; and the figure is a parallelogram.

Again, let the opposite angles be equal. Then since the four angles of the quadrilateral figure $ABCD$ are equal to four right angles, and that BAD, ADC together are equal to DCB, CBA , $\therefore BAD, ADC$ together are equal to two right angles; whence AB is parallel to CD . In the same way it may be shewn that AD is parallel to BC , and $\therefore ABCD$ is a parallelogram.

(3.) *To bisect a parallelogram by a line drawn from a point in one of its sides.*

Let $ABCD$ be a parallelogram, and P a given point in the side AB . Draw the diameter BD , which bisects the parallelogram. Bisect BD in F ; join PF , and produce it to E . PE bisects the parallelogram.

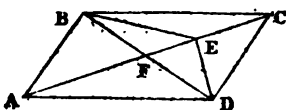


Since the angle PBD is equal to the angle BDE , and the vertically opposite angles at F are equal, and $BF = FD$, $\therefore$ the triangles PBF, DFE are equal. But the triangle ABD is equal to BDC , $\therefore APFD$ is equal to $BFEC$; and to these equals adding the equal triangles DFE, PFB , the figure $APED = PECB$; and AC is $\therefore$ bisected by PE .

COR. Any line drawn through the middle point of the diameter of a parallelogram is bisected in that point.

(4.) *If from any point in the diameter (or diameter produced) of a parallelogram straight lines be drawn to the opposite angle; they will cut off equal triangles.*

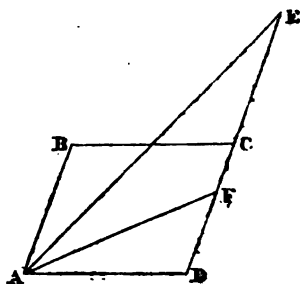
From any point E , in AC the diameter of the parallelogram $ABCD$, let lines EB , ED be drawn; the triangles ABE , AED are equal; as also the triangles BEC , CED .



Draw the diameter BD . The bases BF , FD being equal, the triangles BFA , DFA (Eucl. i. 38.), as also the triangles BFE , DFE are equal, hence $\therefore$ BAE , DAE are equal. And ABC being equal to ADC , the triangles BEC , DEC are also equal.

(5.) From one of the angles of a parallelogram to draw a line to the opposite side, which shall be equal to that side together with the segment of it which is intercepted between the line and the opposite angle.

Let $ABCD$ be the parallelogram, A the angle from which the line is to be drawn. Produce DC to E , making $CE = CD$. Join AE , and at the point A make the angle $EAF = AEF$; AF is the line required.



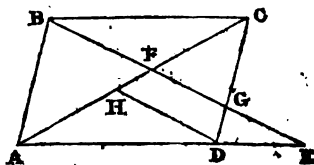
For CE being equal to CD , EF is equal to DC and CF together; and the angles FEA , FAE being equal, $FA = FE$, and $\therefore$ $AF = DC$ and CF together.

COR. In the same manner if $CE = CB$, $AF = EF = BC$ and CF together.

(6.) If from one of the angles of a parallelogram a straight line be drawn cutting the diameter, a side, and

a side produced; the segment intercepted between the angle and the diameter, is a mean proportional between the segments intercepted between the diameter and the sides.

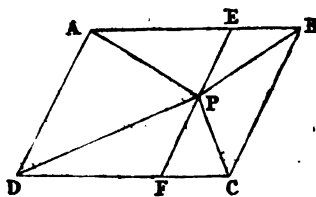
From B , one of the angles of the parallelogram $ABCD$, let any line BE be drawn cutting the diameter AC in F , the opposite side in G , and AD produced in E ; BF is a mean proportional between FG and FE .



Draw DH parallel to BF , and $\therefore$ equal to it; $\therefore$ also $AH = FC$, and $AF = CH$. Since HD is parallel to BG , $FG : DH (:: CF : CH :: AH : AF) :: DH : FE$, or $FG : BF :: BF : FE$.

(7.) The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of two opposite sides, are together half of the parallelogram.

Let P be any point within the parallelogram $ABCD$, from which let lines PA , PD , PB , PC be drawn to the extremities of the opposite sides; the triangles PAD , PBC are equal to half the parallelogram; as also the triangles APB , DPC .

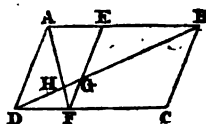


Through E draw EPF parallel to AD or BC ; then (Eucl. I. 41.) the triangle APD is half of $AEFD$, and BPC is half of $BEFC$, $\therefore APD$, BPC are together half

of $ABCD$. In the same manner if a line be drawn through P parallel to AB or DC , it may be shewn that APB , DPC together are half of $ABCD$.

(8.) *If a straight line be drawn parallel to one of the sides of a parallelogram, and one extremity of this line be joined to the opposite one of the parallel side, by a line which also cuts the diameter; the segments of the diameter made by this line will be reciprocally proportional to the segments of that part of it which is intercepted between the side and the parallel line.*

Let EF be drawn parallel to AD one of the sides of the parallelogram $ABCD$, cutting the diameter BD in G . Join AF , cutting it also in H ; then will $BH : HD :: HD : HG$.

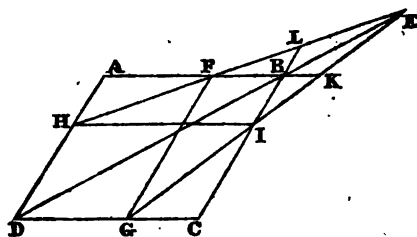


For the angle ABH being equal to HDF , and AHB to DHF , the triangles AHB , DHF are equiangular, and $\therefore BH : HD :: AH : HF :: DH : HG$, since the triangles AHD , FHG are also equiangular.

(9.) *If two lines be drawn parallel and equal to the adjacent sides of a parallelogram; the lines joining their extremities, if produced, will meet the diameter in the same point.*

Let HI , FG be drawn equal and parallel to the adjacent sides AB , BC of the parallelogram $ABCD$. Join HF , GI ; these lines produced will meet the diameter DB in the same point.

Produce AB , CB to K and L . Then the triangles AFH , LBF having the vertically opposite angles at F equal, and the alternate angles AHF , FLB also equal, are equiangular,



whence $AF : FB :: (HA=) IB : BL$;
and in the same manner it may be shewn that

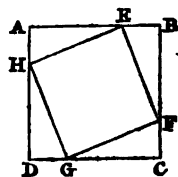
$$(GC=) FB : CI :: BK : BI,$$

$$\therefore AF : CI :: BK : BL.$$

But $AF=DG$, and $CI=DH$, $\therefore DG : DH :: BK : BL$,
and $\therefore HF$, DB , GI converge to the same point.

(10.) *If in the sides of a square, at equal distances from the four angles, four other points be taken, one in each side; the figure contained by the straight lines which join them shall also be a square.*

Let E , F , G , H be four points at equal distances from the angles of the square $ABCD$. Join EF , FG , GH , HE ; $EFGH$ is also a square.

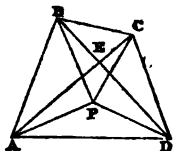


Since $AH=EB$, and $AE=BF$, and the angles at A and B are right angles, $\therefore HE=EF$, and the angle AEH is equal to the angle BFE . In the same way it may be shewn that HG and GF are

each of them equal to HE and EF , $\therefore$ the figure $HEFG$ is equilateral. It is also rectangular; for since the exterior angle FEA is equal to the interior angles EBF , EFB ; parts of which AEH and EFB are equal; $\therefore$ the remaining angle FEH is equal to the remaining angle FBE , and $\therefore$ is a right angle. In the same manner it may be shewn that the angles at F , G , H are right angles, and $\therefore$ $EFGH$ being equilateral and rectangular, is a square.

(11.) *The sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the four angles from any point within the figure, except from the intersection of the diagonals.*

Let $ABCD$ be a trapezium, whose diagonals are AC , BD , cutting each other in E ; they are less than the sum of any four lines which can be drawn to the angles from any other point within the trapezium.



Take any point P , and join PA , PB , PC , PD . Then (Eucl. i. 20.) AC is less than AP , PC ; and BD is less than BP , PD ; $\therefore$ AC , BD are less than AP , PB , PC , PD .

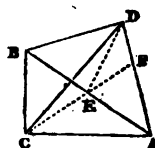
(12.) *Every trapezium is divided by its diagonals into four triangles proportional to each other.*

Let $ABCD$ be a trapezium (see last Fig.) divided by its diagonals AC , BD into the triangles AEB , BEC , AED , DEC ; these are proportional to each other.

For (Eucl. vi. 1.) $AEB : BEC :: AE : EC$,
 and $AED : DEC :: AE : EC$,
 $\therefore AEB : BEC :: AED : DEC$.

(13.) *If two opposite angles of a trapezium be right angles; the angles subtended by either side at the two opposite angular points will be equal.*

Let the two angles ACB , ADB of the trapezium $ACBD$, be right angles. Join AB , CD ; the angles ACD , ABD , subtended by AD , are equal.



Bisect AB in E . Join CE , ED , and produce CE to F . Then (iii. 2.) AE , EB , EC , ED are equal to one another. Also the angle AED is equal to the two EDB , EBD , i. e. to twice EBD ; and DEF is equal to the two DCE , EDC , i. e. to twice DCE ; and AEF is equal to twice ACE ; $\therefore$ twice ACE and twice ECD , or twice ACD will be equal to AED , i. e. to twice EBD , $\therefore ACD = ABD$.

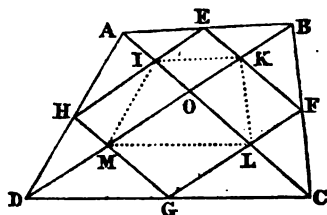
The same may be proved for the angles standing on any of the other sides.

(14.) *To determine the figure formed by joining the points of bisection of the sides of a trapezium; and its ratio to the trapezium.*

Let $ABCD$ be a trapezium, whose sides are bisected in E , F , G , H . Let the points of bisection be joined; and draw the diagonals AC , BD .

Since AB , AD are bisected in E and H , (Eucl. vi. 2.)

EH is parallel to BD ; and for the same reason FG is parallel to BD , and $\therefore$ to EH . In the same way it may be shewn that EF is parallel to HG , and $\therefore$ the figure $EFGH$ is a parallelogram.



Again (Eucl. vi. 19.) the triangle EBF is to the triangle ABC in the duplicate ratio of $EB : AB$, i.e. in the ratio of 1 : 4, $\therefore EBF$ is equal to one fourth of ABC ; for the same reason HDG is one fourth of DAC , whence EBF and HDG are together equal to one fourth of the trapezium. For, the same reason HAE and GFC are together equal to one fourth of the trapezium; therefore the four triangles together are equal to half the trapezium; and consequently $HEFG$ is equal to half of $ABCD$.

COR. 1. Hence two lines, drawn to bisect the opposite sides of a trapezium, will also bisect each other.

COR. 2. If the sides of a square be bisected and the points of bisection joined, the inscribed figure is a square, and equal to half the original square.

(15.) To determine the figure formed by joining the points where the diagonals of the trapezium cut the parallelogram; and its ratio to the trapezium.

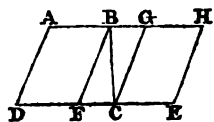
Let I, K, L, M be the points of intersection; (see last Fig.) join IK, KL, LM, MI . And let O be

the intersection of the diagonals. Since EK is parallel to OI ,

$BK : KO :: BE : EA$, i. e. in a ratio of equality. For the same reason $AI = IO$. Whence the sides of the triangle AOB being cut proportionally, IK is parallel to AB . In the same manner it may be shewn that KL , LM , MI are respectively parallel to BC , CD , DA ; whence the figure $IKLM$ will be similar to $ABCD$. Also since the triangle MIK is half the parallelogram HK , and MLK half of GK , $\therefore$ the figure $IKLM$ is half of HF , and $\therefore$ equal to one fourth of the trapezium $ABCD$.

(16.) *If two sides of a trapezium be parallel; its area is equal to half that of a parallelogram, whose base is the sum of those two sides, and altitude the perpendicular distance between them.*

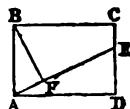
Let $ABCD$ be a trapezium, whose side AB is parallel to DC . Produce DC to E , making $CE = AB$. Draw BF , and CG , EH parallel to AD , meeting AB produced. Then AE is a parallelogram of the same altitude with the trapezium, and its base is equal to the sum of the sides AB , DC ; and $ABCD$ is half of AE .



Since $DF = CE$, the parallelograms AF , GE are equal (Eucl. i. 36.); and the diameter BC bisects the parallelogram FG ; whence $ABCD = BCEH$, and $\therefore$ the trapezium is half of the parallelogram AE .

(17.) *If from any angle of a rectangular parallelogram a line be drawn to the opposite side, and from the adjacent angle of the trapezium thus formed another be drawn perpendicular to the former; the rectangle contained by these two lines, is equal to the given parallelogram.*

From A , one of the angles of the rectangular parallelogram $ABCD$, let any line AE be drawn to the opposite side, cutting off the trapezium $ABCE$; from B let fall the perpendicular BF ; the rectangle contained by AE , BF , is equal to the parallelogram $ABCD$.



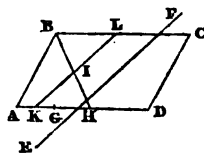
For BA being parallel to ED , the angle $BAF = AED$, and the angles at F and D are right angles, $\therefore$ the triangles BAF , AED are similar; whence

$$BF : BA :: AD : AE,$$

and the rectangle AE , BF is equal to the rectangle BA , AD , i. e. to $ABCD$.

(18.) *To divide a parallelogram into two parts which shall have a given ratio, by a line drawn parallel to a given line.*

Let $ABCD$ be the given parallelogram, and EF the line whose direction is given. Divide AD in G in the given ratio, and make $GH = AG$. Join BH ; bisect it in I , and through I draw KL parallel to EF ; KL divides the parallelogram in the given ratio.



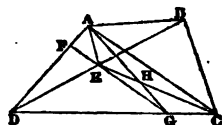
For (Eucl. vi. 1. and i. 41.)

$$ABH : ABCD :: AG : AD.$$

But the triangles BIL , KIH are equal, since $BI=IH$, and BL is parallel to KH ; $\therefore ABH = ABLK$, and $ABLK : ABCD :: AG : AD$,
 $\therefore ABLK : LKDC :: AG : GD$, i. e. in the given ratio.

(19.) To bisect a trapezium by a line drawn from one of its angles.

Let $ABCD$ be the given trapezium, and A the angle from which it is to be bisected. Draw the diagonals AC , BD ; and bisect BD , which is opposite to the angle A , in the point E . Join AE , CE ; and through E draw FEG parallel to AC . Join AG ; AG bisects the trapezium.

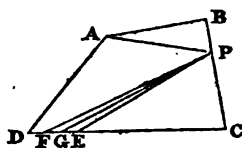


Since DE is equal to EB , the triangles AED , AEB are equal; as also DEC , BEC ; $\therefore$ the figure $AECD$ is equal to the figure $AECB$. Also (Eucl. i. 38.) the triangles AEG , CEG are equal; take away $\therefore$ the common part EHG , and $AEH = GHC$. To the figure $AECD$ add AEH , and take away its equal GHC ; and to $AECB$ add GHC , and take away AEH ; and the triangle AGD is equal to the trapezium $AGCB$, or the given trapezium is bisected by AG .

(20.) To bisect a trapezium by a line drawn from a given point in one of its sides.

Let $ABCD$ be the given trapezium, and P the given point.

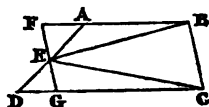
Join PA , and from the angle P bisect the trapezium $APCD$ (iv. 19.) by the line PE . On PE make the triangle PEF equal to ABP . Bisect EF in G ; join PG . PG bisects the trapezium.



Since FG is equal to GE , the triangle PGF is equal to the triangle PGE . But PGE is equal to half the triangle ABP ; and PEC is half the figure $PABC$; whence PGC is half of the trapezium $ABCD$; which is $\therefore$ bisected by PG ,

(21.) *If two sides of a trapezium be parallel; the triangle contained by either of the other sides, and the two straight lines drawn from its extremities to the bisection of the opposite side, is half the trapezium.*

Let $ABCD$ be a trapezium, having the side AB parallel to DC . Let AD be bisected in E ; join BE , CE ; the triangle BEC is half of the trapezium.

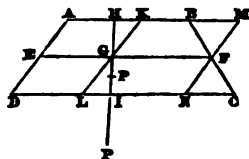


Through E draw FEG parallel to BC , meeting CD in G , and BA produced in F . The alternate angles FAE , EDG being equal, as also the angles at E , and $AE = ED$, $\therefore$ the triangles AEF , DEG are equal; whence the parallelogram $BFGC$ is equal to the trapezium $ABCD$. But $BFGC$ and the triangle BEC , being on the same base BC , and between the same parallels BC , FG , the triangle BEC is half of $BFGC$, and $\therefore$ also half of $ABCD$.

COR. From the demonstration it appears, that a trapezium which has two sides parallel, may be reduced to a parallelogram equal to it, by drawing through the point of bisection of one of the sides, which are not parallel, a line parallel to the other of those sides, and meeting the parallel sides.

(22.) To divide a given trapezium whose opposite sides are parallel, in a given ratio, by a line drawn through a given point, and terminated by the two parallel sides.

Let $ABCD$ be a trapezium whose sides AB, DC are parallel, and P the given point. Bisect AD in E , and draw EF parallel to AB , meeting BC in F . Divide EF in G in the given ratio; and through G and P draw IPH ; IPH will divide the trapezium in the given ratio.



Draw KGL, MFN parallel to AD ; then $EA = KG = MF$, and $ED = GL = FN$; but $AE = ED$, $\therefore KG = GL$, and $MF = FN$; whence (iv. 21. Cor.) $ADIH = AL$, and $HICB = KN$.

Now $AL : KN :: EG : GF$,

$\therefore ADIH : HICB :: EG : GF$,

i. e. in the given ratio.

(23.) If a trapezium, which has two of its adjacent angles right angles, be bisected by a line drawn from the middle of one of those sides which are not parallel; the

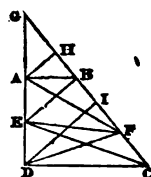
sum of the parallel sides will have to one of them the same ratio, that the side which is not bisected has to that segment of it which is adjacent to the other.

Let $ABCD$ be a trapezium, having the angles at A and D right angles, and $\therefore AB, DC$ parallel; and let the trapezium be bisected by EF ; if AD be bisected in E ,

$$AB + DC : AB :: BC : CF;$$

but if BC be bisected in F ,

$$AB + DC : AB :: AD : DE.$$



Produce DA, CB to meet in G . Join AF, DE, BE, CE , and let fall the perpendiculars AH, DI . Since $AE = ED$, the triangles AFE, DFE are equal, $\therefore$ the triangles DFC, AFB are equal; $\therefore$ the rectangles FC, DI and BF, AH are equal,

$$\text{and } FC : FB :: AH : DI :: AB : CD,$$

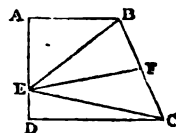
$$\therefore AB + CD : AB :: (FC + FB =) BC : FC.$$

But if BC be bisected, the triangles EBF, ECF being equal, the triangles AEB, EDC are also equal,

$$\therefore (\text{Eucl. vi. 15.}) AB : CD :: DE : EA,$$

$$\text{and } AB + CD : AB :: (DE + EA =) AD : DE.$$

In like manner $AB + DC : DC :: AD : AE$.

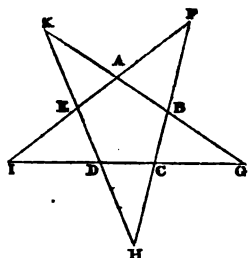


(2A.) *If the sides of an equilateral and equiangular pentagon be produced to meet; the angles formed by these lines are together equal to two right angles.*

Let $ABCDE$ be an equilateral and equiangular

pentagon; and let the sides be produced to meet in F, G, H, I, K ; the angles at these points are together equal to two right angles.

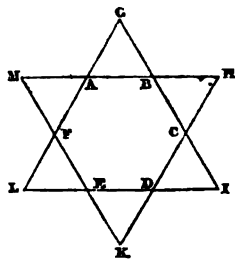
For since BCG is the exterior angle of the triangle FCI , it is equal to the angles at F and I . For the same reason the angle CBG is equal to the angles at K and H ; and $\therefore$ the angles at F, G, H, I, K are equal to the three angles of the triangle BCG , *i. e.* to two right angles.



(25.) *If the sides of an equilateral and equiangular hexagon be produced to meet; the angles formed by these lines are together equal to four right angles.*

Let $ABCDEF$ be an equilateral and equiangular hexagon; and let the sides be produced to meet in G, H, I, K, L, M ; the angles at these points are together equal to four right angles.

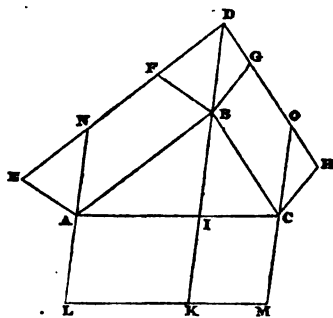
For GLI being a triangle, the angles at G, I, L are equal to two right angles; and for the same reason, the angles at H, K, M are equal to two right angles; $\therefore$ the six angles are equal to four right angles.



(26.) *The area of any two parallelograms described on the two sides of a triangle is equal to that of a paral-*

lelogram on the base, whose side is equal and parallel to the line drawn from the vertex of the triangle to the intersection of the two sides of the former parallelograms produced to meet.

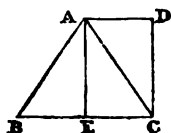
Let BE and CG be parallelograms described on the sides AB , BC of the triangle ABC ; and let EF , HG be produced to meet in D . Join DB ; produce it, and make $IK = DB$; through A draw AL equal and parallel to IK ; and complete the parallelogram AM . AM is equal to AF and CG together.



Produce LA , MC to N and O ; since ND is parallel to AB , and AN to BD , $NABD$ is a parallelogram, and equal to EB , which is on the same base, and between the same parallels. It is also equal to AK ; because they are upon equal bases DB , IK , and between the same parallels; $\therefore AK = EB$. In the same manner $IM = BH$, $\therefore AM$ is equal to AF and CG together.

(27.) *The perimeter of an isosceles triangle is greater than the perimeter of a rectangular parallelogram, which is of the same altitude with, and equal to the given triangle.*

Let ABC be an isosceles triangle, whose base is BC . Draw AE perpendicular to BC , and $\therefore$ bisecting it; and draw AD , CD parallel respectively to BC ,

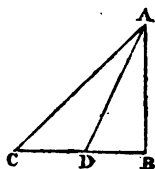


AE ; then DE is a rectangular parallelogram of the same altitude with, and equal to the triangle ABC (Eucl. i. 42.). The perimeter of ABC is greater than that of DE .

Because $AB=AC$, and $BE=EC$, the perimeter of ABC is double of AC and EC together; also the perimeter of DE is double of AE and EC together. But since AEC is a right angle, AC is greater than AE ; and $\therefore$ the perimeter of ABC greater than that of DE .

(28.) *If from one of the acute angles of a right-angled triangle, a line be drawn to the opposite side; the squares of that side and the line so drawn are together equal to the squares of the segment adjacent to the right angle and of the hypotenuse.*

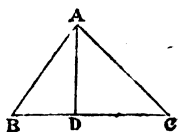
Let ABC be a right-angled triangle, and from A let AD be drawn to the opposite side; the squares of AD and BC are together equal to the squares of AC and BD .



For the squares of AD and BC together are equal to the squares of AB , BD and BC , i. e. to the squares of AC and BD ; since the squares of AB and BC are equal to the square AC .

(29.) *In any triangle if a line be drawn from the vertex at right angles to the base; the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.*

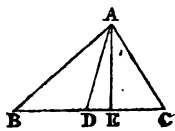
From A the vertex of the triangle ABC , let AD be drawn perpendicular to the base; the difference of the squares of AB , AC is equal to the difference of the squares of BD , DC .



For since ABD is a right-angled triangle, the square of AB is equal to the squares of AD , BD ; and since ADC is a right-angled triangle, the square of AC is equal to the squares of AD , DC ; whence the difference of the squares of AC and AB is equal to the difference of the squares of CD and DB .

(30.) *In any triangle, if a line be drawn from the vertex bisecting the base; the sum of the squares of the two sides of the triangle is double the sum of the squares of the bisecting line and of half the base.*

From the vertex A of the triangle ABC , let AD be drawn to the point of bisection of the base; the squares of AB , AC , are together double the squares of AD , DB .

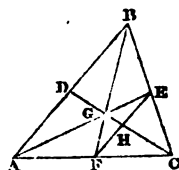


From A draw AE perpendicular to BC ;
 Then (Eucl. ii. 12.) $AB^2 = AD^2 + DB^2 + 2 BD \times DE$,
 and (Eucl. ii. 13.) $AC^2 = AD^2 + DC^2 - 2 CD \times DE$
 $\quad\quad\quad = AD^2 + DB^2 - 2 BD \times DE$,
 whence $AB^2 + AC^2 = 2 AD^2 + 2 DB^2$.

(31.) *If from the three angles of a triangle lines be drawn to the points of bisection of the opposite sides; the squares of the distances between the angles and the*

common intersection are together one third of the squares of the sides of the triangle.

From the angles of the triangle ABC , let lines be drawn to the middle points of the opposite sides, intersecting each other in G ; the sum of the squares of AG , GB , GC is one third of the sum of the squares of AB , BC , CA .



Join EF . Then $AB^2 + AC^2 = 2AE^2 + 2EB^2$,

$$AB^2 + BC^2 = 2AF^2 + 2FB^2,$$

$$AC^2 + BC^2 = 2AD^2 + 2DC^2,$$

$$\therefore AB^2 + BC^2 + CA^2 = AE^2 + BF^2 + CD^2 + AF^2 + [EB^2 + AD^2,$$

Now the sum of the squares of AF , EB , AD is equal to one fourth of the sum of the squares of AB , BC , CA ; whence three fourths of the sum of the squares of AB , BC , CA will be equal to the sum of the squares of AE , BF , CD ; or three times the sum of the squares of AB , BC , CA , is equal to four times the sum of the squares of AE , BF , CD .

Now $BG : GF :: BA : EF :: BC : CE :: 2 : 1$,

$\therefore BG : BF :: 2 : 3$, and $BG^2 : BF^2 :: 4 : 9$, whence $4BF^2 = 9BG^2$. And the same being true of each of the rest, three times the sum of the squares of AB , BC , CA , is equal to nine times the sum of the squares of AG , BG , CG ; $\therefore$ the sum of the squares of AB , BC , CA is three times the sum of the squares of AG , BG , CG .

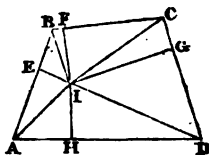
COR. If from the angles of a triangle lines be drawn to the points of bisection of the opposite sides, the squares of those lines together are to the squares of the sides of the triangle as 3 : 4.

(32.) *If from any point within or without any rectilinear figure, perpendiculars be let fall on every side; the sum of the squares of the alternate segments made by them will be equal.*

Let $ABCD$ be any quadrilateral figure (the demonstration being the same whatever be the number of sides).

From any point I let perpendiculars

IE, IF, IG, IH be drawn: $AE^2 + BF^2 + GC^2 + DH^2 = EB^2 + FC^2 + GD^2 + AH^2$.



From I draw lines to each of the angles;

then $AE^2 + EI^2 = (AI^2) = AH^2 + HI^2$,

$BF^2 + FI^2 = (BI^2) = BE^2 + EI^2$,

$CG^2 + GI^2 = (CI^2) = CF^2 + FI^2$,

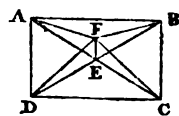
$DH^2 + HI^2 = (DI^2) = DG^2 + GI^2$,

whence,

$AE^2 + BF^2 + CG^2 + DH^2 = EB^2 + FC^2 + GD^2 + HA^2$.

(33.) *If from any point within a rectangular parallelogram lines be drawn to the angular points; the sums of the squares of those which are drawn to the opposite angles are equal.*

Let $ABCD$ be a rectangular parallelogram, and F any point within it; join FA, FB, FC, FD ; the squares of FA and FC are together equal to the squares of FB and FD .

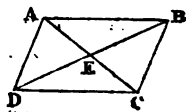


Draw the diagonals AC, BD ; and join FE . Because the triangles ADC, BDC are similar and equal, $AC = BD$; and $\therefore$ their halves, AE and DE , are equal.

Now (iv. 30.) $FD^2 + FB^2 = 2 DE^2 + 2 EF^2$,
 $= 2 AE^2 + 2 EF^2 = AF^2 + FC^2$.

(34.) *The squares of the diagonals of a parallelogram are together equal to the squares of the four sides.*

Let $ABCD$ be a parallelogram, whose diagonals are AC , BD ; the squares of AC , BD are together equal to the squares of AB , BC , CD , DA .



Since DB is bisected by AC ,

$$2 AE^2 + 2 ED^2 = AD^2 + AB^2,$$

and for the same reason,

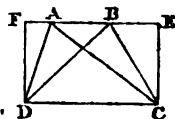
$$2 CE^2 + 2 ED^2 = CD^2 + CB^2,$$

$$\therefore 4 AE^2 + 4 ED^2 = AD^2 + AB^2 + CB^2 + CD^2,$$

$$\text{i. e. } AC^2 + BD^2 = AD^2 + AB^2 + CB^2 + CD^2.$$

(35.) *If two sides of a trapezium be parallel to each other; the squares of its diagonals are together equal to the squares of its two sides which are not parallel and twice the rectangle contained by its parallel sides.*

Let the sides AB , DC of the trapezium $ABCD$ be parallel; draw the diagonals AC , BD ; the squares of AC and BD , are together equal to the squares of AD and BC , and twice the rectangle AB , DC .



Let fall the perpendiculars CE , DF .

Then (Eucl. ii. 12.), $DB^2 = DA^2 + AB^2 + 2 AB \times AF$,
 and $AC^2 = CB^2 + AB^2 + 2 AB \times BE$,

whence,

$$AC^2 + DB^2 = AD^2 + CB^2 + 2AB^2 + 2AB \times BE + 2AB \times AF,$$

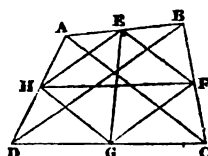
Now (Eucl. ii. i.),

$$AB \times FE = AB \times FA + AB \times AB + AB \times BE,$$

$$\therefore AC^2 + DB^2 = AD^2 + CB^2 + 2AB \times DC.$$

(36.) *The squares of the diagonals of a trapezium are together double the squares of the two lines joining the bisections of the opposite sides.*

Let $ABCD$ be a trapezium, whose sides are bisected in E, F, G, H . Join EG, FH ; and draw the diagonals AC, BD . The squares of AC, BD are together double of the squares of EG, FH .



Join EF, FG, GH, HE . Then (iv. 14.) $EFGH$ is a parallelogram, and BD is double of EH ;

$$\therefore BD^2 = 4EH^2 = 2EH^2 + 2FG^2,$$

and for the same reason $AC^2 = 2EF^2 + 2HG^2$,

$$\therefore AC^2 + BD^2 = 2EF^2 + 2FG^2 + 2GH^2 + 2HE^2, \\ = 2EG^2 + 2HF^2. \text{ (iv. 34.)}$$

(37.) *The squares of the diagonals of a trapezium are together less than the squares of the four sides, by four times the square of the line joining the points of bisection of the diagonals.*

Let $ABCD$ be a trapezium whose diagonals AC, BD are bisected in E, F ; join EF ; the squares of $AC,$

BD are less than the squares of the four sides by four times the square of EF .

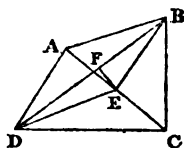
Since BE bisects AC the base of the triangle ABC ,

$$AB^2 + BC^2 = 2AE^2 + 2EB^2;$$

and for a similar reason,

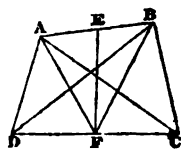
$$AD^2 + DC^2 = 2AE^2 + 2ED^2;$$

$$\begin{aligned} \therefore AB^2 + BC^2 + CD^2 + DA^2 &= 4AE^2 + 2EB^2 + 2ED^2 \\ &= AC^2 + 2EB^2 + 2ED^2 \\ &= AC^2 + 4BF^2 + 4FE^2 \\ &= AC^2 + BD^2 + 4FE^2. \end{aligned}$$



(38.) In any trapezium, if two opposite sides be bisected; the sum of the squares of the two other sides, together with the squares of the diagonals, is equal to the sum of the squares of the bisected sides together with four times the square of the line joining those points of bisection.

Let AB , DC , two opposite sides of the trapezium $ABCD$, be bisected in E , and F ; join EF ; and draw the diagonals AC , BD . The squares of AD , BC , AC , BD are equal to the squares of AB , DC , and four times the square of EF .



Join AF , BF . Since AF bisects DC the base of the triangle ADC ,

$$AD^2 + AC^2 = 2DF^2 + 2FA^2;$$

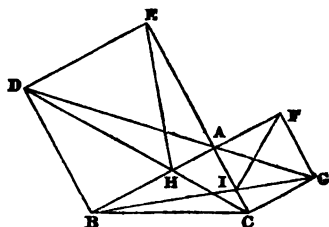
and in the same manner,

$$BC^2 + BD^2 = 2DF^2 + 2FB^2;$$

$$\begin{aligned}
 \text{whence } AD^2 + BC^2 + AC^2 + BD^2 &= 4DF^2 + 2FA^2 + 2FB^2 \\
 &= DC^2 + 2FA^2 + 2FB^2 = DC^2 + 4AE^2 + 4EF^2 \\
 &= DC^2 + AB^2 + 4EF^2.
 \end{aligned}$$

(39.) *If squares be described on the sides of a right-angled triangle; each of the lines joining the acute angles and the opposite angle of the square, will cut off from the triangle an obtuse-angled triangle, which will be equal to that cut off from the square by a line drawn from the intersection with the side to that angle of the square which is opposite to it.*

From the angles B , C of the right-angled triangle BAC , let lines BG , CD be drawn to the angles of the squares described upon the sides, and from the intersections H and I let HE , IF

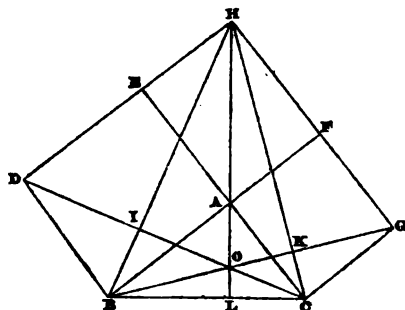


be drawn to the opposite angles of the squares; the triangle $BIC = AIF$, and $CHB = AHE$.

Join AG , AD . Then (Eucl. i. 37.) the triangle $AFI = AIG$; to each of which add ABI , $\therefore$ the triangle $BIF = BAG = BCA$ (Eucl. i. 37.) From each of these equals take away the triangle BIA , and $BIC = AIF$. In the same manner it may be shewn that $CHB = AHE$.

(40.) *If squares be described on the two sides of a right-angled triangle; the lines joining each of the acute angles of the triangle and the opposite angle of the square will meet the perpendicular drawn from the right angle upon the hypotenuse, in the same point.*

Let BE , CF be squares described on the sides BA , AC containing the right angle. Join DC , BG ; they



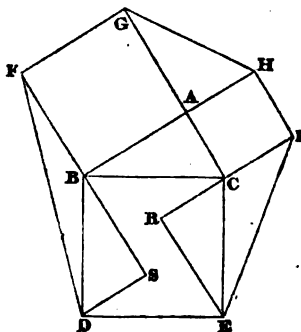
intersect AL , which is perpendicular to BC , in the same point O .

Produce DE , GF , to meet in H . Join HA , HB , HC . Let BH , CH respectively meet DC , BG in I and K . Since $EH = AF = AC$, and $EA = AB$, and the angles HEA , BAC are right angles, the triangles HEA , BAC are equal, and the angle $EHA = BCA = BAL$, i.e. since EH and BA are parallel, HAL is a straight line, or LA produced passes through H , and HL is perpendicular to BC . Again, since $AC = CG$, $AH = BC$, and the angle $HAC = BCG$, $\therefore$ the triangles HAC , BCG are equal; $\therefore$ the angle $CBK = CHL$; but $BCK = HCL$; $\therefore BKC = HLC$, i.e. is a right angle, and BK is perpendicular to HC . In the same manner it may be shewn that CI is perpendicular to BH . Hence $\therefore HL$, CI , BK are perpendicular to the sides of the triangle HBC , and $\therefore$ they intersect each other in the same point.

(41.) If squares be described on the three sides of a right-angled triangle, and the extremities of the ad-

adjacent sides be joined; the triangles so formed are equal to the given triangle and to each other.

On the sides of the right-angled triangle ABC let squares be described, and join GH , FD , IE . The triangles AGH , BFD , ECI are equal to ABC , and to each other.



It is evident that $AGH = ABC$. Produce FB , and from D draw DS perpendicular to it. Since ABS and CBD are right angles, $\therefore$ the angles ABC , SBD are equal; and BAC , BSD are also right angles, and $BC = BD$, $\therefore DS = AC$. And the triangles ABC , FBD being upon equal bases AB , FB are as their altitudes AC , DS (Eucl. vi. 1.); and $\therefore$ are equal. In the same manner if IC be produced, and ER drawn perpendicular to it, it may be shewn that ER is equal to AB , and the triangle ECI to ABC . And since each of the triangles is equal to ABC , they are equal to one another.

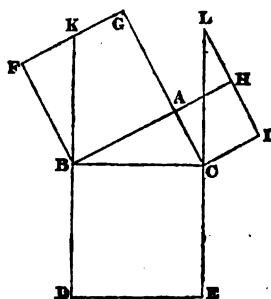
(42.) *If the sides of the square described upon the hypotenuse of a right-angled triangle be produced to meet the sides (produced if necessary) of the squares described upon the legs; they will cut off triangles equiangular and equal to the given triangle.*

Let DB , EC , the sides of the square described on BC the hypotenuse of the right-angled triangle ABC , be produced to meet the sides of the squares described

upon BA , AC , in K and L ; the triangles BFK , CIL , cut off by them, are equal and equiangular to ABC .

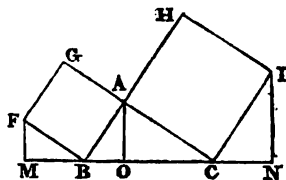
Since FBA and KBC are right angles, the angles FBK , ABC are equal; also the angles at F and A are right angles, and $FB = BA$, $\therefore FK = AC$, and the triangles FKB , ABC are equiangular and equal.

In like manner it may be proved that the triangles ABC , LCI are equiangular and equal.



(43.) *If from the angular points of the squares described upon the sides of a right-angled triangle perpendiculars be let fall upon the hypotenuse produced; they will cut off equal segments; and the perpendiculars will together be equal to the hypotenuse.*

Let FM , IN be drawn from the angles F , I of the squares described upon BA , AC , perpendicular to BC the hypotenuse produced; MB will be equal to NC ; and FM , IN together equal to BC .



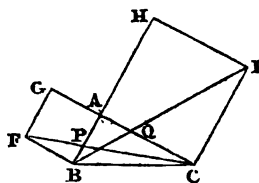
From A draw AO perpendicular to BC . Since FBA is a right angle, the angles FBM and ABO together are equal to FBM and BFM , $\therefore ABO$ is equal to BFM ; and the angles at M and O are right angles, and $AB = BF$, $\therefore BM = AO$, and $FM = BO$. In the same manner it may be shewn that $CN = AO$, and $IN = CO$; $\therefore MB$

$= NC$; and FM and IN together are equal to BO and CO together, i. e. to BC .

COR. The triangles FBM , ICN are together equal to ABC .

(44.) *If on the two sides of a right-angled triangle squares be described; the lines joining the acute angles of the triangle and the opposite angles of the squares will cut off equal segments from the sides; and each of these equal segments will be a mean proportional between the remaining segments.*

On AB , AC the sides of the right-angled triangle BAC , let squares be described, and BI , CF joined; the segments AP , AQ are equal, and each of them is a mean proportional between BP and CQ .



Since AQ is parallel to HI , and AP to FG ,

$$BH : HI :: BA : AQ,$$

$$\text{and } (CA =) HI : CG :: AP : (FG =) AB,$$

$$\therefore BH : CG :: AP : AQ;$$

and BH being equal to CG , $AP = AQ$.

Again, the triangles BPF , ACP being similar, as also ABQ , ICQ ,

$$BP : (BF =) AB :: AP : AC,$$

$$\text{and } BA : AQ :: (IC =) AC : CQ,$$

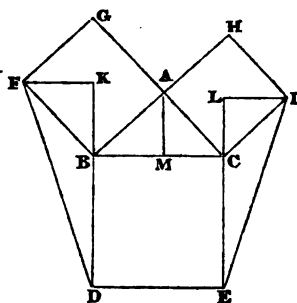
$$\therefore \text{ex æquo } BP : (AQ =) AP :: AP : CQ.$$

(45.) *If squares be described on the hypotenuse and sides of a right-angled triangle, and the extremities of*

the sides of the former and the adjacent sides of the others be joined; the sum of the squares of the lines joining them will be equal to five times the square of the hypotenuse.

Let squares be described on the three sides of the right-angled triangle ABC ; join DF , EI ; the squares of DF and EI together are equal to five times the square of BC .

Draw FK , IL perpendicular to DB , EC produced, and AM to BC . The angle FBK is equal to ABC , and the angle at K to the right angle AMB , and $FB = BA$, $\therefore BK = BM$. In the same way, $CL = CM$.



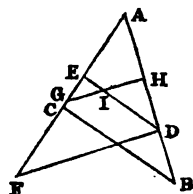
$$\begin{aligned}\text{Now (Eucl. ii. 12.) } FD^2 &= DB^2 + BF^2 + 2DB \times BK \\ &= BC^2 + BA^2 + 2BC \times BM, \\ \text{and } EI^2 &= BC^2 + CA^2 + 2BC \times CM,\end{aligned}$$

$$\begin{aligned}\therefore FD^2 + EI^2 &= 2BC^2 + BA^2 + AC^2 + 2BC \times BM + \\ &\quad [2BC \times CM] \\ &= 2BC^2 + BC^2 + 2BC^2 = 5BC^2.\end{aligned}$$

(46.) *If a line be drawn parallel to the base of a triangle, and terminated in the sides; to draw a line cutting it, and terminated also by the sides, so that the rectangles contained by their segments may be equal.*

Let ED be parallel to CB the base of the triangle ABC ; from D draw DF , making with AC (produced if

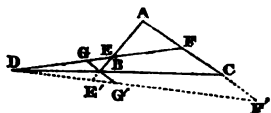
necessary) the angle DFE equal to ABC , and draw any line GH parallel to FD , cutting ED in I ; the rectangle EI, ID is equal to the rectangle GI, IH .



For the angle $AGH = AFD = ABC = ADE$, and the vertical angles at I are equal, $\therefore$ the triangles GEI, HID are equiangular;
 and $HI : ID :: IE : IG$,
 $\therefore$ the rectangle EI, ID is equal to the rectangle HI, IG .

(47.) *If the sides, or sides produced, of a triangle be cut by any line; the solids formed by the segments which have not a common extremity are equal.*

Let ABC be a triangle having the sides (produced if necessary) cut by the line DEF ; then $AF \times CD \times BE = AE \times DB \times CF$,

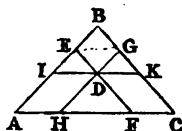


Draw BG parallel to AC ; the triangles AEF, BEG will be similar, as also CDF, BDG ;

$$\begin{aligned} \therefore AF : AE &:: BG : BE, \\ \text{and } CD : CF &:: BD : BG, \\ \therefore AF \times CD : AE \times CF &:: BD : BE, \\ \therefore AF \times CD \times BE &= AE \times DB \times CF. \end{aligned}$$

(48.) *If through any point within a triangle, three lines be drawn parallel to the sides; the solids formed by the alternate segments of these lines are equal.*

Through any point D within the triangle ABC , let HG , EF , IK , be drawn parallel to the sides; then $ID \times DG \times DF = ED \times DK \times DH$.



Since the lines are drawn parallel to the sides, the triangles IED , GDK , HDF are similar to ABC , and to one another;

$$\therefore ID : DE :: AC : CB$$

$$GD : DK :: AB : AC$$

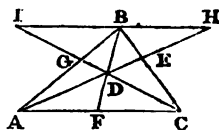
$$DF : DH :: BC : AB,$$

whence $ID \times DG \times DF : DE \times DK \times DH :: AC \times AB \times BC : BC \times AC \times AB$,

i. e. in a ratio of equality.

(49.) *If through any point within a triangle lines be drawn from the angles to cut the opposite sides; the segments of any one side will be to each other in the ratio compounded of the ratios of the segments of the other sides.*

Through any point D within the triangle ABC , let lines AE , BF , CG be drawn from the angles to the opposite sides; the segments of any one of them as AC , will be in the ratio compounded of the ratios $AG : GB$, and $BE : EC$.



Draw IBH parallel to AC , meeting AE and CG produced in H and I . Then the triangles GCA , GBI , and EAC , EBH , as also ADF , BDH , and FDC , IDB , are respectively equiangular,

$$\text{whence } BH : AC :: BE : CE,$$

$$\text{and } AC : BI :: AG : BG,$$

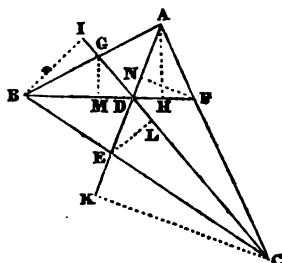
$$\therefore BH : BI :: AG \times BE : GB \times CE.$$

$$\text{But } BH : BI :: AF : FC,$$

$$\therefore AF : FC :: AG \times BE : GB \times CE.$$

(50.) *If from each of the angles of any triangle, a line be drawn through any point within the triangle, to the opposite side; the solid contained by the segments thereof, intercepted between the angles and the point, will have to the solid contained by the three remaining segments, the same ratio that the solid contained by the three sides of the triangle, has to either of the (equal) solids contained by the alternate segments of the sides.*

Let ABC be the given triangle, and through any point D within it, let AE , BF , CG be drawn from the angles to the opposite sides; then will $AD \times DB \times DC : ED \times DF \times DG :: AB \times BC \times CA : AF \times CE \times BG$.



Let fall the perpendiculars AH , BI , CK ; EL , GM , FN .

Since EL is parallel to BI , $CB : CE :: BI : EL$, and GM being parallel to AH , $BA : BG :: AH : GM$, also FN and CK being parallel, $AC : AF :: CK : FN$,
 $\therefore AC \times AB \times BC : CE \times BG \times AF :: BI \times AH \times CK :$
 $[EL \times GM \times FN.]$

Again, since EL is perpendicular to DC , and CK to DK , the triangles DEL , DCK are equiangular,
 and $\therefore DC : DE :: CK : EL$.

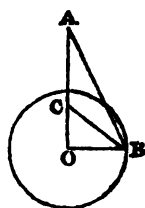
In the same manner, $DB : DG :: BI : GM$,
 and $DA : DF :: AH : FN$,
 $\therefore DA \times DB \times DC : DE \times DF \times DG :: BI \times AH \times$
 $CK : EL \times GM \times FN :: AB \times BC \times CA : AF \times CE \times$
 $BG.$

SECT. V.

(1.) *A straight line of given length being drawn from the centre at right angles to the plane of a circle ; to determine that point in it, which is equally distant from the upper end of the line and the circumference of the circle.*

From O the centre of the circle, let OA be drawn at right angles to its plane ; draw OB perpendicular to OA ; join AB , and make the angle ABC equal to BAC . C is the point required.

Since the angle $ABC = CAB$, $\therefore AC = CB$.

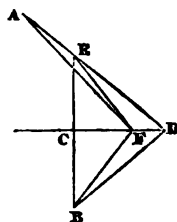


(2.) *To determine a point in a line given in position, to which lines drawn from two given points may have the greatest difference possible.*

Let A and B be the given points, and CD the line given in position. Let fall the perpendicular BC , and

U

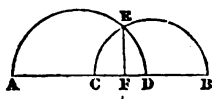
produce it, so that CE may be equal to CB ; join AE , and produce it to meet CD in D . Join BD . D is the point required.



For $DE = DB$; and $\therefore AE$ is equal to the difference between AD and DB . If then any other point F be taken, $BF = EF$; and the difference between AF and BF is equal to the difference between AF and EF , which is less than AE (iii. 1.). The same may be proved for every other point in CD .

(3.) *A straight line being divided in two given points; to determine a third point such that its distances from the extremities may be proportional to its distances from the given points.*

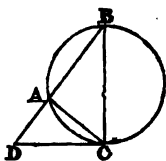
Let AB be the given line, divided in C and D . On AD and CB let semicircles be described intersecting in E . From E let fall the perpendicular EF ; F is the point required.



For (Eucl. vi. 8. Cor.) $AF : FE :: FE : FD$,
and $FE : FB :: FC : FE$,
 $\therefore AF : FB :: FC : FD$.

(4.) *In a straight line given in position, to determine a point, at which two straight lines, drawn from given points on the same side, will contain the greatest angle.*

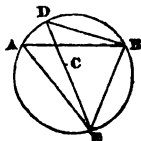
Let A and B be the given points, and CD the given line. Join BA , and produce it to meet CD in D . Take DC a mean proportional between DA and DB . C is the point required.



Join AC , BC ; and about the triangle ABC describe a circle; DC is a tangent at the point C (Eucl. iii. 37.), and $\therefore$ the angle is the greatest (ii. 62.).

(5.) *To determine the position of a point, at which lines drawn from three given points, shall make with each other angles equal to given angles.*

Let A , B , C be the three given points; join AB , and on it describe a segment of a circle containing an angle equal to that which the lines from A and B are to include. Complete the circle, and make the angle ABD equal to that which the lines from A and C are to include. Join DC , and produce it to the circumference in E . E is the point required.

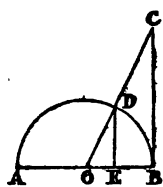


Join AE , BE . Then the angle $AEC = ABD$, and AEB is of the given magnitude, by construction.

(6.) *To divide a straight line into two parts such, that the rectangle contained by them may be equal to the square of their difference.*

Let AB be the given line; upon it describe a semi-circle ADB . From B draw BC at right angles and

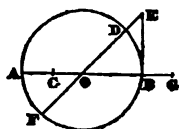
equal to AB . Take O the centre, and join OC ; and from D draw DE perpendicular to AB ; AB is divided in the point E , as was required.



Since BC is double of BO , DE is double of OE (Eucl. vi. 2.), and OE being half the difference between AE and EB , DE is equal to the difference. Also (Eucl. vi. 13.) the rectangle AE, EB is equal to the square of DE .

(7.) *If a straight line be divided into any two parts; to produce it, so that the rectangle contained by the whole line so produced, and the part produced may be equal to the rectangle contained by the given line and one segment.*

Let AB be the given line divided into two parts in the point C . On AB as a diameter describe a circle ADB . From B draw BE at right angles to AB , and $\therefore$ a tangent to the circle; and make BE a mean proportional between AB and AC . Take O the centre; join EO , and produce it to F . Produce AB to G , making BG equal to ED . Then will the rectangle AG, GB be equal to the rectangle BA, AC .



Since $DE = BG$, the rectangle BG, GA is equal to the rectangle DE, EF , i. e. to the square of EB , or to the rectangle AB, AC , by construction.

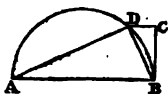
COR. 1. If it be required to produce the line, so that the rectangle contained by the whole line produced and the part produced, may be equal to the rectangle con-

tained by two given lines; find BE a mean proportional between the two given lines, and proceed as in the proposition.

COR. 2. If it be required to produce the line, so that the rectangle contained by the whole line produced and the part produced, may be equal to a given square; take BE equal to a side of the square, and proceed as in the proposition.

(8.) *To determine two lines such that the sum of their squares may be equal to a given square, and their rectangle equal to a given rectangle.*

Let AB be equal to a side of the given square. Upon it describe a semicircle ADB ; and from B draw BC perpendicular to AB , and equal to a fourth proportional to AB and the sides of the given rectangle. From C draw CD parallel to BA . Join AD , DB ; they are the lines required.



Since CB touches the circle at B , the angle CBD is equal to DAB , and the angles DCB , ADB are right angles; $\therefore$ the triangles DCB , ADB are equiangular,

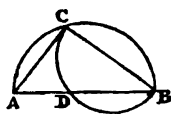
$$\text{and } AB : AD :: DB : BC,$$

whence the rectangle AD , DB is equal to the rectangle AB , BC , *i. e.* to the given rectangle. Also the squares of AD , DB are equal to the square of AB , *i. e.* to the given square.

(9.) *To divide a straight line into two parts, so that*

the rectangle contained by the whole and one of the parts may be equal to the square of a given line, which is less than the line to be divided.

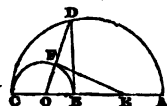
Let AB be the given line to be divided. Upon it describe a semicircle, in which place the line $AC =$ to the given line. Join CB ; and on it describe a semicircle CDB , cutting AB in D ; D is the point required.



Since the angle ACB is in a semicircle, it is a right angle, $\therefore AC$ touches the circle CDB (Eucl. iii. 16. Cor.); whence the rectangle BA, AD is equal to the square of AC , i. e. to the square of the given line.

(10.) *To divide a given line into two such parts that the rectangle contained by the whole line and one of the parts may be (m) times the square of the other part, m being whole or fractional.*

Let AB be the given line, and in it produced, take $BC =$ an m^{th} part of AB . On AC describe a semicircle, and from B draw BD perpendicular to AC . Bisect CB in O ; join OD , and take $OE = OD$; and AB will be divided in E , as required.



On BC describe a semicircle, cutting OD in F ; join FE . Then the angle DOE being common to the triangles DOB , EOF , and DO , OB respectively equal to EO , OF , the triangles will be similar and equal, and $\therefore$ the angle OFE equal to OBD , and $\therefore$ a right angle; whence FE is a tangent to the circle CFB . Hence the rectangle AB, BC is equal to the square of DB , i. e. to the square of FE , or the rectangle CE, EB .

From each of these equals take away the rectangle CB , BE ; and the rectangle AE , CB is equal to the square of BE , $\therefore (m)$ times the rectangle AE , CB , i. e. the rectangle AB , AE is equal to (m) times the square of BE .

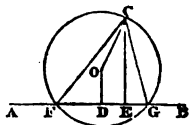
(11.) *To divide a given line into two such parts that the square of the one shall be equal to the rectangle contained by the other and a given line.*

Let AB be the given line to be divided, (see last Fig.) and BC the other given line. Let them be placed so as to be in the same straight line. On AC describe a semicircle and draw the lines, as in the last proposition; and E is the point required.

For the rectangle AE , CB is equal to the square of BE .

(12.) *A straight line being given in magnitude and position; to draw to it from a given point, two lines, whose rectangle shall be equal to a given rectangle, and which shall cut off equal segments from the given line.*

Let AB be the given line, and C the given point. Bisect AB in D , and from D draw DO at right angles to AB , and let fall the perpendicular CE . With the centre C , and radius equal to a fourth proportional to $2CE$ and the sides of the given rectangle, describe a circle cutting DO in O . Join OC ; and with the centre O ,

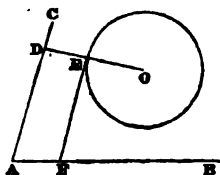


and radius OC , describe a circle CFG , cutting AB in F and G ; join CF , CG ; they are the lines required.

For (Eucl. vi. C.) the rectangle CF , CG is equal to the rectangle contained by $2CO$ and CE , i. e. to the given rectangle. And since $AD = DB$, and $FD = DG$, $\therefore AF = GB$.

(13.) *To draw a straight line which shall touch a given circle, and make with a given line, an angle equal to a given angle.*

Let AB be the given line, and O the centre of the given circle. From any point A in the given line, draw AC making with it an angle equal to the given angle; from O draw OD perpendicular to AC , and through the point E where it meets the circle, draw EF parallel to DA ; EF is the line required.



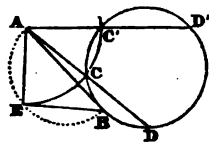
For being parallel to AC it is perpendicular to OD , and $\therefore$ a tangent to the circle; and the angle $EFB = DAB =$ the given angle.

(14.) *Through a given point to draw a line terminating in two lines given in position, so that the rectangle contained by the two parts may be equal to a given rectangle.*

Let AB , CD be the lines given in position, E the given point; from E draw EF perpendicular to AB , and

(16.) *From a given point without a circle, to draw a straight line cutting the circle, so that the rectangle contained by the part of it without and the part within the circle shall be equal to a given square.*

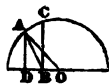
Let A be the given point, and BCD the given circle. From A draw AB touching the circle; and on it as a diameter describe a semicircle AEB , in which place BE equal to a side of the given square. Join AE ; and with the centre A and radius AE , describe the circle EC , cutting BCD in C . Join AC , and produce it to D . ACD is the line required.



For the rectangle AC , AD is equal to the square of AB , i. e. to the squares of AE and EB or to the squares of AC and EB ; take away from each the square of AC , $\therefore$ the rectangle AC , CD is equal to the square of EB , i. e. to the given square.

(17.) *From a given point in the circumference of a semicircle, to draw a straight line meeting the diameter, so that the difference between the squares of this line and a perpendicular to the diameter from the point of intersection may be equal to a given rectangle.*

Let A be the given point in the circumference of the semicircle; from it draw AD perpendicular to the diameter. Take O the centre, and divide DO in B , so that the rectangle contained by $2OB$, BD may be equal to the given rectangle. Join AB ; and draw BC perpendicular to DB . AB , BC are the lines required.

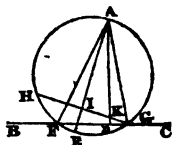


For (Eucl. ii. 12.) the square of AB together with

twice the rectangle OB, BD is equal to the difference of the squares of OA and OB , i. e. to the square of BC ; $\therefore$ the difference between the squares of AB and BC is equal to twice the rectangle OB, BD , i. e. to the given rectangle.

(18.) *From a given point to draw two lines to a third given in position, so that the rectangle contained by those lines may be equal to a given rectangle, and the difference of the angles which they make with that part of the third which is intercepted between them may be equal to a given angle.*

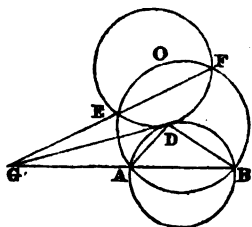
Let A be the given point, and BC the line given in position. From A draw AD perpendicular to BC ; make the angle DAE equal to the given angle; and produce AE , till the rectangle DA, AE , is equal to the given rectangle. On AE as a diameter describe the circle AFG , cutting BC in F and G . Join AF, AG ; they are the lines required.



Draw GH perpendicular to the diameter AE ; then the arc HA is equal to the arc AG , and the angle AGH to AFG ; $\therefore$ the angle HGF is equal to the difference of the angles AGF, AFG . Now the right-angled triangles AIK, KDG have the angles at K equal, $\therefore$ the angle $KAI = KGD$; but KAI was made equal to the given angle; $\therefore$ the difference of the angles AFG, AGF is equal to the given angle. And (Eucl. vi. C.) the rectangle AF, AG is equal to the rectangle DA, AE , i. e. to the given rectangle.

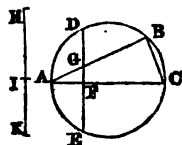
(19.) *Two points being given without a given circle; to determine a point in the circumference, from which lines drawn to the two given points shall contain the greatest possible angle.*

Let A and B be the given points, and EDF the given circle whose centre is O . Describe a circle through A, B, O . Join EF, BA , and produce them to meet in G . From G draw GD touching the given circle in D . Through D, A, B describe another circle; then since the square of GD is equal to the rectangle EG, GF , i. e. to the rectangle AG, GB , $\therefore GD$ touches the circle ABD . Join AD, DB . D is the point required, as is evident from (ii. 62.)



(20.) *From the bisection of a given arc of a circle to draw a straight line such that the part of it intercepted between the chord of that and the opposite circumference shall be equal to a given straight line.*

Let DAE be the given arc of the circle ABC , bisected in A ; AFC the diameter, and HI the given straight line. Produce HI to K , so that the rectangle HK, KI may be equal to the rectangle FA, AC . From A place in the circle $AB = HK$; AB is the line required.



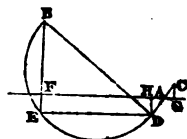
Join BC ; then the angle AFG being a right angle is equal to the angle ABC , and the angle at A is common, $\therefore$ the triangles AGF, ABC are equiangular,

and $AF : AG :: AB : AC$,

$\therefore$ the rectangle GA, AB is equal to the rectangle FA, AC , i. e. to the rectangle HK, KI . But $AB = HK$,
 $\therefore AG = KI$, and consequently $GB = HI$.

(21.) *To draw a straight line through a given point, so that the sum of the perpendiculars to it from two other given points may be equal to a given line.*

Let A, B, C be the three given points, A being that through which the line is to be drawn. Join AC , and produce it, making $AD = AC$. Join BD , and on it describe a semicircle; in which place BE equal to the given line. Join DE ; and through A draw FAG parallel to DE ; it is the line required.

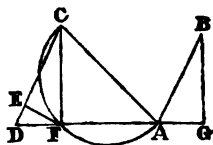


For let fall the perpendicular CG , and draw DH parallel to BE ; then the triangles ACG, AHD being similar, and $AC = AD$, $\therefore CG = HD = FE$, FD being a parallelogram; $\therefore BF$ and CG together are equal to BE , i. e. to the given line; and FH being parallel to ED , BF is perpendicular to FG .

(22.) *To draw a straight line through one of three points given in position; so that the rectangle contained by the perpendiculars let fall upon it from the other two may be equal to a given square.*

Let A, B, C be the three given points, and A the point through which the line is to be drawn. Join AB ,

AC ; and draw CD parallel to BA , and take CE a third proportional to AB and a side of the given square. On AC describe a semicircle; and from E draw EF at right angles to CD , and meeting the semicircle in F . Join AF , and produce it; it is the line required.



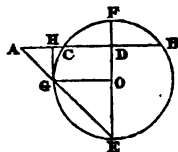
Join CF , which will be perpendicular to AD ; and from B draw BG perpendicular to AG . Since CE is parallel to BA , and CF to BG , the triangles ARG , CEF will be similar,

$$\therefore AB : BG :: CF : CE,$$

$\therefore$ the rectangle BG , CF , is equal to the rectangle AB , CE . But since the side of the given square is, by construction, a mean proportional between AB and CE , the rectangle AB , CE , is equal to the given square; $\therefore$ the rectangle BG , CF is equal to the given square.

(23.) *A given straight line being divided into two parts; to cut off a part which shall be a mean proportional between the two remaining segments.*

Let AB be divided into two parts in the point C ; bisect CB in D , and draw DE perpendicular, and equal to AD ; and through the points B , C , E describe a circle; produce ED to F .

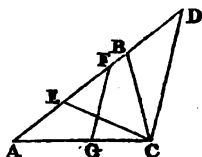


Join AE , and bisect EF in O ; and from O draw OG parallel to AB , meeting AE in G ; and since $AD = DE$, $\therefore GO = OE$, and G is a point in the circumference: From G draw GH perpendicular to AC ; H is the point required.

For HG , being perpendicular to AD , is perpendicular also to GO , and $\therefore$ is a tangent at G ; $\therefore$ the square of HG is equal to the rectangle CH, HB . But since $AD=DE$, $\therefore AH=HG$, and consequently the square of AH is equal to the rectangle CH, HB ; and AH is a mean proportional between the two remaining segments CH and HB .

(24.) To draw a straight line making a given angle with one of the sides of a given triangle, so that the triangle cut off may be to the whole in a given ratio.

Let ABC be the given triangle; make the angle ACD equal to the given angle which the cutting line is to make with AC . Produce AB to D ; and make $AE : AB$ in the ratio of the part to be cut off to the whole. Take AF a mean proportional between AE and AD ; draw FG parallel to CD ; FG is the line required.

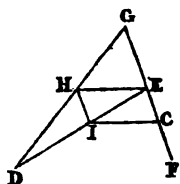


Join EC . Then the triangle $ADC : AFG :: AD^2 : AF^2 :: AD : AE :: ACD : ACE$, and $\therefore AFG = ACE$.

But $ACE : ACB :: AE : AB$,
 $\therefore AFG : ACB :: AE : AB$, i. e. in the given ratio.

(25.) Between two given straight lines containing a given angle, to place a straight line of given length, and subtending that angle, so that the segment of the one of them adjacent to the angle may be to the segment of the other which is not adjacent, in the ratio of two given lines.

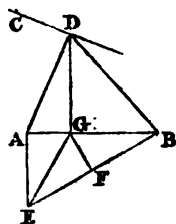
Let ED , EF be the lines given in length and position. Produce one of them FE , till $EG : ED$ in the given ratio. Join DG ; and with the centre E , and radius equal to the given line to be placed, describe a circle cutting DG in H ; join EH , and draw HI parallel to EF , and IC parallel to HE . IC is the line required.



For (Eucl. vi. 2.) $HI : ID :: GE : ED$,
and HC being a parallelogram, $HI = EC$,
 $\therefore EC : ID :: GE : ED$, i. e. in the given ratio;
and $IC = EH =$ the given line.

(26.) From two given points to draw two lines to a point in a third, such that the difference of their squares may be equal to a given square.

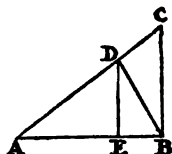
Let A and B be the given points; join AB ; and from A draw AE perpendicular to it, and equal to a side of the given square. Join BE , and bisect it in F ; from F draw the perpendicular FG , meeting AB in G ; and from G draw GD perpendicular to AB , meeting CD in D ; join AD , DB ; these are the lines required.



Join GE , it is equal to GB . And (iv. 30.) the difference between the squares of BD and AD is equal to the difference between the squares of BG and GA , i. e. between the squares of EG and GA , or it is equal to the square of AE , i. e. to the given square.

(27.) *To divide a given straight line into two such parts, that the square of the one may be to the excess of a given rectangle above the square of the other, in a given ratio.*

Let AB be the given straight line. From B draw BC at right angles to AB , and make $AB^2 : BC^2$ in the given ratio. Join AC . Find a mean proportional between the sides of the given rectangle; and with it as radius, and B as centre describe a circle cutting AC in D . Join BD , and draw DE parallel to BC ; E is the point required.

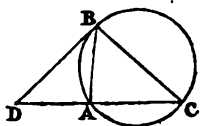


For (Eucl. vi. 2.) $AE^2 : ED^2 :: AB^2 : BC^2$. Now the square of ED is equal to the difference of the squares of BD and BE , i. e. to the difference of the given rectangle and the square of BE ; $\therefore$ the square of AE is to the difference between the given rectangle and the square of BE as $AB^2 : BC^2$, i. e. in the given ratio.

N.B. The given rectangle must not be less than the square of the perpendicular from B upon AC ; and when BD is less than BC , there are two points E .

(28.) *From any angle of a triangle, not isosceles about the angle, to draw a line without the triangle to the opposite side produced, which shall be a mean proportional between the segments of the side.*

Let ABC be the triangle, and B the angle from which the mean proportional is to be drawn. About the triangle describe a circle, and to the point B

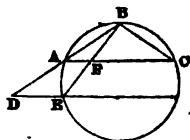


draw a tangent BD meeting the side CA produced in D .
 BD is a mean proportional between AD and DC .

(Eucl. iii. 36.) the rectangle AD, DC is equal to the square of DB ; and $\therefore AD : DB :: DB : DC$.

(29.) *From the obtuse angle of any triangle, to draw a line within the triangle to the opposite side, which shall be a mean proportional between the segments of the side.*

Let ABC be a triangle having the obtuse angle ABC . Describe a circle about it, and produce BA to D , making $AD=AB$. From D draw DE parallel to AC , meeting the circle in E ; join BE , cutting AC in F ; BF will be a mean proportional between AF and FC .



For (Eucl. vi. 2.) $BF : FE :: BA : AD$,

and since $BA=AD$, $\therefore BF=FE$.

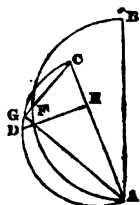
Now the rectangle AF, FC is equal to the rectangle BF, FE , i. e. to the square of BF ;

$\therefore AF : FB :: FB : FC$.

(30.) *From the common extremity of the diameters of two semicircles given in magnitude and position; to draw a line meeting the circumferences, so that the rectangle contained by the two chords may be equal to a given square.*

Let AB, AC be the diameters drawn from A , and given in magnitude and position. With the centre A ,

and radius equal to a side of the given square, describe a circle, cutting the lesser semicircle in D . Draw DE perpendicular to AC , and meeting the other semicircle in F . Join AF , and produce it to G ; AG is the line required.



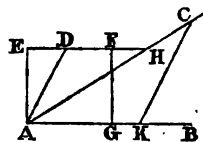
For joining GC , the triangles AGC , AFE are similar,

$$\therefore AC : AG :: AF : AE,$$

and $\therefore$ the rectangle FA , AG is equal to the rectangle CA , AE , i. e. to the square of AD , which is equal to the given square.

(31.) To draw a line parallel to a given line, which shall be terminated by two others given in position, so as to form with them a triangle equal to a given rectilineal figure.

Let AB , AC be the lines given in position, AD the line to which it is required to draw a parallel. Describe a rectangular parallelogram $AEFG$ equal to the given figure. Produce EF



to H ; and take AK a mean proportional between DH and $2EF$; draw KC parallel to AD ; KC is the line required.

For the angles DHA , CAK being equal, as also DAH , ACK , the triangles DAH , AKC are equiangular, and similar; whence

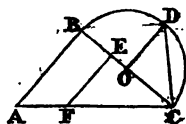
$$AKC : AHD :: AK^2 : DH^2 :: 2EF : DH :: 2EF$$

$$[\times AE : DH \times AE.$$

Now the rectangle DH , AE is double of the triangle AHD , $\therefore AKC$ is equal to the rectangle EF , AE , i. e. to the given rectilineal figure.

(32.) *To bisect a triangle by a line drawn parallel to one of its sides.*

Let ABC be a triangle to be bisected by a line parallel to its side AB . On BC describe a semicircle; bisect BC in O , and draw the perpendicular OD ; join CD ; and with C as centre, and radius CD , describe a circle cutting CB in E ; draw EF parallel to AB ; EF bisects the triangle.



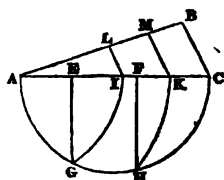
(Eucl. vi. 8.) $BC : CD :: CD : CO$,

$\therefore BC^2 : (CD^2 =) CE^2 :: BC : CO :: 2 : 1$;

but the triangles ABC , FEC are in the duplicate ratio of $BC : CE$, and $\therefore$ in the ratio of $2 : 1$, i. e. EFC is half of ABC , and EF bisects the triangle.

(33.) *To divide a given triangle into any number of parts having a given ratio to each other, by lines drawn parallel to one of the sides of the triangle.*

Let ABC be the given triangle; divide AC into parts AE , EF , FC having the same ratio to one another that the parts of the triangle are to have. On AC describe a semicircle, and draw the perpendiculars EG , FH ; and with the centre A , and radii AG , AH , describe circles meeting AC in I and K , from which points draw IL , KM parallel to BC ; these will divide the triangle in the ratio required.

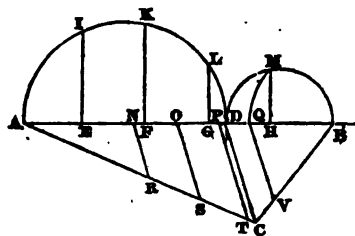


For the triangles ALI , AKM , ABC are to one another in the duplicate ratio of the sides AI , AK , AC , i. e. in the ratio of the rectangles AC , AE ; AC , AF ; and the

square of AC ; or in the ratio of the lines AE, AF, AC ; whence $ALI, LIKM, MKCB$ are in the ratio of AE, EF, FC , i. e. in the given ratio.

(34.) To divide a given triangle into any number of equal parts by lines drawn parallel to a given line.

Let ABC be the given triangle; from the angle C draw CD parallel to the given line; and let it be required to divide the triangle into five equal parts. On AD, BD describe semi-circles AID, BMD ; divide AB into five equal parts in the points E, F, G, H ; draw EI, FK, GL, HM perpendicular to AB ; and make AN, AO, AP respectively equal to AI, AK, AL , and $BQ = BM$; and draw NR, OS, PT, QV , parallel to DC ; they divide the triangle as required.



(Eucl. vi. 1.) the triangle $ABC : ADC :: AB : AD$,
(Eucl. vi. 19.) $ACD : ANR :: AD^2 : AN^2 :: AD : AE$.

$\therefore$ ex æquo, $ABC : ANR :: AB : AE :: 5 : 1$,
i. e. ANR is one fifth of ABC .

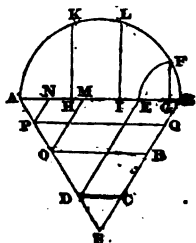
In the same manner $ABC : AOS :: 5 : 2$,
whence $NRSO$ is also one fifth of ABC .

And by a similar manner, $OPTS$ and BQV , may each be shewn to be one fifth of ABC , $\therefore TPQV$ will also be one fifth of ABC .

COR. In nearly the same manner the triangle may be divided into any number of parts having a given ratio.

(35.) To divide a trapezium which has two sides parallel into any number of equal parts, by lines drawn parallel to those sides.

Let $ABCD$ be the given trapezium having the sides AB , DC parallel. On AB the longer side describe a semi-circle AFB ; from D draw DE parallel to BC ; with the centre B , and radius BE , describe the arc EF , and from F let fall the perpendicular FG ; and divide AG into the given number of equal parts, e. g. three, in H and I ; and draw HK , IL at right angles to AB . Make BM , BN respectively equal to BL , BK ; and draw MO , NP parallel to BC ; and PQ , OR parallel to AB ; and produce AD , BC to S .



Since $DC = BE = BF$, and $OR = BM = BL$, and $PQ = BN = BK$, the triangle ORS is to DSC in the duplicate ratio of OR to CD , or of BL to BF , i. e. in the ratio of $BI : BG$;

whence $ODCR : DSC :: GI : GB$.

In the same manner $PDCQ : DSC :: GH : GB$,

$\therefore ODCR : PDCQ :: GI : GH$,

and $ODCR : PORQ :: GI : IH$,

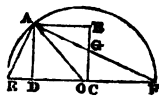
i. e. in a given ratio of equality.

And in a similar manner $APQB$ may be shewn to be equal to $PORQ$. And so on, whatever be the number of equal parts.

COR. In nearly the same manner, the trapezium might be divided into parts having any given ratio.

(36.) *From one of the angular points of a given square to draw a line meeting one of the opposite sides, and the other produced, in such a manner, that the exterior triangle formed thereby may have a given ratio to the square.*

Let $ABCD$ be the given square, and $M : N$ the given ratio. From A to DC (produced if necessary) draw a line AO , such that $M : M+N :: DC : AO$. With the centre O and radius OA , describe a semicircle meeting DC produced in E and F . Join AF ; which will be the line required.



Join AE . Then $M : M+N :: DC : AO :: ABCD : \text{the rectangle } AO, AD$. Now the triangle ADE is similar to ABG , and equal to it, since $AB = AD$; $\therefore$ the trapezium $AECG$ is equal to $ABCD$; and the rectangle AO, AD is equal to the triangle AEF ,

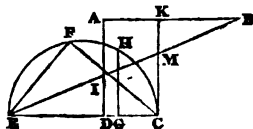
whence $M : M+N :: AECG : AEF$,

$\therefore M : N :: AECG : GCF$

$:: ABCD : GCF$.

(37.) *From a given point in the side produced, of a given rectangular parallelogram, to draw a line which shall cut the perpendicular sides and the other side produced, so that the trapezium cut off, which stands on the aforesaid side, may be to the triangle which stands upon the produced part of the opposite side, in a given ratio.*

Let $AKCD$ be the given rectangle, and E the given point in the side CD produced. On EC describe a semicircle, and in it place



$EF=ED$; join FC ; and divide EC in G , so that $EG : GC$ in the given ratio, and draw GH at right angles to EC . In AK produced take BK a fourth proportional to EG , GH and FC . Join BE ; it is the line required.

For (Eucl. vi. 19.) the triangle $ECM : EDI :: EC^2 : ED^2$,

$$\therefore \text{div. } CDIM : ECM :: EC^2 - ED^2 : EC^2,$$

$$\text{but } ECM : BMK :: EC^2 : BK^2,$$

$$\therefore \text{ex æquo } CDIM : BMK :: EC^2 - EF^2 : BK^2$$

$$:: FC^2 : BK^2$$

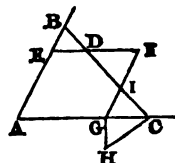
$$:: EG^2 : GH^2, \text{ by construction,}$$

$$:: EG : GC,$$

i. e. in the given ratio.

(38.) *Through a given point, between two straight lines containing a given angle, to draw a line which shall cut off a triangle equal to a given figure.*

Let AB, AC be the lines containing the given angle BAC , and D the given point. Through D draw DE parallel to AC , and describe a parallelogram EG equal to the given figure. Draw GH perpendicular to AC , and equal to DE ; and make $HC = DF$; join CD , and produce it to meet AB in B ; CB is the line required.

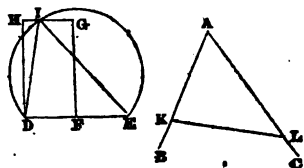


For the triangles EBD, DIF, GIC being similar, are to one another in the duplicate ratio of the sides DE, DF, GC ; but the square of HC is equal to the squares of HG, GC ; and $\therefore$ the square of DF is equal to the squares of DE, GC ; whence the triangle DIF is

equal to the triangles DBE , GIC ; $\therefore$ the triangle ABC is equal to $AEFG$, i. e. to the given figure.

(39.) *Between two lines given in position, to draw a line equal to a given line, so that the triangle thus formed may be equal to a given rectilineal figure.*

Let AB , AC be the lines given in position, and DE the line whose magnitude is given. Bisect it in F , and on DF describe a rectangular parallelogram equal to the given figure. On DE describe a segment of a circle containing an angle equal to the angle at A , and cutting HG in I . Join DI , IE ; and make $AK=ID$, and $AL=IE$. Join KL ; it is the line required.



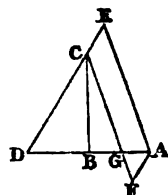
Since $AK=ID$, and $AL=IE$, and the angle at $A=DIE$, $\therefore KL=DE$, and the triangle $AKL=IDE=HGFD$ =the given figure.

(40.) *From two given lines to cut off two others, so that the remainder of one may have to the part cut off from the other a given ratio; and the difference of the squares of the other remainder and part cut off from the first may be equal to a given square.*

Perpendicular to AB one of the given lines, draw BC equal to a side of the given square; and take AD to the other given line in the given ratio of the part remaining

Z.

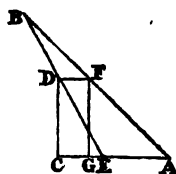
from the first to the part cut off from the second. Join DC ; and with the centre A , and radius equal to the second given line, describe a circle cutting DC in E ; join AE , and draw CGF parallel to it, meeting AF , drawn parallel to EC , in F . Then BG and GF are equal to the parts to be cut off.



For the difference between the squares of CG , GB is equal to the square of BC , i. e. to the given square; and $AG : GF :: AD : AE$, i. e. in the given ratio.

(41.) *From two given lines to cut off two others which shall have a given ratio, so that the difference of the squares of the remainders may be equal to a given square.*

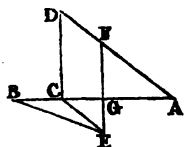
Let AC be one of the two given lines. From C draw CD perpendicular to AC , and equal to a side of the given square. Take AE to the other given line in the given ratio of the parts to be cut off. Join ED , and produce it; and with the centre A , and radius equal to that other given line, describe a circle cutting ED in B . Join AB ; and let it meet DF , which is parallel to AC , in F . Draw FG parallel to CD . CG and BF are the parts required to be cut off.



For $(DF=) CG : FB :: EA : AB$, i. e. in the given ratio of the parts to be cut off; and the difference between the squares of FA and AG is equal to the square of GF , i. e. to the square of CD , or the given difference of the squares of the remainders.

(42.) *From two given lines to cut off two others so that the remainders may have a given ratio, and the sum of the squares of the parts cut off may be equal to the square of a given line.*

Let AB be one of the given lines, and in it take AC to the other given line, in the given ratio of the remainders. From C draw CD perpendicular to AB , and equal to the second given line. Join AD , and draw CE parallel to AD ; and with the centre B , and radius equal to the side of the given square, describe a circle, cutting CE in E . Draw EF parallel to DC . Then BG , GE will be equal to the parts to be cut off.



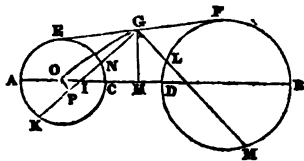
Join BE . The squares of BG , GE are equal to the square of BE , i. e. to the given square;

and $AG : GF :: AC : CD$,

i. e. in the given ratio of the remainders.

(43.) *Two points being given in a given straight line; to determine a third such that the rectangles contained by its distances from each extremity and the given point adjacent to that extremity may be equal.*

Let AB be the given straight line, C and D the given points in it. On AC and DB as diameters let circles be described, and let EF touch them in E and F . Bisect EF in G , and let fall the perpendicular GH ; H is the point required.



From G draw any lines GNK , GLM cutting the circles. Take O the centre of the circle ACE , and draw

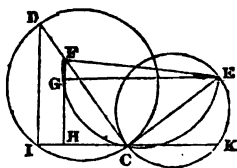
OP perpendicular to GK . (Eucl. ii. 6.) the rectangle NG, GK together with the square of PN is equal to the square of PG ; to each of these add the square of PO ; and the rectangle NG, GK together with the squares of OP, PN (i. e. the square of OC) is equal to the squares of OP, PG , i. e. to the square of OG , or to the squares of OH, HG . But the square of OH is equal to the rectangle CH, HA together with the square of OC ; whence the rectangle NG, GK is equal to the rectangle CH, HA together with the square of HG . In the same manner it may be shewn that the rectangle LG, GM is equal to the rectangle DH, HB together with the square of HG . But since the rectangle NG, GK is equal to the rectangle LG, GM , the rectangle CH, HA is equal to the rectangle DH, HB .

COR. If IH be a mean proportional between CH and HA ; $IG = GE$.

(44.) *Through the point of intersection of two given circles, to draw a line in such a manner that the sum of the respective rectangles contained by the parts thereof which are intercepted between the said point and their circumferences, and given lines A and B, may be equal to a given square.*

Let the two circles CID, CEK cut each other in the point C ; from C draw the diameters CD, CE . In CD take the point F such, that $CD : CF :: A : B$. Join EF ; and on it as a diameter describe a semicircle, in which place EG a third proportional to A and the side of the given square. Draw ICK parallel to EG ; it will be the line required.

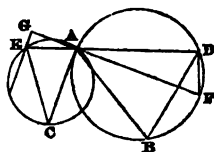
Join FG , and produce it to H . The angle DIC is



equal to FGE , i. e. to FHC , $\therefore FH$ is parallel to DI ;
 and $CI : CH :: CD : CF :: A : B$,
 $\therefore$ the rectangle A, CH , is equal to the rectangle B, CI .
 Now since EG is a third proportional to A and the side
 of the given square, the rectangle A, EG will be equal to
 the given square. But the rectangle A, EG , is equal to
 the rectangles A, HC , and A, CK , i. e. to the rectangles
 B, IC , and A, CK ; $\therefore$ the rectangles A, KC , and $B,$
 IC , are equal to the given square.

(45.) *Through a given point, to draw an indefinite line, such, that if lines be drawn from two other given points, and forming given angles with it, the rectangle contained by the segments intercepted between the given point and the two lines so drawn, shall be equal to the square of a given line.*

Let A be the given point through which the line is to be drawn; B and C the other given points. Join AB , AC ; and on them describe segments of circles ADB , AEC , containing angles equal to the given angles. Draw either diameter AF , on which produced take AG such, that the rectangle FA, AG , may be equal to the given square. Draw GE perpendicular to GF ; join EA , and produce it both ways; it is the line required.



Join DF . The angles at G and D being right angles, the triangles AGE , ADF are similar,

$$\therefore EA : AG :: FA : AD,$$

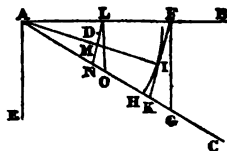
$\therefore$ the rectangle EA, AD is equal to the rectangle $FA,$

AG, i. e. to the given square; and *CE*, *BD* form with *ED* angles equal to the given angles.

If *GE* does not meet the circle, the problem is impossible.

(46.) *Through a given point between two straight lines containing a given angle, to draw a line such that a perpendicular upon it from the given angle may have a given ratio to a line drawn from one extremity of it, parallel to a line given in position.*

Let *AB*, *AC* be the lines forming the given angle *BAC*, and *D* a point between them, and *AE* the line given in position. Draw any line *FG* parallel to *AE*, and take *AH* : *FG* in the given ratio; and with the centre *A*, and radius *AH*, describe a circle, to which draw *FIK* a tangent. Join *AI*; and through *D* draw *LMN* parallel to *FK*, and *LO* parallel to *FG*. *LN* is the line required.

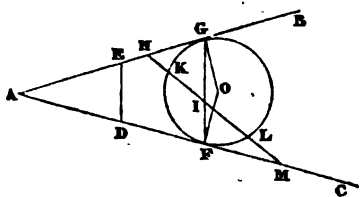


For *AI* is perpendicular to *FK*, and $\therefore$ *AM* to *LN*; and *LO* is parallel to *AE*,

and $FG : LO :: AF : AL :: (AI =) AH : AM$,
 $\therefore AM : LO :: AH : FG$, i. e. in the given ratio.

(47.) *Through a given point between two indefinite straight lines not parallel to one another, to draw a line which shall be terminated by them, so that the rectangle contained by its segments shall be less than the rectangle contained by the segments of any other line drawn through the same point.*

Let AB, AC be the given lines meeting in A . In AC take any point D , and make $AE = AD$. Join DE ; and through I the given point draw FIG parallel to DE . FIG is the line required.



Draw the perpendiculars FO, GO meeting in O . Then since ED is parallel to FG , and the angles AED, ADE are equal, $\therefore AFG$ and AGF are equal. But $AFO = AGO$, each being a right angle, $\therefore OGF = OFG$, and $OF = OG$; a circle $\therefore$ described from the centre O , and radius OG , will pass through F , and touch AB, AC in G and F , since the angles at G and F are right angles. Let now any other line $HKLM$ be drawn through I , and terminated by AB, AC . Since all other points in AB but G are without the circle, H is without the circle; $\therefore HM$ cuts the circle in K ; and for the same reason also in L . Now the rectangle KI, IL is equal to the rectangle GI, IF . But the rectangle KI, IL is less than the rectangle HI, IM ; $\therefore$ the rectangle GI, IF is less than the rectangle HI, IM . In the same manner it may be shewn that the rectangle GI, IF is less than the rectangle contained by the segments of any other line drawn through I , and terminated by AB, AC .

SECT. VI.

(1.) *To describe an isosceles triangle on a given finite straight line.*

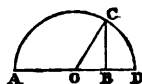
Let AB be the given straight line. Produce it, if necessary; and make AC and BD , each equal to one of the equal sides of the triangle. With A and B as centres, and radii AC , BD , describe circles, cutting each other in E ; join AE , BE ; AEB is the triangle required.

For $AE = AC = BD = BE$.



(2.) *To describe a square which shall be equal to the difference of two squares, whose sides are given.*

Take a straight line AB terminated at A , and cut off AO equal to a side of the greater, and OB equal to a side of the lesser square. With O as centre, and radius OA , describe a circle OCD ; and from B draw BC at right angles to AD . The square described upon BC is the square required.



Join OC . (Eucl. i. 48.), the square described upon BC is equal to the difference of the squares on OC and OB , i. e. on AO and OB .

COR. Hence a mean proportional between the sum and difference of two given lines may be determined.

(3.) *To describe a rectangular parallelogram which shall be equal to a given square, and have its adjacent sides together equal to a given line.*

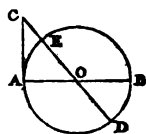
Let AB be equal to the given line. Upon it describe a semicircle ADB . From A draw AC perpendicular to AB , and equal to a side of the given square. Through C draw CD parallel to AB , and let fall the perpendicular DE . The rectangle contained by AE , EB will be the rectangle required.



For the rectangle AE , EB is equal to the square of ED , which is equal to the square of AC , i.e. to the given square; and AB is the sum of the adjacent sides AE , EB .

(4.) *To describe a rectangular parallelogram which shall be equal to a given square, and have the difference of its adjacent sides equal to a given line.*

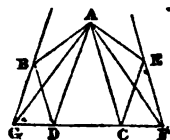
Let AB be equal to the given line. On it as diameter describe a circle. From A draw AC at right angles to AB , and $\therefore$ a tangent to the circle at A ; make AC equal to a side of the given square. Take O the centre; join CO , and produce it to D . The rectangle contained by EC , CD is the rectangle required.



For the rectangle EC , CD , is equal to the square of AC , i.e. to the given square; and the difference of the sides containing the rectangle is $ED = AB =$ the given line.

(5.) *To describe a triangle which shall be equal to a given equilateral and equiangular pentagon, and of the same altitude.*

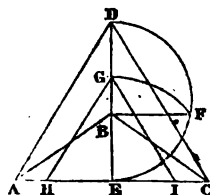
Let $ABDCE$ be the given pentagon. Join AC , AD ; and produce CD indefinitely both ways. Through B and E draw BG , EF respectively parallel to AD and AC . Join AF , AG . AFG is the triangle required.



Since AD is parallel to BG , (Eucl. i. 37.) the triangles ABD , AGD are equal; and for a similar reason, $AEC = AFC$; $\therefore$ the triangles ABD , AEC are equal to AGD , AFC ; to these equals add the triangle ADC ; and the pentagon $ABDCE$ is equal to the triangle AGF ; and they have the same altitude, viz. the perpendicular from A upon DC .

(6.) *To describe an equilateral triangle equal to a given isosceles triangle.*

Let ABC be the given isosceles triangle. On AC describe an equilateral triangle ADC , and from D draw DE perpendicular to AC ; it will also bisect AC and pass through B . On DE describe a semicircle; and from B draw BF perpendicular to DE , meeting the circle in F . With the centre E , and radius EF , describe a circle meeting ED in G ; draw GH , GI parallel to DA , DC respectively; the triangle GHI is equilateral, and equal to ABC .



Since GH is parallel to AD , and GI to DC , the triangles GHI , ADC are similar; but ADC is equilateral, and $\therefore$ also GHI is equilateral.

Also (Eucl. vi. 8. Cor.) $ED : EG :: EG : ER$,

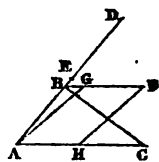
and (Eucl. vi. 2.) $ED : EG :: EA : EH$,

$\therefore EG : EB :: EA : EH$,

and $\therefore$ (Eucl. vi. 15.) the triangles EGH , EBA are equal. But $GHE = GIE$, and $BAE = BCE$, $\therefore$ also $GHI = BAC$.

(7.) *To describe a parallelogram, the area and perimeter of which shall be respectively equal to the area and perimeter of a given triangle.*

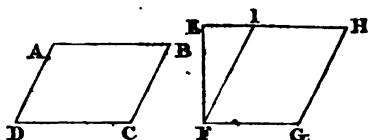
Let $\triangle ABC$ be the given triangle. Produce AB to D , making $BD = BC$; bisect AD in E ; draw BF parallel to AC ; and with the centre A , and radius AE , describe a circle cutting BF in G . Join AG ; and bisect AC in H . Draw HF parallel to AG . $AGFH$ is the parallelogram required.



For $HF = AG = AE$, $\therefore HF$ and AG together are equal to AD , i.e. to AB and BC together; and $GF = AH = HC$, $\therefore$ the perimeter of $AGFH$ is equal to the perimeter of ABC ; and $AGFH$ is double of a triangle on the base AH and between the same parallels, and $\therefore$ is equal to the triangle ABC .

(8.) *To describe a parallelogram which shall be of given altitude, and equiangular and equal to a given parallelogram.*

Let $ABCD$ be the given parallelogram, and EF the given altitude. Draw EH and FG at right angles to FE ; and at the

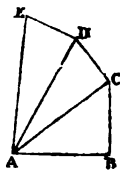


point F , in the line GF , make the angle GFI equal to CDA ; take FG a fourth proportional to FI , AD and DC ; and from G draw GH parallel to FI , meeting EH produced in H ; $IFGH$ is the parallelogram required.

For its altitude is EF ; and the angle $GFI = CDA$, $\therefore FIH = DAB$; whence the parallelograms are equiangular; and they are equal; since the sides about the equal angles are reciprocally proportional (Eucl. vi. 14.).

(9.) To describe a square which shall be equal to the sum of any number of given squares.

Let AB be a side of one of the given squares. From B draw BC perpendicular to AB , and equal to a side of the second square. Join AC ; and from C draw CD perpendicular to it, and equal to a side of the third square. Join AD ; and from D draw DE perpendicular to AD , and equal to a side of the fourth. Join AE . The square of AE is equal to the squares of AB , BC , CD , DE .



Since the angles ADE , ACD , ABC are right angles, the square of AE is equal to the squares of AD , DE , i. e. to the squares of AC , CD , DE ; and $\therefore$ to the squares of AB , BC , CD , DE .

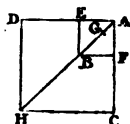
And by proceeding in the same manner whatever be

the number of given squares, one equal to their sum may be found.

COR. Hence lines may be found, which have the same ratio as the square roots of the natural numbers.

(10.) *Having given the difference between the diameter and side of a square ; to describe the square.*

Let AB be the given difference. Draw AC , AD , each making half a right angle with AB ; and complete the square EF . Take $AG =$ the difference between BA and BF . Since the ratio between the side of a square and its diameter is given, that of their difference to the diameter is also given. Take $\therefore AH : AB :: AB : AG$; and through H draw HC , HD perpendicular to AC , AD ; CD is the square required.

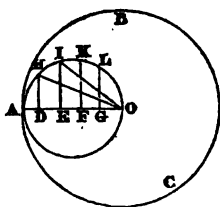


For DC being a parallelogram is also (Eucl. i. 46. Cor.) rectangular; and the angle DAH being half a right angle, is equal to DHA , $\therefore DA = DH$; whence the sides are equal; and the figure is a square. And since $BG = BF$, $HB = HC$; and AB is the difference between the diameter and side.

(11.) *To divide a circle into any number of concentric equal annuli.*

Let ABC be the given circle, and O its centre. Draw any radius OA , and divide into the given number

of equal portions in the points $D, E, F, G, \&c.$ On OA describe a semi-circle, and draw the perpendiculars $DH, EI, FK, GL, \&c.$ Join $OH, OI, \&c.$ and with the centre O and radii $OH, OI, \&c.$ let circles be described; they will divide the circle ABC as is required.

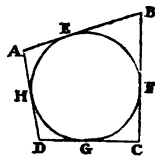


Since the areas of circles are in the duplicate ratio of their radii; the area of the circle whose radius is OA is to that whose radius is OH in the duplicate ratio of $OA : OH$, i. e. in the ratio of $OA : OD$; $\therefore$ the area of the first annulus will be to the area of the circle whose radius is $OD :: AD : OD$. And in the same manner the area of the second annulus, will be to the area of the circle whose radius is OD , as $DE : OD$; and since $AD = DE$, the annuli will be equal. The same may be proved of all the rest.

COR. The construction will be nearly the same, if it be required to divide the circle into annuli which shall have a given ratio; by dividing the radius AO in that proportion.

(12.) *In any quadrilateral figure circumscribing a circle, the opposite sides are equal to half the perimeter.*

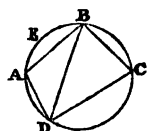
Let $ABCD$ be a quadrilateral figure circumscribing the circle EFG ; its opposite sides are equal to half the perimeter.



For (Eucl. iii. 36. Cor.) $AE = AH$, and $DH = DG$, $\therefore AD$ is equal to AE and DG together. In the same way BC is equal to BE and GC together, $\therefore AD$ and BC together are equal to AB and DC together.

(13.) *If the opposite angles of a quadrilateral figure be equal to two right angles, a circle may be described about it.*

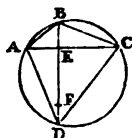
Let $ABCD$ be a quadrilateral figure, whose opposite angles are equal to two right angles.



Join BD ; then if a circle be described about the triangle BCD it will pass through A . For the angle BCD and the angle in the segment BED , are together equal to two right angles, and $\therefore$ equal to BCD , BAD ; whence BAD is equal to the angle in the segment BED ; and $\therefore A$ must be a point in the circumference; or the circle will be described about $ABCD$.

(14.) *A quadrilateral figure may have a circle described about it, if the rectangles contained by the segments of the diagonals be equal.*

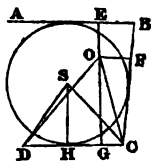
Let $ABCD$ be a quadrilateral figure, the rectangles contained by the segments of whose diagonals are equal, viz, the rectangle AE, EC , equal to BE, ED .



Describe a circle about the triangle ABC ; if it does not pass through D , let it cut BD in F ; then (Eucl. iii. 35.) the rectangle BE, EF , is equal to the rectangle AE, EC , i. e. to the rectangle BE, ED , by the supposition; whence EF is equal to ED , the less to the greater, which is impossible; $\therefore$ the circle must pass through D .

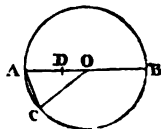
(15.) *If from any point within a regular figure circumscribed about a circle perpendiculars be drawn to the sides; they will together be equal to that multiple of the semidiameter, which is expressed by the number of the sides of the figure.*

Let $ABCD$ be a regular figure circumscribed about the circle; and, from any point O , let perpendiculars OE, OF, OG , &c. be drawn. Take S the centre of the circle. Join SD, SC, SH . Then the figure will be divided into as many triangles round S and O , as there are sides of the figure; now the triangle $SCD : OCD :: SH : OG$; and the same being true of the triangles on each side, the sum of the triangles round S , will be to the sum of the triangles round O , as the sum of the lines SH to the sum of the perpendiculars from O . And the first term of the proportion being equal to the second, the sum of the perpendiculars from O is equal to that multiple of the radius which is expressed by the number of the sides; each perpendicular from S being a radius of the circle.



(16.) *If the radius of a circle be cut in extreme and mean ratio; the greater segment will be equal to the side of an equilateral and equiangular decagon inscribed in that circle.*

Let AO , the radius of the circle ABC , be cut in extreme and mean ratio in D ; AD is equal to the side of an equilateral and equiangular decagon inscribed in the

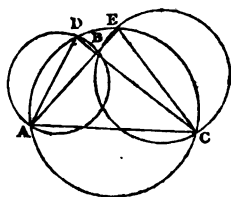


circle. In the circle place AC equal to AD ; join CO . Then (Eucl. iv. 10.) the angles at A and C are double the angle at O ; whence AOC is one fifth part of two right angles, or one tenth part of four right angles, i. e. of the angles at O ; and $\therefore AC$ is the side of a regular decagon inscribed in the circle.



(17.) *Any segment of a circle being described on the base of a triangle; to describe on the other sides segments similar to that on the base.*

Let ABC be a triangle, on the base AC of which a segment of a circle ADC is described. Produce AB , CB to E and D . Join AD , CE ; and through A , D , B , and C , E , B let circles be described; the segments ADB , BEC are similar to ADC .

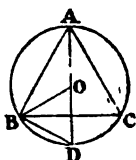


For the angle ADC being in the segments ADB , ADC , those segments are similar. For the same reason the segments ADC , BEC are similar. And since the angles ADC , AEC are equal, $\therefore$ the segments ADB , BEC are similar.

(18.) *If an equilateral triangle be inscribed in a circle; the square described on a side thereof is equal to three times the square described upon the radius.*

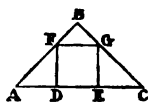
Let ABC be an equilateral triangle inscribed in a circle. From A draw the diameter AD , and take O the centre;

join BD , BO . Then the angle $BOD = BAC = BCA = BDO$, $\therefore BD = BO$; and the squares of AB , BD are equal to the square of AD , i. e. to four times the square of BO , or BD ; and $\therefore$ the square of AB is equal to three times the square of BD or BO .



(19.) To inscribe a square in a given right-angled isosceles triangle.

Let ABC be a right-angled isosceles triangle, having the side $BA = BC$. Trisect the hypotenuse AC in the points D , E ; and from D , E draw DF , EG perpendicular to AC ; join FG ; $DFGE$ is the square required.

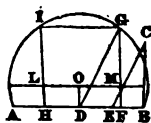


Since the angle DAF is half a right angle, and the angle at D a right angle, $\therefore$ the angle DFA is half a right angle, and equal to DAF ; whence $DF = DA$. In the same manner it may be shewn that $EG = EC$. But $AD = EC$; and $\therefore FD$, DE and EG are equal; and (Eucl. i. 33.) $FG = DE$; $\therefore$ the figure is equilateral. And it is rectangular, (Eucl. i. 46.) since the angles at D and E are right angles; $\therefore$ it is a square.

(20.) To inscribe a square in a given quadrant of a circle.

Let AOB be the given quadrant, whose centre is O . Bisect the angle AOB by the line OC . Draw CE , CD parallel to OA , OB . DE is a square.

Let AIB be the segment of a circle, whose base AB is bisected in D . From B draw BC perpendicular to BA and equal to BD . Bisect BD in E , and join CE .



Draw DG parallel to CE , and GF to CB . Take $DH = DF$; draw HI perpendicular to AH , and $\therefore$ parallel to GF ; Join GI . FI is the square required.

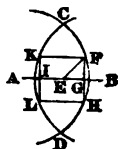
Since GD and GF are respectively parallel to CE and CB ,

$$GF : FD :: CB : BE :: 2 : 1,$$

$\therefore GF = 2FD = FH$. Take O the centre, and draw OL , OM perpendiculars to HI , FG ; then since $HD = DF$, $OL = OM$, $\therefore$ (Eucl. iii. 14.), $IL = GM$; but $LH = FM$, $\therefore IH = GF$; whence $IG = HF$, and the figure is equilateral; and since the angle at F is a right angle, the figure is rectangular, and $\therefore$ is a square.

(23.) *Having given the distance of the centres of two equal circles which cut each other; to inscribe a square in the space included between the two circumferences.*

Let A and B be the centres of two equal circles, which cut each other in C and D . Join AB , and bisect it in E ; and at the point E make the angle $GEF =$ half a right angle; and from F draw FGH perpendicular to AB .



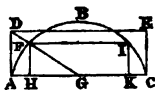
Make $EI = EG$; and through I draw KL perpendicular to AB ; join KF , LH . KH is a square.

Since $EI = EG$, $BI = AG$, and $\therefore$ (Eucl. iii. 14.) $KL = FH$; and they are parallel, $\therefore KF$ is equal and parallel

to LH , $\therefore KH$ is a parallelogram. Also since GEF is half a right angle, and EGF a right angle, $\therefore EFG$ is half a right angle, and $\therefore$ equal to GEF ; whence $EG = GF$, and $FH = IG$. But KF is equal and parallel to IG (Eucl. i. 33); $\therefore$ the four sides are equal; and GFK is a right angle, $\therefore$ the figure is rectangular (Eucl. i. 46. Cor.), and consequently is a square.

(24.) *In a given segment of a circle to inscribe a rectangular parallelogram, whose sides shall have a given ratio.*

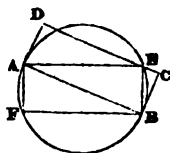
Let ABC be the given segment of a circle. From A draw AD perpendicular to AC , and make $AD : AC$ in the ratio of the sides. Complete the parallelogram AE . Bisect AC in G , and join DG ; and from F draw FH perpendicular, and FI parallel to AC . Draw IK parallel to FH ; HI is the rectangular parallelogram required.



Since FH is perpendicular to AC , it is parallel to AD ;
 and $\therefore FH : HG :: AD : AG$,
 whence $FH : HK :: AD : AC$,
i.e. in the given ratio. And FHG being a right angle
 all the angles of the figure are right angles.

(25.) *In a given circle to inscribe a rectangular parallelogram equal to a given rectilineal figure.*

Let AEB be the given circle; on the diameter AB describe a rectangular parallelogram $ABCD$ equal to the given rectilineal figure; and let the side DC



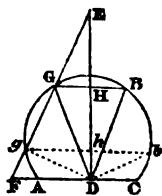
cut the circumference in E . Join AE , EB . Draw BF parallel to AE , and join AF . FE is the rectangular parallelogram required.

For the angle EBF is equal to the two angles EBA , ABF , i. e. to EBA , BAE , and $\therefore$ is a right angle; and the angles BEA and BFA are right angles, by construction; $\therefore$ also EAF is a right angle; the figure $AFBE$ is therefore rectangular; and it is double of ABE , and (Eucl. i. 41.) $\therefore$ equal to $ABCD$, i. e. to the given rectilineal figure.

The given figure must not exceed the square of half the diameter.

(26.) *In a given segment of a circle to inscribe an isosceles triangle, such that its vertex may be in the middle of the chord, and the base and perpendicular together equal to a given line.*

Let ABC be the given segment. Bisect AC in D , and draw DE at right angles to AC , and equal to the given line. Make DF the half of DE ; and join EF , meeting the circle in G . Draw GB parallel to AC ; join GD , DB . GDB is the triangle required.



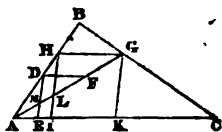
Since GB is parallel to AC , it is bisected by DE . Also (Eucl. vi. 2.) EH is double of GH , and $\therefore$ equal to GB ; $\therefore$ GB and HD together are equal to ED , i. e. to the given line; and since $GH = HB$, and the angles at H are right angles, $GD = DB$, $\therefore$ the triangle is isosceles.

If EF does not meet the segment, the problem is impossible. When the line FE cuts the segment, there

are two isosceles triangles DGB , Dgb that will answer the conditions ; when it touches the segment, only one.

(27.) *In a given triangle to inscribe a parallelogram similar to a given parallelogram.*

Let ABC be the given triangle. In AB take any point D , and draw DF parallel to AC ; and make the angle FDE equal to one angle of the parallelogram, and take $DF : DE$ in the ratio of the sides. Join AF , and produce it to G ; draw GH , HI respectively parallel to FD , DE ; and GK parallel to HI . HK is the parallelogram required.



For HI being parallel to DE , and HG to DF ,

$$HI : DE :: HA : DA :: HG : DF,$$

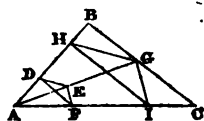
$$\therefore HI : HG :: DE : DF,$$

i. e. in the ratio of the sides.

Also the angle $GHI = FDE =$ one of the angles of the parallelogram, $\therefore HIK$ will be equal to the adjacent angle of the parallelogram, and HK is similar to the given parallelogram.

(28.) *In a given triangle to inscribe a triangle similar to a given triangle.*

Let ABC be the given triangle, in which the triangle is to be inscribed. In AB take any point D , and draw any line DF to the opposite side; and at the points D and F make the angles FDE , DFE equal to two of the angles of the given triangle to which



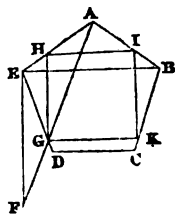
the inscribed one is to be similar, $\therefore$ the angle at E will be equal to the third angle. Join AE , and produce it to G ; and from G draw GH , GI respectively parallel to ED , EF ; join HI . HIG is the triangle required.

Since DE and EF are respectively parallel to HG , GI , the angle DEF is equal to HGI .

Also $DE : HG :: AE : AG :: EF : GI$, whence (Eucl. vi. 6.) the triangles HGI , DEF are similar, and $\therefore HGI$ is similar to the given triangle.

(29.) *In a given equilateral and equiangular pentagon, to inscribe a square.*

Let $ABCDE$ be the given pentagon. Join EB ; and from E draw EF perpendicular and equal to EB . Join AF ; and from G , where it cuts ED , draw GH parallel to FE . Draw HI , GK parallel to EB . Join IK . HK is the square required.



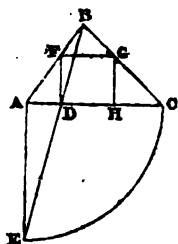
Since HG is parallel to EF and HI to EB ,

$$HG : EF :: AH : AE :: HI : EB,$$

but $EF = EB$, $\therefore HG = HI$. And since $AE = AB$, $\therefore$ (Eucl. vi. 2.) $HE = IB$; also GK and DC being parallel to EB , and $DE = BC$, $\therefore EG = BK$. The triangles EHG , IKB , therefore have two sides in each and the included angles equal, and $\therefore HG = IK$, and the angle $EHG = BIK$, whence HG and IK are also parallel; therefore also GK is equal to HI ; hence the four sides are equal; and the angle at H being a right angle, all the angles are right angles, and consequently HK is a square.

(30.) *In a given triangle to inscribe a rhombus, one of whose angles shall be in a given point in the side of the triangle.*

Let ABC be the given triangle, and D the given point. Join BD , and produce it; and with the centre A , and radius AC , describe a circle cutting it in E . Join AE ; and draw DF parallel to it, FG parallel to AC , and GH to FD . FH is the rhombus required.



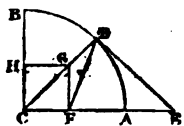
Since FD is parallel to AE , $BF : FD :: BA : AE$; and since FG is parallel to AC ,

$$BF : FG :: BA : AC :: BA : AE,$$

$\therefore FD = FG$; and the sides opposite to these are equal, $\therefore$ the figure $FDHG$ is a rhombus.

(31.) *To inscribe a circle in a given quadrant.*

Let ABC be the given quadrant. Bisect the angle ACB by the line CD ; and at D draw DE touching the quadrant, and meeting CA produced in E . Make $CF = AE$. From F draw FG at right angles to AC . G is the centre of the circle required.

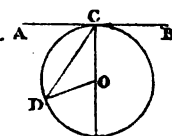


From G draw GH perpendicular to BC . Join DF . Since the angle DCE is half a right angle, and the angle at D a right angle, $DE = DC = AC = FE$, $\therefore$ the angle $EDF = EFD$; whence also $GDF = GFD$, and $GD = GF$; and since the angles FCG, GCH are equal, and GC common to the right-angled triangles GFC, GHC , $\therefore GF = GH$; $\therefore$ the three lines GD, GF, GH

are equal and the circle described from the centre G , and distance of any one of them, will pass through the extremities of the other two, and touch the arc and sides in the points D, F, H , because the angles at those points are right angles.

(32.) To describe a circle, the circumference of which shall pass through a given point, and touch a given straight line in a given point.

Let AB be the given straight line, C the given point, in which the circle is to touch it, D the point through which it must pass. Draw CO perpendicular to AB . Join CD ; and at the point D make the angle $CDO = DCO$; the intersection of the lines CO and DO is the centre of the circle required.

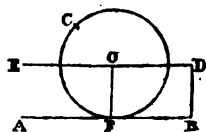


Since the angle $DCO = CDO$, $CO = DO$, and $\therefore$ a circle described from the centre O , at the distance OD , will pass through C , and touch the line AB in C , because OC is perpendicular to AB .

(33.) To describe a circle which shall pass through a given point, have a given radius, and touch a given straight line.

Let AB be the given straight line, and C the given point through which the circle must pass.

In AB take any point B ; and from



it draw BD at right angles to AB , and equal to the given radius; through D draw DE parallel to AB ; and with the centre C , and radius equal to the given radius, describe a circle cutting DE in O . O is the centre of the circle required.

From O draw OF perpendicular to AB , it is equal to DB , i. e. to the given radius; and the circle described from the centre O , and radius OF , will touch (Eucl. iii, 16. Cor.) the line AB in F , and pass through C .

(34.) To describe a circle which shall pass through two given points, and touch a given straight line.

Let A, B be the given points, and CD the given straight line. Join AB . And

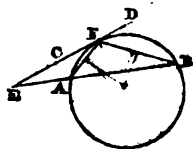
1. let CD be parallel to AB .

Bisect AB in E , and draw EF perpendicular to AB , and $\therefore$ to CD . Join FA , and make the angle $FAO = AFO$; then will O be the centre of the circle required.



Since the angle $FAO = AFO$, $AO = OF$. But $AE = EB$, and EO is common to the triangles AEO, BEO , and the angles at E right angles, $\therefore AO = OB$. Whence AO, OB, OF are all equal; and the circle described from the centre O , at the distance of any one of them, will pass through the extremities of the other two, and touch the line CD , since OF is perpendicular to CD .

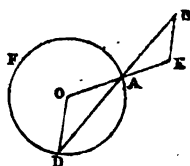
2. But if AB is not parallel to CD , let them be produced to meet in E ; and take EF a mean proportional between EA and EB . Join FA, FB ; and describe a circle about the triangle AFB ; it will be the circle required.



Since EF is a mean proportional between EA and EB , EF touches the circle (Eucl. iii. 37.), which passes through A and B .

(35.) *To describe a circle, the circumference of which shall pass through a given point, and touch a circle in a given point; the two points not being in a tangent to the given circle.*

Let A be the given point in the circumference of the circle whose centre is O ; B the given point without. Join BA , and produce it to D . Join OD ; and through A draw OAE ; and draw BE parallel to OD , cutting OAE in E . E is the centre of the circle required.

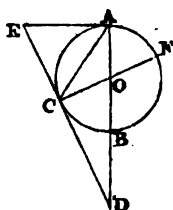


Since (Eucl. i. 29.) the angle ODA is equal to ABE , and OAD to BAE , $\therefore$ the triangles ODA , ABE are similar, and OD being equal to OA , AE will be equal to EB ; a circle $\therefore$ described with the centre E , and radius EA , will pass through B , and touch the circle ADF in the point A , since the line joining the centres passes through A .

(36.) *To describe a circle the centre of which may be in the perpendicular of a given right-angled triangle, and the circumference pass through the right angle and touch the hypotenuse.*

Let EAD be the given right-angled triangle, having the angle at A a right angle. Make $EC = EA$. Join

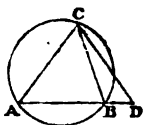
CA ; and draw CO at right angles to ED . The circle described with O as centre, and radius OA , will be the circle required.



Since $EA = EC$, the angle $ECA = EAC$, and ECO , EAO are equal, being right angles; $\therefore OCA = OAC$ and $OA = OC$. The circle $\therefore$ described from the centre O , and radius OA , will pass through the extremity of OC , and touch ED in C , because CO is at right angles to ED .

(37.) *To describe a circle which shall pass through the extremities of a given line, so that if from any point in its circumference a line be drawn making a given angle with the given line; the rectangle contained by the segment it cuts off and the given line, may be equal to the square of the line drawn from the same point to the farther extremity of the given line.*

Let AB be the given line. On it describe a segment of a circle containing an angle equal to the given angle. Complete the circle; it will be the one required.



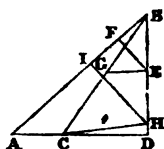
From any point C draw CD , making with AB the angle ADC equal to the given angle; join CA , CB . Since the angle $CDA = ACB$, and the angle at A is common, the triangles ACD , ABC are equiangular, and therefore

$$AB : AC :: AC : AD,$$

whence the rectangle contained by AB , AD , is equal to the square of AC .

(38.) *To determine a point in the perpendicular let fall from the vertical angle of a triangle on the base; about which as a centre a circle may be described touching the longer side, and passing through the opposite angular point.*

Let ABC be a triangle, and from B the vertex let BD be drawn perpendicular to AC . In DB take any point E , and from it draw EF perpendicular to AB ; and from E to BC , draw $EG = EF$; from C draw CH parallel to GE , and from H draw HI perpendicular to AB ; H is the point required.



Since EF is parallel to HI ,

$$FE : HI :: BE : BH,$$

and since GE is parallel to HC ,

$$GE : HC :: BE : BH,$$

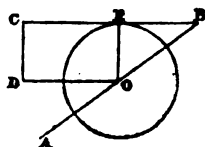
$$\therefore FE : HI :: GE : HC;$$

but, by construction, $FE = EG$, $\therefore HI = HC$; and a circle described from the centre H at the distance HI , will pass through C , and touch AB in I , since the angle HIB is a right angle.

(39.) *To describe a circle which shall have a given radius, and its centre in a given straight line, and shall also touch another given straight line inclined at a given angle to the former.*

Let AB be the given line, in which the centre is to be; BC the line which the circle is to touch.

In BC take any point C , and draw CD at right angles to it; and make

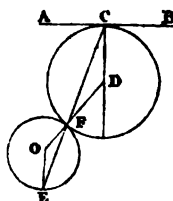


CD equal to the given radius. Through D draw DO parallel to CB ; O is the centre of the circle required.

Through O draw OE parallel to DC ; $\therefore CO$ is a parallelogram; whence OE is equal to DC , i. e. to the given radius. With the centre O , and radius OE , describe a circle; it will touch CB in E , because CO being a parallelogram, and ECD a right angle, CEO is also a right angle.

(40.) *To describe a circle, which shall touch a straight line in a given point, and also touch a given circle.*

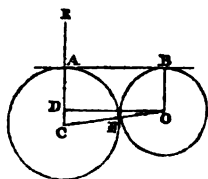
Let AB be the given line, and C the given point in it, O the centre of the given circle. Draw CD perpendicular to AB , and OE parallel to CD . Join CE , meeting the circumference in F . Join OF , and produce it to meet CD in D . D is the centre of the circle required.



Since the triangles OEF , CFD are similar, and $OE = OF$, $\therefore FD = DC$; consequently a circle described with the centre D , and radius DF , will pass through C , and touch AB in C , because the angles at C are right angles; and it will touch the given circle in F , since the line joining the centres passes through F .

(41.) *To describe two circles, each having a given radius, which shall touch each other, and the same given straight line on the same side of it.*

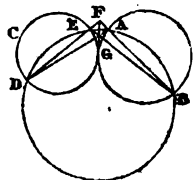
Let AB be the given straight line. From any point A in it, draw AC at right angles to it, and make AC, AD , equal to the given radii. Produce CA to E , making $AE = AD$. Draw DO parallel to AB ; and with the centre C , and radius CE , describe a circle cutting DO in O . C and O will be the centres of the circles required.



Join CO ; and draw OB perpendicular to AB ; then DAB being a right angle, as also ABO , $\therefore AD$ is parallel to BO ; and DO was drawn parallel to AB , $\therefore AO$ is a parallelogram, and $OB = AD$. With the centres C and O , and radii CA, OB describe circles, they will touch AB , since the angles at A and B are right angles; they will also touch each other, for CO is equal to CE , or to CA and AE , i. e. to CA and AD , or the sum of the radii.

(42.) To describe a circle passing through two given points, and touching a given circle.

Let A and B be the given points, and CDE the given circle. Describe a circle through A and B , and cutting the given circle in D and E . Join DE, EB, DA, AB . Then the angle $EDA = EBA$; if $\therefore DE$ and BA be produced to meet in F , the triangle FDA will be similar to the triangle FBE ;



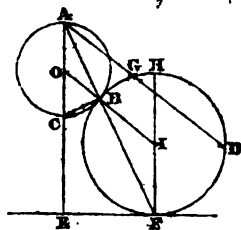
and $\therefore DF : FA :: BF : FE$,

or the rectangle DF, FE is equal to the rectangle AF, FB . Draw FG a tangent to the given circle; then the

square of FG is equal to the rectangle EF, FD , and $\therefore$ to the rectangle BF, FA ; whence a circle described through the points A, G, B , will touch the given circle, since it touches FG .

(43.) *To describe a circle, which shall pass through a given point, and touch a given circle and a given straight line.*

Let ABC be the given circle, D the given point, and EF the given straight line. Through O draw AOE perpendicular to EF . Join AD ; and divide it in G , so that the rectangle AG, AD , may be equal to the rectangle AC, AE . Through G and D describe a circle touching EF in F ; this will also touch the circle ABC .

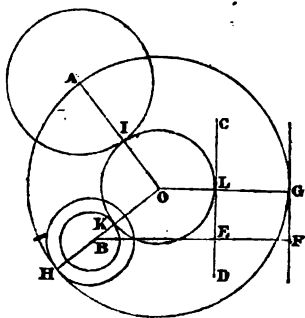


Draw the diameter FH ; it is (Eucl. iii. 18.) parallel to AE . Join AF , meeting the circle in B . Join CB . The triangles ABC, AEF having the angle at A common, and the angles ABC, AEF right angles, are similar; whence

$$AC : AB :: AF : AE,$$

$\therefore$ the rectangle AB, AF is equal to the rectangle AC, AE , i. e. to the rectangle AG, AD ; $\therefore B$ is a point in the circle HDF . Take I the centre; join OB, BI . Since AC is parallel to FI , the angle $OAB = BFI$; but $OAB = OBA$, and $IFB = IBF$, $\therefore OBA = IBF$; and OBI is a straight line, which joins the centres of the two circles, which $\therefore$ touch each other.

Let A and B be the centres of the two circles, and CD the line given in position. From B let fall the perpendicular BE , and produce it, making $EF =$ the radius of the circle whose centre is A . Through F draw FG parallel to CD . With the centre B , and radius equal to the difference of the radii of the two circles, describe a circle; through A let a circle be described, touching the line GF and the last described circle (vi. 43.); and let G and H be the points of contact. The centre of this circle will also be the centre of the circle required.



Let O be the centre; join OA, OG, OH ; and with the centre O , and radius OI , describe the circle IKL . Since $LG = KH = AI$, $\therefore OL = OK = OI$; the circle IKL $\therefore$ touches CD in L , and the circle, whose centre is A , in I ; and since OB is equal to the difference between OH and HB , *i.e.* between OA and $(IA - BK)$, or is equal to OK and KB together, $\therefore$ it touches the circle whose centre is B , in K .

Let AB, CD be the given lines, and E the given

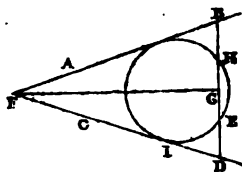
Rectangle = product of (-) sides

Sect. 6.]

GEOMETRICAL PROBLEMS.

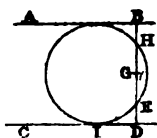
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point. Produce the lines to meet in F . Bisect the angle BFD by the line FG ; and from E draw EG perpendicular to FG , and produce it both ways to B and D . Take $GH = GE$; and make DI a mean proportional between DE and DH ; a circle described through the points H, E, I , will touch CD .



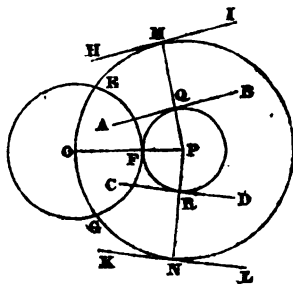
For the rectangle DE, DH , is equal to the square of DI . And for a similar reason it will touch AB ; since the rectangle BH, BE , is equal to the rectangle ED, DH .

If the lines AB, CD be parallel; through the given point E , draw $DEHB$ perpendicular to AB or CD ; bisect it in G ; and make $GH = GE$. Take DI a mean proportional between DE and DH ; and a circle described through I, E and H will be the circle required.



(46.) To describe a circle which shall touch two given straight lines, and also touch a given circle.

Let AB, CD be the given straight lines, EFG the given circle, whose centre is O . Draw HI, KL parallel to the given lines, so that their perpendicular distances from those lines may be equal to OF the radius of the given circle. By the last problem describe a circle touching HI, KL ,

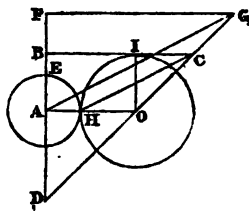


and passing through O the centre of the given circle. Let P be the centre of this circle; it will also be the centre of the circle required.

Join PM, PN, PO . Since these lines are equal, and MQ, RN, OF are also equal by construction, $\therefore PQ, PR, PF$ are also equal; and a circle described from the centre P at the distance of any one of them, will pass through the extremities of the other two, and touch the lines AB, CD , in Q and R ; since the angles at those points are equal to the angles at M and N , and $\therefore$ right angles; and it will also touch the circle EFG in F , since OP the line joining the centres passes through F .

(47.) *To describe a circle which shall touch a circle and straight line, both given in position, and have its centre also in a given straight line,*

Let the circle whose centre is A , and the straight line BC be given in position; and let CD be the line, in which the centre of the required circle is to be. On BC let fall the perpendicular AB ; and make $BF = AE$; through F draw FG parallel to BC , meeting DC in G .



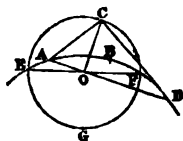
Join GA ; and draw CH parallel to it, meeting the given circle in H , (if the problem be possible). Join AH , and let it meet DC in O . O is the centre of the circle required.

Let fall the perpendicular OI . Then (Eucl. vi. 2.)

$HO : OC :: AH : GC :: FB : GC$ by construction,
 $:: BD : DC$, (Eucl. vi. 2.)
 $:: IO : OC$, by sim. triangles;
 $\therefore HO = IO$; and a circle described with the centre O , and
 radius OI or OH , will pass through the extremity of the
 other, and touch the line BC in I , and the circle in H ;
 because the angles at I are right angles; and AO the
 line joining the centres of the circles passes through H .

(48.) *Through two given points within a given circle, to describe a circle, which shall bisect the circumference of the other.*

Let A and B be the given points within the circle whose centre is O . Join AO ; and produce it indefinitely; and from O draw OC at right angles to it. Join AC ; and draw CD at right angles to it, meeting AO produced in D ; and through A, B, D describe a circle; it will bisect the other in the points E , and F .



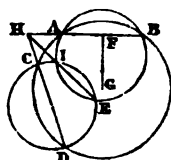
Join EO, OF . Then (Eucl. vi. 8.)

$$AO : OC :: OC : OD,$$

$\therefore$ the rectangle AO, OD is equal to the square of OC ,
i. e. to the rectangle EO, OF ; whence (Eucl. iii. 35.)
 EOF is a straight line; and since it passes through the
 centre of the circle ECF , it will be a diameter of that
 circle; $\therefore$ the circumference ECF is equal to the cir-
 cumference EGF , or the circumference of the given
 circle is bisected.

(49.) *Through two given points without a given circle, to describe a circle, which shall cut off from the given one an arc equal to a given arc.*

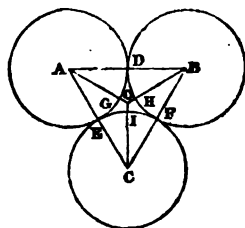
Let A and B be the given points, and CDE the given circle. Join AB ; and bisect it in F ; from F draw FG at right angles to AB ; and from any point G in it, at the distance GA or GB , describe a circle ABD , cutting the given circle in C and D . Join DC ; and produce it to meet BA in H . From H draw HIE (ii. 20.), so that IE may be equal to the chord of the given arc. Through A , B and E describe a circle; it will also pass through I , and cut off the arc required.



For the rectangle HI , HE is equal to the rectangle HC , HD , and $\therefore$ also to the rectangle HA , HB , whence I is a point in the circle ABE .

(50.) *To describe three circles of equal diameters, which shall touch each other.*

Take any straight line AB , which bisect in D ; and from the centres A and B , with the equal radii AD , BD describe two circles. Upon AB describe an equilateral triangle ABC , cutting the circles in E and F ; and with the centre C , and radius CE or CF , describe another circle; these circles touch each other as required.



Since $AB = AC$, and AD is half of AB , $\therefore AE$, which is equal to it, is half of AC , and $\therefore AE = EC$. In the

same manner $CF=FB$, $\therefore$ the radii of the circles are all equal. And the circles touch each other in D, E, F , because the lines joining the centres pass through those points.

(51.) *Every thing remaining as in the last proposition; to describe a circle which shall touch the three circles.*

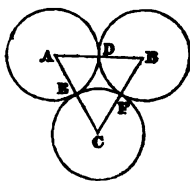
Bisect the angles CAB, CBA (see last Fig.) by the line AO, BO meeting in O . O is the centre of the circle required.

Join OC . Since the angle $CAB=CBA$, $\therefore$ their halves are equal, or $OAB=OBA$, $\therefore OA=OB$. Also since $CA=AB$, and AO is common, and the angle $CAO=BAO$, $\therefore CO=OB$; and the three lines OA, OB, OC are equal; parts of which GA, HB, CI are equal; $\therefore OG, OH, OI$ are equal; and a circle described from O as a centre, with a radius equal to any one of those lines, will pass through the extremities of the other two, and touch the circles in the points G, H, I ; because the lines joining the centres pass through those points.

In nearly the same manner a circle may be described which shall touch the three circles on the opposite circumference.

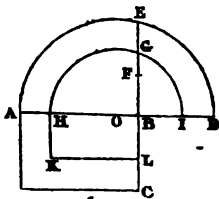
(52.) *To determine how many equal circles may be placed round another circle of the same diameter, touching each other and the interior circle.*

Let A be the centre of the interior circle, and AD its radius. Describe (vi. 50.) the circles DF , EF touching the circle DE , and each other. Then the angle at A being one third part of two, or one sixth part of four right angles, subtends an arc ED equal to one sixth of the whole circumference. And the same being true of every other contiguous circle, the number of circles which can be described touching each other and the interior one will be six.



(53.) To draw two lines parallel to the adjacent sides of a given rectangular parallelogram, which shall cut off a portion, whose breadth shall be every where the same, and whose area shall be to that of the parallelogram in any given ratio.

Let AC be the given parallelogram. Produce AB to D making $BD = BC$. On AD describe a semicircle, and produce CB to E ; and let the ratio of the part to be cut off, to the whole, be that of $1 : n$. Make $BE : BF :: n : n-1$; and take BG a mean proportional between BE and BF . Bisect AD in O ; and with the centre O , and radius OG , describe a semicircle HGI ; $AH = ID$, will be the breadth of the part to be cut off.



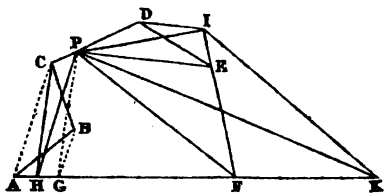
Make $BL = BI$, and draw HK , LK parallel to the sides of the parallelogram; then $AC : HL$ in the ratio compounded of the ratios of $AB : BH$ and $BD : BI$,

i. e. in the duplicate ratio of $BE : BG$, or the ratio of $BE : BF$, *i. e.* in the ratio of $n : n - 1$,

$\therefore$ the portion AKC is to AC in the ratio of $1 : n$.

(54.) *To describe a triangle equal to a given rectilinear figure, having its vertex in a given point in a side of the figure, and its base in the base (produced if necessary) of the figure.*

Let $ABCDEF$ be the given rectilinear figure, and P a given point in CD , which is to be the vertex of the triangle, the base being in AF . Join CA , and draw BG



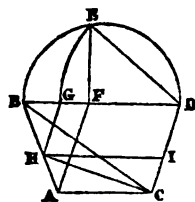
parallel to it; join CG , PG , PF , PE . Draw CH parallel to PG . Join PH . Draw DI parallel to PE , meeting FE produced in I . Join PI ; and draw IK parallel to PF , meeting AF in K . Join PK ; HPK will be equal to $ABCDEF$.

Since BG is parallel to CA , the triangles BAG , BCG are equal; the figure therefore is equal to $GCDEF$. And since GP is parallel to CH , the triangles GCP , GHP are equal. Again, since DI is parallel to PE , the triangles PIE , PDE are equal; $\therefore$ $PDEF$ is equal to the triangle PIF , *i. e.* to the triangle PKF , since IK is parallel to PF ; whence the whole figure

ABCDEF is equal to the triangles *PHG*, *PGF*, *PKF*, i. e. to the triangle *PHK*.

(55.) *On the base of a given triangle, to describe a quadrilateral figure equal to the triangle, and having two of its sides parallel, one of them being the base of the triangle; and one of its angles being an angle at the base, and the other equal to a given angle.*

Let *ABC* be the given triangle, *AC* its base. At the point *C* make the angle *ACD* equal to the given angle; and let *CD* meet *BD* drawn parallel to *AC*, in the point *D*. On *BD* describe a semicircle *BED*; draw *AF* parallel to *CD*, and *FE* perpendicular to *BD*; and with the centre *D*, and radius *DE*, describe the arc *EG*. Draw *GH* parallel to *AF*, and *HI* to *AC*; *AHIC* will be the figure required.



Join *HC*. Since *DG = DE*,

$$BD : DG :: DG : DF,$$

$$\therefore (\text{Eucl. v. 19.}) \quad BG : GF :: DG : DF :: HI : AC.$$

$$\text{Now } BG : GF :: BH : HA,$$

$$\therefore BH : HA :: HI : AC.$$

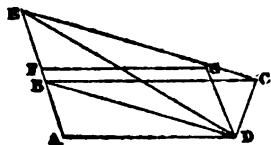
But the triangles *HCI*, *AHC* are in the proportion of *HI : AC*, and the triangles *BHC*, *AHC* in the proportion of *BH : HA*,

$$\therefore HCI : AHC :: BHC : AHC,$$

or *HCI = BHC*; $\therefore$ *ACH*, and *HCI* together are equal to *ACH*, and *BCH* together, or *AHIC = ABC*.

(56.) *A trapezium being given, two of whose sides are parallel; to describe on one of those sides another trapezium, having its opposite side also parallel to this, and one of the angles at the base the same as the former, and the other equal to a given angle.*

Let $ABCD$ be the given trapezium whose sides AD , BC are parallel. Join BD ; and draw CE parallel to it, meeting AB produced in E . Then the triangles BCD , BED are equal; and $\therefore$ the triangle AED is equal to $ABCD$. Hence (vi. 55.) a figure $ADGF$ may be described equal to ADE , and $\therefore$ to $ADCB$.



(57.) *If with any point in the circumference of a circle as centre, and distance from its centre as radius, a circular arc be described; and any two chords be drawn, one from the centre of the circular arc, and the other through the point where this cuts the arc, and parallel to the line joining the centres; the segments of each chord intercepted between the circumferences which are concave to each other, will be equal respectively to those of the other between the other circumferences.*

With any point C in the circumference of the circle ABC as centre, and radius CE equal to the distance from the centre E , let a circle DFE be described. Join CE , and draw any chord CFA ; and through F draw HFG parallel to CE ; then will $CF = FH$, and $GF = FA$.



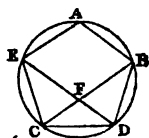
Produce CE to B , and join HE . And since HG is

parallel to BC , the angle FHE is equal to HEB . Also since the circles are equal, the arc HB is equal to the arc FE (ii. 1. Cor. 1.), $\therefore$ the angle HEB is equal to FCE , $\therefore FCE = FHE$, and HC is a parallelogram; whence $HF = EC = CF$. Also since the rectangle CF, FA is equal to the rectangle HF, FG , and $HF = CF$, $\therefore FA = FG$.

COR. Hence if any number of lines be drawn parallel to BE , and terminated by the two circumferences, each of them will be equal to BE .

(58.) *If two diagonals of an equilateral and equiangular pentagon be drawn to cut one another, the greater segments will be equal to the side of the pentagon; and the diagonals cut one another in extreme and mean ratio.*

Let $ABDCE$ be an equilateral and equiangular pentagon; draw the diagonals ED, BC cutting each other in F ; EF and FB will be each equal to a side of the pentagon; and ED, BC are cut in F , in extreme and mean ratio.



About the pentagon describe a circle. And since $AB = CE$, the arcs AB, CE are equal; $\therefore AE$ is parallel to BC . For the same reason, AB is parallel to EF ; $\therefore$ the figure $ABFE$ is a parallelogram; whence $AB = FE$, and $AE = FB$; but $AB = AE$, $\therefore EF = FB$, and each is equal to a side of the pentagon.

Also the angle $DCF = CDF = DEC$, $\therefore$ the triangles DCF, DEC are similar,

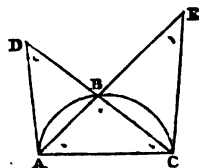
and $ED : CD :: CD : DF$,

or $ED : EF :: EF : FD$;

$\therefore ED$ is cut in extreme and mean ratio. The same may also be proved of BC .

(59.) *If the sides of a triangle inscribed in the segment of a circle be produced to meet lines drawn from the extremities of the base, forming with it angles equal to the angle in the segment; the rectangle contained by these lines will be equal to the square described on the base.*

Let the sides AB , CB of the triangle ABC , inscribed in the segment ABC , be produced to meet CE , AD , which make with AC , angles equal to the angle ABC in the segment; the rectangle AD , CE is equal to the square of AC .



Since the angle $ABC = DAC$, and the angle at C is common to the triangles ABC , ADC , the triangles are similar. In the same manner it may be shewn that ABC , AEC are similar; and $\therefore ADC$, AEC are also similar; whence

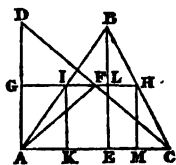
$$AD : AC :: AC : CE,$$

and the rectangle AD , CE is equal to the square of AC .

(60.) *If two triangles (one of them right angled) have the same base and altitude, and the hypotenuse intersect a line which is drawn bisecting the right angle; a*

line passing through this point of intersection parallel to the base, and terminated by the sides of the other triangle, shall be a side of the square inscribed within it.

Let $\triangle ADC$ be a right-angled triangle, and $\triangle ABC$ on the same base, have its altitude $BE = AD$; and let the hypotenuse DC , meet AF which bisects the angle DAC in F ; through which draw GH parallel to AC ; IH will be the side of a square inscribed in the triangle ABC .



From I draw IK perpendicular to AC ; then (Eucl. vi. 3.)

$$DA : AC :: DF : FC :: DG : (GA =) IK,$$

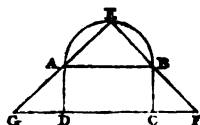
$$\text{also } AC : BE :: IH : (BL =) DG,$$

$$\therefore \text{ex æquo } DA : BE :: IH : IK;$$

But $DA = BE$, $\therefore IH = IK$; and if HM be drawn perpendicular to AC , $IKMH$ is a parallelogram, whose sides are equal; and the angles at K and M being right angles (Eucl. i. 46. Cor.) it is a square.

(61.) *If on the side of a rectangular parallelogram as a diameter, a semicircle be described, and from any point in the circumference lines be drawn through its extremities to meet the opposite side produced; the altitude of the parallelogram will be a mean proportional between the segments cut off.*

On AB , the side of the rectangular parallelogram $ABCD$, let a semicircle AEB be described; and from any point E , draw EA , EB , and produce



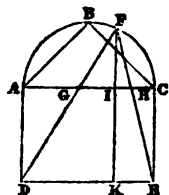
them to meet CD produced; AD will be a mean proportional between GD and CF .

Since DG is parallel to BA , the angle DGA is equal to BAE , and the angles at D and E are right angles, $\therefore$ the triangles BAE , DGA are equiangular. In the same manner it may be shewn that FCB is equiangular to BAE , and $\therefore$ to DGA ; whence

$$GD : DA :: (CB =) DA : CF.$$

(62.) *If on the diameter of a semicircle a rectangular parallelogram be described, whose altitude is equal to the chord of half the semicircle, and lines drawn from any point in the circumference to the extremities of the base intersect the diameter; the squares of the distances of each point of section from the farthest extremity of the diameter will be together equal to the square of the diameter.*

Let ABC be a semicircle, on the diameter of which describe the rectangular parallelogram AE , whose side AD is equal to AB a chord of half ABC ; and from any point F in the semicircle draw FD , FE cutting the diameter in G and H ; the squares of AH and CG are together equal to the square of AC .



Draw the perpendicular FK ; the triangles DGA , DFK being similar,

$$DA : AG :: FK : KD,$$

and ECH , FKE being similar,

$$(CE =) DA : CH :: FK : KE,$$

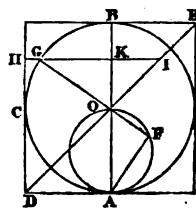
$$\begin{aligned} \therefore DA^2 : AG \times CH &:: FK^2 : (KE \times KD =) IF^2 \\ &:: DE^2 : GH^2. \end{aligned}$$

Now the square of DE is double of the square of DA ,
 $\therefore$ the square of GH is double of the rectangle AG, CH .
 But the square of AH is equal to the squares of AG ,
 GH and twice the rectangle AG, GH , *i. e.* to the square
 of AG and twice the rectangle AG, GC ; $\therefore$ the squares
 of AH and GC are together equal to the squares of AG ,
 GC , and twice the rectangle AG, GC , *i. e.* to the square
 of AC , (Eucl. ii. 4.).

COR. The square of the part of the diameter inter-
 cepted between the two lines drawn from the point in
 the semicircle is double of the rectangle contained by the
 two extreme segments.

(63.) *If on the radius drawn from the point of con-
 tact of a circle and its circumscribed square, another
 circle be described; and from any point in the outer cir-
 cumference a line be drawn through its centre to the inner
 circumference, and through the same point another line
 be drawn parallel to the common tangent to the circles,
 and terminated by the side of the square and its diag-
 onal; these two lines are equal.*

Let O be the centre of the circle,
 circumscribed by a square, whose di-
 agonal is DE . On AO describe a circle
 AOF ; and from any point F draw a
 line FOG ; and through G draw HI
 parallel to AD ; FG is equal to HI .

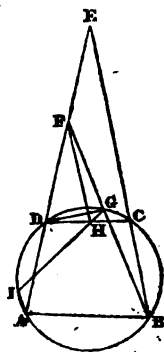


Join AF ; and let HI cut AB in K . Since IG is

parallel to AD , it is perpendicular to AB ; $\therefore$ the angle GKO is a right angle, and equal to AFO ; and the vertical angles at O are equal, and $GO = OA$; $\therefore$ the triangles GKO , OFA are equal, and $OF = OK$. But since $OB = BE$, $\therefore$ (Eucl. vi. 2.) $OK = KI$; and $\therefore$ $OF = KI$, and $OG = KH$; $\therefore$ $FG = IH$.

(64.) *If two sides of a trapezium inscribed in a circle be produced, and from the same point in one side produced a line be drawn parallel to the other, intersecting the adjacent side of the trapezium, and a second line to the extremity of that other intersecting the circumference; the line joining the two points of intersection, will pass through the same point.*

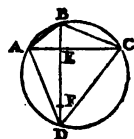
Let the two sides AD , BC , of the trapezium $ABCD$ inscribed in the circle ABC , be produced, and let them meet in E ; and from any point in AD produced, draw FH parallel to BE , meeting the side DC in H ; and join FB , meeting the circumference in G ; the line joining G , H will always pass through the same point.



Let GH produced meet the circle in I . Join AI , DG . The angle $GDH = GBC$ in the same segment, and $\therefore$ is equal to the alternate angle GFH ; whence a circle may be described through the points G , H , D , F ; and $\therefore$ the angle $DGH = DFH = DEB$. But the angle DEB being always the same, DGI , and $\therefore$ DAI , and also the arc DI will be invariable, and D being a fixed point, I must be also; i. e. GH will always pass through I .

(65.) *If the diagonals of a quadrilateral figure inscribed in a circle cut each other at right angles, the rectangles contained by the opposite sides are together double of the quadrilateral figure.*

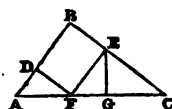
Let $ABCD$ be a quadrilateral figure inscribed in a circle, whose diagonals AC , BD cut each other at right angles in E ; the rectangles contained by AB , CD , and AD , BC are together double of the figure.



For (Eucl. vi. *D.*) the rectangles contained by AB , CD , and AD , BC , are together equal to the rectangle AC , BD , i. e. to the rectangles contained by AC , BE and AC , ED . But the rectangle contained by AC , BE is double of the triangle ABC , and the rectangle contained by AC , ED , is double of ADC ; hence the rectangles contained by AB , CD and AD , BC are together double of $ABCD$.

(66.) *If a rectangular parallelogram be inscribed in a right-angled triangle, and they have the right angle common; the rectangle contained by the segments of the hypotenuse is equal to the sum of the rectangles contained by the segments of the sides about the right angle.*

Let ABC be a right-angled triangle, in which the rectangular parallelogram $DBEF$ is inscribed, having one of its angles at B ; the rectangle AF , FC is equal to the rectangles AD , DB and BE , EC together.



Draw EG perpendicular to FC . The triangles ADF , EFG being similar,

$$AD : AF :: FG : (EF =) BD,$$

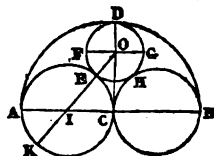
$\therefore$ the rectangle AD, DB is equal to the rectangle AF, FG . In the same manner,

$$AF : (FD=) EB :: EC : CG,$$

$\therefore$ the rectangle BE, EC is equal to the rectangle AF, GC , whence the rectangles AD, DB and BE, EC are together equal to the rectangles AF, FG and AF, GC , i. e. to the rectangle AF, FC (Eucl. ii. 1.).

(67.) *If on the diameter of a semicircle, two equal circles be described, and in the curvilinear space included by the three circumferences a circle be inscribed; its diameter will be to that of the equal circles in the proportion of two to three.*

On AB the diameter of the semicircle ADB let two equal circles ACE, BCH be described; and in the curvilinear space let the circle DEG be inscribed; its diameter $FG : AC ::$



$2 : 3$.

Let O and I be the centres of the circles. Join OI , which will pass through the point of contact E ; and produce it to K . From C draw CD perpendicular to AB , which will pass through O . Then the rectangle KO, OE is equal to the square of OC ;

$$\text{and } OE : OC :: OC : OK,$$

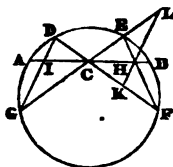
$$\therefore OE : OC :: OE + OC : OC + OK :: CD : KE + [CD :: 1 : 2;$$

$$\text{and } OE : CD :: 1 : 3,$$

$$\therefore FG : (CD=) AC :: 2 : 3.$$

(68.) *If through the middle point of any chord of a circle two chords be drawn; the lines joining their extremities will intersect the first chord at equal distances from the middle point.*

Let ABD be a chord of the circle ABD , bisected in C ; and let DCF , ECG be any chords drawn through C . Join DG , EF cutting AB in I and H ; then will $CI = CH$.



Through H draw KHL parallel to DG , meeting DF in K , and GE produced in L . Because LH is parallel to GI , the angle $HLE = CGI = HFK$, and the vertical angles at H are equal, $\therefore$ the triangles LEH , HKF are equiangular,

$$\therefore LH : HE :: HF : HK,$$

and the rectangle LH, HK is equal to the rectangle HE, HF , i. e. to the rectangle AH, HB or the difference of the squares of AC and CH . The triangles CID , CHK may in like manner be proved to be equiangular, as also the triangles CHL , CIG ; hence

$$KH : HC :: DI : IC,$$

$$\text{and } LH : HC :: GI : IC,$$

$$\therefore KH \times LH : HC^2 :: DI \times IG : IC^2.$$

But $KH \times LH = AC^2 - HC^2$, and $DI \times IG = AC^2 - IC^2$,

$$\therefore AC^2 - HC^2 : HC^2 :: AC^2 - IC^2 : IC^2$$

$$\text{comp. } AC^2 : HC^2 :: AC^2 : IC^2,$$

$$\therefore HC^2 = IC^2, \text{ and } HC = IC.$$

(69.) *The longest side of a trapezium being given, and made the diameter of the circumscribed circle; also*

the distance between its extremity and the intersection of the opposite side, produced to meet it; and the angle formed by the intersection of the diagonals; to construct the trapezium.

With a diameter equal to the given longest side, describe a circle, and from O its centre draw the radii

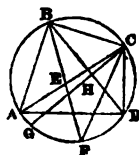


OC, OD making with each other an angle equal to twice the complement of the given angle formed by the intersection of the diagonals. Join CD , and produce it; and with the centre O , and radius equal to the radius of the circle together with the given intercepted distance, describe a circle cutting it in E . Join EO , and produce it to B ; AE is equal to the given intercepted distance. Join BC, AD ; $ABCD$ is the trapezium required.

Join AC, BD . Then the angle ACB in a semi-circle being a right angle, FBC is the complement of CFB ; but FBC is half of DOC , and $\therefore BFC$ is equal to the given angle to be made by the diagonals.

(70.) *The diagonals of a quadrilateral figure inscribed in a circle are to one another as the sums of the rectangles of the sides which meet their extremities.*

Let $ABCD$ be a quadrilateral figure inscribed in a circle; join AC, BD ; AC is to BD , as the rectangles AB, AD and CB, CD together, to the rectangles AB, BC and AD, DC together.



Make the angle ABF equal to the angle DBC ; to each of which add the angle FBD , $\therefore$ the angle ABD is

equal to EBC ; and ADB is equal to ECB , being in the same segment; $\therefore$ the triangles ABD , EBC are equiangular, and

$$AB : BD :: BE : BC,$$

$\therefore$ the rectangle BD , BE is equal to the rectangle AB , BC . Join FC ; and since the angle $ABD = FBC$, $\therefore AD = FC$. Also since the angle EFC is equal to BDC in the same segment, and ECF equal to ABF , i. e. to DBC ; $\therefore$ the triangles ECF , BDC are equiangular,

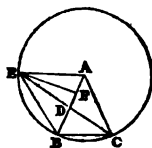
$$\text{and } BD : DC :: CF : FE,$$

$\therefore$ the rectangle FE , BD is equal to the rectangle CF , CD , i. e. to the rectangle AD , DC ; whence the rectangles BE , BD and FE , BD or the rectangle BF , BD is equal to the rectangles AB , BC and AD , DC together. In the same manner if the angle BCG be taken equal to the angle ACD , it may be shewn that the rectangle CG , CA is equal to the rectangles AB , AD and CB , CD together. And since the angle $BCG = ACD$, the arc BG is equal to AD , i. e. to FC ; to each of these add GF ; $\therefore$ the arc BAF is equal to GFC , and consequently the line $BF = GC$; $\therefore$ the rectangle AC , CG is equal to the rectangle AC , BF . And $AC : BD$ as the rectangle AC , BF to the rectangle BD , BF , i. e. as the rectangles AB , AD and CB , CD together, to the rectangles AB , BC and AD , DC together.

(71.) *The square described on the side of an equilateral and equiangular pentagon inscribed in a circle, is equal to the sum of the squares of the sides of a regular hexagon and decagon inscribed in the same circle.*

Let ABC be an isosceles triangle having each of the

angles at the base double of the angle at A . With the centre A , and radius AB , describe a circle BCE . Draw CE bisecting the angle ACB . Join EB , EA ; and draw EF perpendicular to AB . Then the angle EAB is double of ECB , and therefore is equal to CDB (Eucl. iv. 10.), and consequently is equal to the vertically opposite angle ADE ; whence $AE = ED$. Hence EB , ED and BC are equal to the sides of a regular pentagon, hexagon and decagon, respectively inscribed in the circle; and the squares of BC and DE are together equal to the square of BE .

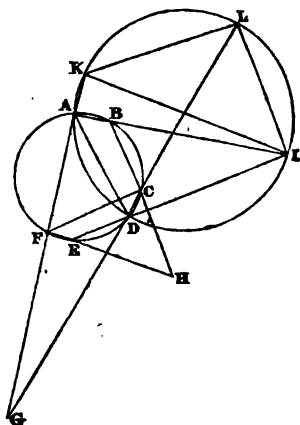


For the angles at F being right angles, the squares of AF , FE are equal to the square of AE , i. e. to the square of ED or to the squares of EF , FD ; whence AF is equal to FD . And since AD is bisected in F , and produced to D , the rectangle AB , BD together with the square of DF is equal to the square of BF ; $\therefore$ the rectangle AB , BD together with the squares of DF and FE , is equal to the squares of BF and FE ; or the rectangle AB , BD together with the square of DE is equal to the square of BE . But (Eucl. iv. 10.) the rectangle AB , BD is equal to the square of AD , i. e. to the square of BC ; $\therefore$ the squares of BC , DE are together equal to the square of BE .

(72.) *If the opposite sides of an irregular hexagon inscribed in a circle be produced till they meet; the three points of intersection will be in the same straight line.*

Let $ABCDEF$ be the hexagon inscribed in the circle; and let its opposite sides meet in G , H , I . Join

two opposite angles as A, D ; and about ADI describe a circle meeting GA, GD produced if necessary, in K, L . Join FC, KI, LI, LK . Then because $AKID$ is a quadrilateral figure inscribed in a circle, the angle AKI is equal to ADE : and for the same reason ADE is equal to GFE , $\therefore$ the angle AKI is equal to GFE , and KI is parallel to FH . In the same manner it may be shewn that the angle AIL is equal to ADL and consequently to CBI , and LI parallel to HB . Again the angle KLD is equal to DAF , i. e. to FCD , and KL is parallel to FC .



Hence $GF : FC :: GK : KL$,

and $FC : FH :: KL : KI$,

$\therefore GF : FH :: GK : KI$;

whence G, H, I will be in a straight line.

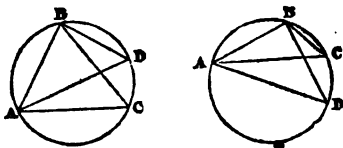
SECT. VII.

(1.) *THE vertical angle of any oblique-angled triangle inscribed in a circle, is greater or less than a right angle, by the angle contained by the base and the diameter drawn from the extremity of the base.*

Let ABC be a triangle inscribed in a circle. From A draw the diameter AD ; join BD ; the angle ABC

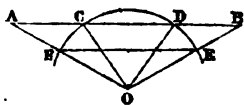
is greater or less than a right angle, by the angle CAD .

For the angle ABD in a semicircle is a right angle; and ABC is equal to the sum of ABD and DBC in one case; and is equal to their difference in the other; and in each case $DBC = DAC$ in the same segment.



(2.) *If from the vertex of an isosceles triangle a circle be described with a radius less than one of the equal sides, but greater than the perpendicular; the parts of the base cut off by it, will be equal.*

From the vertex O of the isosceles triangle AOB , with a radius less than AO , but greater than the perpendicular from O on AB , let a circle be described, cutting AB in C and D ; $AC = BD$.

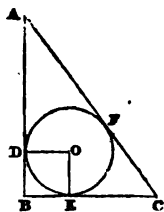


Join EF , OC , OD .

Then $OE : OB :: OF : OA$,
and $\therefore EF$ is parallel to AB ; and (ii. 1.) the arc FC equal to the arc DE , or the angle $FOC = DOE$; but AO , OC are equal to BO , OD each to each; $\therefore AC = DB$.

(3.) *If a circle be inscribed in a right-angled triangle; the difference between the two sides containing the right angle and the hypotenuse, is equal to the diameter of the circle.*

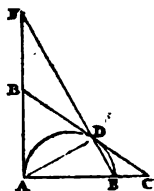
Let DEF be a circle, inscribed in the right-angled triangle ABC . The difference between AC , and AB , BC , is equal to the diameter of the circle.



Find O the centre, and join OD , OE . Then the angles at D , B , and E being right angles, and $OD = OE$, OB is a square; and DB , BE are equal to OD , OE , i. e. are together equal to the diameter of the circle. Now (Eucl. iii. 36. Cor.) $CE = CF$, and $AD = AF$; i. e. AC is equal to AD and CE ; whence it is less than the sides containing the right angle, by DB and BE , or by the diameter of the circle.

(A.) If a semicircle be inscribed in a right-angled triangle, so as to touch the hypotenuse and perpendicular, and from the extremity of its diameter a line be drawn through the point of contact to meet the perpendicular produced; the part produced will be equal to the perpendicular.

Let the semicircle ADE touch the hypotenuse BC of the right-angled triangle ABC in D , and the perpendicular in A ; and from E let ED be drawn to meet AB produced in F ; $AB = BF$.



Join AD . Since ADE is a right angle, ADF is also a right angle, and $\therefore$ equal to DAF , DFA together. But DAF is equal to BDA , since $BD = BA$, being tangents from the same point B without the circle; and $\therefore$ the angle $BFD = BDF$, and $BF = BD = BA$.

(5.) *If the base of any triangle be bisected by the diameter of its circumscribing circle, and from the extremity of that diameter a perpendicular be let fall upon the longer side; it will divide that side into segments, one of which will be equal to half the sum, and the other to half the difference of the sides.*

Let the base BC of the triangle ABC be bisected in E , by the diameter of the circumscribing circle ACD ; and from D draw DF perpendicular to AB the longer side; BF will be equal to half the sum, and AF to half the difference of AB , BC .



Join DA , DB , DC ; and make $BG = BC$; join DG . Since $BG = BC$, and BD is common, and the angle $GBD = CBD$, since the arc $AD = DC$; $\therefore DG = DC = DA$, and DF is at right angles to AG , $\therefore AF = FG$. Whence the sum of AB and BC is equal to AG and $2BG$, i. e. to $2BF$; and the difference of AB and BC is equal to the difference of AB and BG , i. e. to $2AF$.

(6.) *The same supposition being made, as in the last proposition; if from the point, where the perpendicular meets the longer side, another perpendicular be let fall on the line bisecting the vertical angle; it will pass through the middle of the base.*

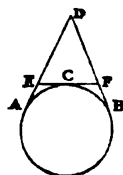
The same construction being made as before; (see last Fig.) let FH be drawn perpendicular to BD , which bisects the vertical angle; FH will pass through E .

Because $CB = BG$, and the angle $CBD = GBD$,

$\therefore BD$ is perpendicular to CG ; and $\therefore FH$ is parallel to CG . But since $AF = FG$, $\therefore$ (Eucl. vi. 2.) AC is bisected by FH ; which $\therefore$ passes through E .

(7.) *If a point be taken without a circle, and from it tangents be drawn to the circle, and another point be taken in the circumference between the two tangents, and a tangent be drawn to it; the sum of the sides of the triangle thus formed is equal to the sum of the two tangents.*

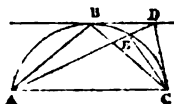
From a given point D let two tangents DA, DB be drawn; and to C any point in the circumference between them, let a tangent ECF be drawn. The sum of the sides of the triangle is equal to the two tangents DA and DB .



Since $AE = EC$, and $FC = FB$, $\therefore DE, EF, FD$ together are equal to AD and DB together. In the same manner, if through any other point in the arc ACB a tangent be drawn, it will be equal to the two segments of DA, DB intercepted between it, and the points of contact A and B ; and the three sides of the triangle so formed will be equal to DA , and DB together.

(8.) *Of all triangles on the same base and between the same parallels, the isosceles has the greatest vertical angle.*

Let ABC be an isosceles triangle on the base AC , and between the parallels AC, BD . It has a greater vertical angle



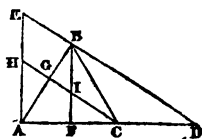
than any other triangle ADC on the same base, and between the same parallels.

About ABC describe a segment of a circle ABC ; and since B is the middle point of the arc, and BD is parallel to AC , BD is a tangent at B . Let the arc cut AD in E ; join EC . Then the angle $ABC = AEC$, and $\therefore$ is greater than ADC .

COR. Of all triangles on the same base and having the same vertical angle, the isosceles is the greatest. For the triangle AEC has the same vertical angle with ABC , and $ABC = ADC$ on the same base and between the same parallels; but ADC is greater than AEC , $\therefore ABC$ is greater than AEC .

(9.) *If through the vertex of an equilateral triangle a perpendicular be drawn to the side, meeting a perpendicular to the base drawn from its extremity; the line intercepted between the vertex and the latter perpendicular is equal to the radius of the circumscribing circle.*

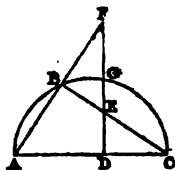
Let BE perpendicular to AB meet AE , which is perpendicular to the base AC , in E ; BE is equal to the radius of the circle described about ABC .



Draw BF , CG perpendicular to the sides; and produce CG to H . Then CI is equal to the radius of the circle described about ABC ; and $EBIH$ is a parallelogram. And since CF is equal to FA , (Eucl. vi. 2.) CI is equal to IH , i. e. to the opposite side BE ; and $\therefore BE$ is equal to the radius of the circumscribing circle.

(10.) *If a triangle be inscribed in a semicircle, and a perpendicular drawn from any point in the diameter, meeting one side, the circumference, and the other side produced; the segments cut off will be in continued proportion.*

Let ABC be a triangle in the semicircle ABC ; and from any point D in the diameter, let DF be drawn perpendicular to AD , meeting BC , the circumference, and AB produced, in E, G, F ;



$$DE : DG :: DG : DF.$$

For the angles at E being equal, and the angles at B right angles, $\therefore$ the angle ECD is equal to BFD ; and the angles at D are right angles; $\therefore$ the triangles EDC, ADF are similar, and therefore

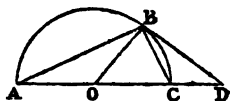
$$DF : DA :: DC : DE,$$

$$\text{but } DA : DG :: DG : DC;$$

$$\therefore \text{ex æquo } DF : DG :: DG : DE.$$

(11.) *If a triangle be inscribed in a semicircle, and one side be equal to the semi-diameter; the other side will be a mean proportional between that side and a line equal to that side and the diameter together.*

Let ABC be a triangle inscribed in the semicircle, and let BC be equal to the semi-diameter; then will



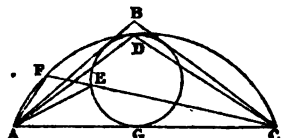
$$BC : BA :: BA : BC + CA.$$

Produce AC to D , making CD equal to the semi-diameter. Take O the centre. Join BD, BO . Since

$BO = BC$, the angle BCO is equal to BOC , i. e. to OAB and OBA together, or to $2BAC$. But BCA is equal to CBD and CDB together, i. e. to $2CDB$, since $CB = CD$; hence the angle $BAC = BDC$, and $BA = BD$; also the triangles BAD , BCD are similar;
 $\therefore BC : (BD =) BA :: BA : AD$, which is equal to BC and CA together.

(12.) *If a circle be inscribed in a right-angled triangle; to determine the least angle that can be formed by two lines drawn from the extremity of the hypotenuse to the circumference of the circle.*

Let ABC be a right-angled triangle, in which a circle DEG is inscribed. On AC describe a segment of a circle ADC , which may touch the inscribed circle in some point, as D . The lines AD , DC , drawn to this point from A and C , contain an angle less than the lines drawn to any other point in the circumference of the circle DEG .

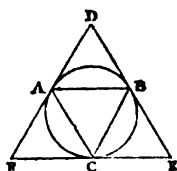


For take any other point E , and join AE , EC ; produce CE to F , and join AF . The exterior angle AEC is greater than AFC , i. e. than ADC , which is in the same segment. And the same may be proved of lines drawn to every other point in the circumference of the circle DEG .

(13.) *If an equilateral triangle be inscribed in a*

circle, and through the angular points another be circumscribed; to determine the ratio which they bear to each other.

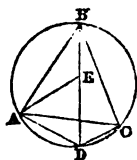
Let ABC be an equilateral triangle inscribed in the circle, about which another DEF is circumscribed, touching the circle in the points A, B, C .



Since DA touches the circle, the angle $DAB = ACB$ (Eucl. iii. 32.); but $ACB = ABC$; $\therefore DAB = ABC$, and they are alternate angles, $\therefore DF$ is parallel to BC . In the same manner it may be shewn that AB is parallel to FE , $\therefore ABCF$ is a parallelogram; and the triangle ABC is equal to AFC . In the same manner ABC may be shewn to be equal to each of the triangles ABD, BCE ; and $\therefore$ it is one fourth of the circumscribing triangle.

(14.) *A straight line drawn from the vertex of an equilateral triangle inscribed in a circle to any point in the opposite circumference, is equal to the two lines together, which are drawn from the extremities of the base to the same point.*

Let ABC be an equilateral triangle inscribed in a circle; from B draw BD to any point D in the circumference. Join AD, CD . BD is equal to AD and CD together.

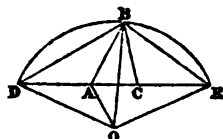


Make $DE = DA$, and join AE . The angle DAE is equal to the angle DEA ; but $ADE = ACB$ in the same segment, $\therefore DAE$ and DEA together are equal to CBA and CAB together; whence DAE

$= CAB$; and taking away the common angle CAE , $DAC = EAB$; but $DCA = EBA$, and $AC = AB$, $\therefore BE = DC$; and BD is equal to AD and CD together.

(15.) *If the base of a triangle be produced both ways so that each part produced may be equal to the adjacent side, and through the extremities of the parts produced and the vertex a circle be described; the line joining its centre and the vertex of the triangle will bisect the angle at the vertex.*

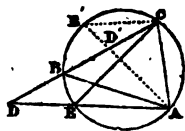
Let AC a side of the triangle ABC be produced both ways till $AD = AB$, and $CE = CB$; and through D, B, E let a circle be described, whose centre is O . If OB be joined, it will bisect the angle ABC .



Join BD, BE, OA, OD, OE . Since $DA = AB$, the angle ABD is equal to ADB ; but the angle OBD is equal to ODB , and $\therefore$ the angle OBA is equal to ODA . In the same manner it may be shewn that the angle OBC is equal to OEC ; and since ODA is equal to OEC , OBA is equal to OBC ; or ABC is bisected by OB .

(16.) *If an isosceles triangle be inscribed in a circle, and from the vertical angle a line be drawn meeting the circumference and the base; either equal side is a mean proportional between the segments of the line thus drawn.*

Let ABC be an isosceles triangle inscribed in the circle AEC , the side AB being equal to AC ; and from A draw any line AED , meeting the circumference in E , and CB produced in D ; AB is a mean proportional between DA and AE .

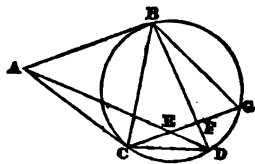


Join EC . Since $AB = AC$, the angle $ACB = ABC = AEC$ in the same segment; and the angle at A is common to the triangles AEC , ACD , $\therefore$ the triangles are equiangular and similar;

$$\therefore AD : AC :: AC : AE.$$

(17.) *If from the extremities of one of the equal sides of an isosceles triangle inscribed in a circle, tangents be drawn to the circle, and produced to meet; two lines drawn to any point in the circumference from the point of concurrence and one point of contact will divide the base (produced if necessary) in geometrical proportion.*

Let CBG be an isosceles triangle inscribed in a circle, the side CB being equal to BG ; and at B and C let tangents BA , CA be drawn, meeting in A . From A and B draw AD , BD to any point D in the circumference, cutting the base CG in E and F ;



$$CE : CF :: CF : CG.$$

Join CD . The angle ABC being equal to BGC , is equal to BCG , and $\therefore CG$ is parallel to AB ; and the triangles ABC , CBG are equiangular;

$$\therefore BC : CG :: AB : BC;$$

$$\text{but (vii. 16.) } BF : BC :: BC : BD.$$

$\therefore$ *ex æquo* $BF : CG :: AB : BD :: EF : FD$,
since CG is parallel to AB ;

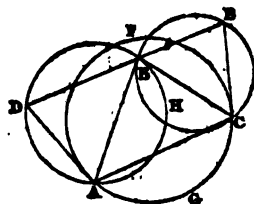
but $CF : BF :: FD : FG$,

$\therefore CF : CG :: EF : FG$,

whence (Eucl. v. 19.) $CE : CF :: CF : CG$.

(18.) *If on the sides of a triangle, segments of circles be described similar to a segment on the base, and from the extremities of the base tangents be drawn intersecting their circumferences; the points of intersection and the vertex of the triangle will be in the same straight line.*

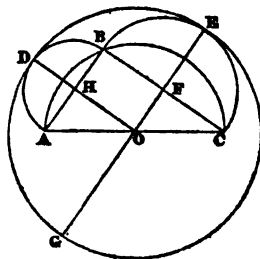
On AB , BC , the sides of the triangle ABC , let the segments ADB , BEC be described, similar to AFC the segment on AC . At A and C let tangents AD , CE be drawn. Join DB , BE ; they are in the same straight line.



Since DA touches the circle AFG , the angle DAC is equal to the angle in the alternate segment AGC , i. e. to the angle in the segment AHB . But the angle ADB , together with the angle in the segment AHB , will be equal to two right angles; $\therefore$ the angles CAD , ADB are equal to two right angles; $\therefore AC$, DB are parallel. In the same manner AC , BE may be shewn to be parallel; $\therefore BD$ and BE being drawn from the same point, parallel to the same lines will also be in the same straight line.

(19.) *The centre of the circle, which touches the semicircles described on the two sides of a right-angled triangle, is in the middle point of the hypotenuse.*

On the sides AB , BC of the right-angled triangle ABC , let semicircles ADB , BEC be described. Bisect AC in O ; O is the centre of a circle which will touch both the semicircles.



From O draw OFE , OHD perpendicular to the sides. Then OH being parallel to BC ,

(Eucl. vi. 2.) $AO : OC :: AH : HB$,

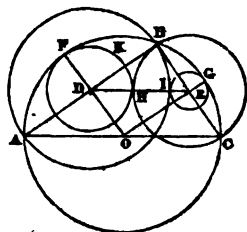
$\therefore AH = HB$, and H is the centre of the semicircle ADB . Hence the centre of a circle touching ADB in D is in the line DHO . For the same reason, the centre of a circle touching BEC in E is in the line EFO . Also since OD is equal to OH and HD together, i. e. to BF and HB , or EF and FO together, i. e. to EO , O is the centre of the circle, which will touch both.

COR. The diameter of this circle will be equal to the sides of the triangle together.

(20.) *If on the three sides of a right-angled triangle semicircles be described, and with the centres of those described on the sides, circles be described touching that described on the base; they will also touch the other semicircles.*

On the sides AB , BC of the right-angled triangle ABC let semicircles be described; and with the centres

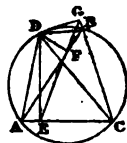
D and *E*, let circles be described touching that described on the base in *F* and *G*; each of these circles will touch the semicircle described on the other side.



Join *ODF*, *OEG*, *DE*. Since *AB*, *BC* are bisected in *D* and *E*, *DE* is parallel to *AC*, and equal to half *AC*, i.e. to *AO* or *OC*. In the same manner *OD* is parallel to *BC*, and *OE* to *AB*; $\therefore$ *ODBE* is a parallelogram, and *EB*, i.e. *EH* = *OD*; but *OF* = *DE*, $\therefore$ *DH* = *DF*, and *H* is a point in the circumference of the circle *FHK*; and being in the circumference of *BHC*, it will be the point of contact, since *DE* joins the centres. In the same manner it may be shewn that the circle *GI* touches the circle *ABI* in *I*.

(21.) *If from any point in the circumference of a circle perpendiculars be drawn to the sides of the inscribed triangle; the three points of intersection will be in the same straight line.*

From *D* any point in the circumference of the circle *ABC*, let *DE*, *DF*, *DG* be drawn perpendicular to the sides of the inscribed triangle *ACB*; join *EF*, *FG*; *EFG* is a straight line.

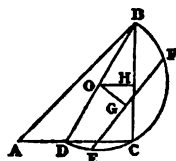


Join *AD*, *BD*, *CD*. Since the angles *DFB*, *DGB* are right angles, a circle may be described about the quadrilateral figure *DGBF* (vi. 13.); and the angle *DFG* is equal to *DBG*. Also since the angles *DFA*, *DEA* are right angles, a circle may be described

about the quadrilateral figure $DFEA$; whence the angles DFE , DAE are together equal to two right angles. But $ADBC$ being a quadrilateral figure inscribed in a circle, the angle DAC is equal to DBG , i.e. to DFG ; $\therefore DFE$, DFG are equal to two right angles; and EFG is a straight line.

(22.) *The base of a right-angled triangle not being greater than the perpendicular; if on any line drawn from the vertex to the base a semicircle be described, and a chord equal to the perpendicular placed in it, and bisected; the point of bisection will always fall within the triangle.*

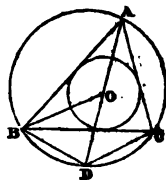
Let ABC be a right-angled triangle, of which the side AC is not greater than BC . From B let any line BD be drawn to the base; on which describe a semicircle BCD , and in it place $EF = BC$, which bisect in G ; the point G is within the triangle ABC .



Take O the centre of the semicircle; draw OH perpendicular to BC ; join OG . Since BC is equal to EF , OH is equal to OG ; and the angles at G and H being right angles, a circle described with the centre O , and radius OG , will touch BC in H , and $\therefore G$ is within the angle DBC . Also since AC is not greater than BC , DC is less than BC or EF , $\therefore EF$ is nearer to the centre O , than DC is; or G falls above DC and within the angle DCB .

(23.) *The straight line bisecting any angle of a triangle inscribed in a given circle, cuts the circumference in a point, which is equidistant from the extremities of the side opposite to the bisected angle, and from the centre of a circle inscribed in the triangle.*

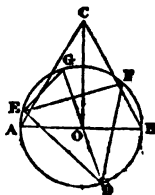
Let ABC be a triangle inscribed in the circle ACD . Bisect the angles BAC , ABC by the lines AD , BO , which meet in O ; O is the centre of the circle inscribed in the triangle. Join BD , DC . The lines DB , DC , DO are equal to each other.



Because the angles DAB , DAC are equal, $BD = DC$; and because the angle $CBD = CAD = DAB$, to each of these add the angle CBO or its equal ABO ; and the whole angle OBD is equal to the two ABO , OAB , i. e. to BOD (Eucl. i. 32.); and $\therefore OD = DB$.

(24.) *The perpendicular from the vertex on the base of an equilateral triangle is equal to the side of an equilateral triangle inscribed in a circle, whose diameter is the base.*

From C the vertex, let CO be drawn perpendicular to AB , the base of the equilateral triangle ABC ; upon AB describe a circle ADB , and let DEF be an equilateral triangle inscribed in it; CO will be equal to a side of this triangle.

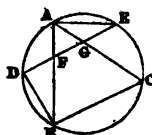


Draw DG bisecting the angle at D , and $\therefore$ bisecting EF at right angles, consequently passing through the centre. Join EG . The angles ACO , ADG being

each equal to half the angle of an equilateral triangle, are equal to each other, and $\angle AOC = \angle DEG$, each being a right angle, and $AC = AB = DG$, $\therefore CO = DE$.

(25.) *If an equilateral triangle be inscribed in a circle, and the adjacent arcs cut off by two of its sides be bisected; the line joining the points of bisection will be trisected by the sides.*

Let ABC be an equilateral triangle inscribed in a circle; bisect the arcs AB , AC in D and E ; join DE ; it is divided into three equal parts in the points F and G .



Since DE and BC cut off equal arcs BD , CE , they are parallel, and $\therefore$ (Eucl. vi. 2.) $AF = AG$. Join BD , AE . The angle $BFD = AFE$, and $DBF = AEF$ in the same segment, and $BD = AE$, since they subtend equal arcs; $\therefore DF = FA$. In the same manner it may be shewn that $AG = GE$. Now the triangle AFG being similar to ABC is equilateral, $\therefore DF$, FG , GE are all equal, and DE is trisected.

(26.) *If any triangle be inscribed in a circle, and from the vertex a line be drawn parallel to a tangent at either extremity of the base; this line will be a fourth proportional to the base and two sides.*

Let ABC be a triangle inscribed in the circle ABC ; and from B let BD be drawn parallel to AE a tan-

gent at A ; then will $AC : AB :: BC : BD$.

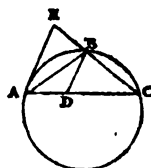
Produce CB to meet the tangent in E . Since the angle EAB is equal to the angle in the alternate segment ACB , and the angle AEB is equal to CBD , $\therefore$ the triangle ABE is similar to CBD ,

and $AE : AB :: CB : CD$;

but from similar triangles BDC , EAC ,

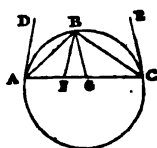
$AC : AE :: DC : DB$,

$\therefore$ ex æquo $AC : AB :: CB : DB$.



(27.) *If a triangle be inscribed in a circle, and from its vertex lines be drawn parallel to tangents at the extremities of its base; they will cut off similar triangles.*

Let ABC be a triangle inscribed in a circle, and AD , CE tangents at the points A and C . From B draw BF , BG respectively parallel to them; these lines will cut off the triangles ABF , CBG , which are similar.



For (Eucl. iii. 32.) the angle ACB is equal to DAB , i. e. to the alternate angle ABF ; and the angle BAC is equal to BCE , i. e. to CBG ; whence the triangles ABF , CBG having two angles in each equal, will be equiangular and similar.

COR. 1. The rectangle contained by the segments of the base adjacent to the angles will be equal to the square of either line drawn from the vertex.

For if AD and CE be produced, they will meet and

form with AC an isosceles triangle, to which BFG is similar, $\therefore BF = BG$.

Now $AF : BF :: BG : GC$

$\therefore$ the rectangle AF, GC is equal to the rectangle BF, BG , i. e. to the square of BF .

COR. 2. Those segments are also in the duplicate ratio of the adjacent sides.

For the triangles ABF and CBG are each of them similar to ABC , whence $AC : AB :: AB : AF$,

$\therefore AC : AF$ in the duplicate ratio of $AC : AB$;
for the same reason,

$AC : CG$ in the duplicate ratio of $AC : CB$,

$\therefore AF : CG$ in the duplicate ratio of $AB : CB$.

(28.) *If one circle be circumscribed and another inscribed in a given triangle, and a line be drawn from the vertical angle to the centre of the inner, and produced to the circumference of the outer circle; the whole line thus produced has to the part produced the same ratio that the sum of the sides of the triangle has to the base.*

Let ABD be a circle circumscribed about the triangle ABC ; O the centre of the inscribed circle. Join AO , and produce it to D ; then AOD bisects the angle BAC . Join BD, DC ; and draw BO, CO to the centre of the inscribed circle; then

$AD : DO :: AB + AC : CB$.

Draw OF, OG parallel to AB, AC , meeting BD, CD in F and G . The angle $DBC = DAC = DAB = DOF$, and the angle at D is common to the triangles

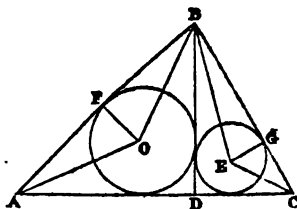


BED, *OFD*, and (vii. 20.) $BD = DO$, $\therefore OF = BE$.
In the same manner it may be shewn that $OG = EC$.
Now the trapeziums *BACD*, *FOGD* being similar, and similarly situated,

$$\begin{aligned} AD : OD &:: AB + AC : FO + OG \\ &:: AB + AC : BC. \end{aligned}$$

(29.) *If in a right-angled triangle, a perpendicular be drawn from the right angle to the hypotenuse, and circles inscribed within the triangles on each side of it; their diameters will be to each other as the subtending sides of the right-angled triangle.*

Let *ABC* be a right-angled triangle; from the right angle *B* let fall the perpendicular *BD*; and in the triangles *ABD*, *BDC* let circles be inscribed; their diameters are to one another as *AB* to *BC*.



Bisect the angles *BAD*, *ABD* by the lines *AO*, *BO*, they will meet in the centre *O*; in the same manner lines bisecting *DBC*, *DCB* meet in the centre *E*; draw *OF*, *EG* to the points of contact. Now the triangles *ABD*, *BDC* being similar (Eucl. vi. 8.), $\therefore$ the triangles *ABO*, *BCE* are similar; whence

$$AB : BC :: BO : CE;$$

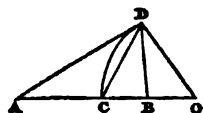
but the triangles *OBF*, *EGC* are similar,

$$\therefore BO : CE :: OF : EG :: 2OF : 2EG,$$

$$\therefore AB : BC :: 2OF : 2EG.$$

(30.) *To find the locus of the vertex of a triangle, whose base and ratio of the other two sides are given.*

Let AB be the given base; divide it in C so that $AC : CB$ may be in the given ratio of the sides. Produce AB to O ; and take CO a mean proportional between AO and BO . With the centre O , and radius CO , describe a circle; it will be the locus required.



In the arc CD take any point D ; join DA , DB , DC , DO . Since $OD = OC$,

$$AO : OD :: OD : OB,$$

$\therefore$ the sides about the common angle O are proportional, and the triangles ADO , BDO are equiangular;

$$\therefore AD : DB :: DO : OB :: CO : OB :: AO : CO \\ :: AO - CO : CO - OB :: AC : CB,$$

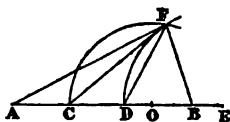
i.e. in the given ratio. In the same manner, if any other point be taken in the circumference of the circle, and lines drawn to it, they will be in the same given ratio, and $\therefore$ the circumference is the locus required.

COR. Since in any triangle, if from the vertex a line be drawn cutting the base in the ratio of the sides, it will bisect the angle, $\therefore$ the angle $ADC = BDC$.

(31.) *A given straight line being divided into any three parts; to determine a point such, that lines drawn to the points of section and to the extremities of the line shall contain three equal angles.*

Let AB be the given line, and AC , CD , DB the

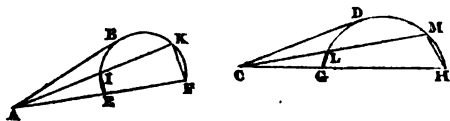
given parts. Take CO a mean proportional between AO and DO ; and with the centre O and radius OC describe a circle. Produce CB ; and make DE a mean proportional between CE and BE ; and with the centre E , and radius ED , describe a circle cutting the former in F ; F is the point required.



For, as was proved in the last proposition,
 $AF : FD :: AC : CD$,
 and $\therefore$ the angle $AFC = CFD$; and
 $CF : FB :: CD : DB$,
 $\therefore$ the angle $CFD = DFB$;
 and $\therefore$ the three angles AFC, CFD, DFB are equal.

(32.) *If two equal lines touch two unequal circles, and from the extremities of them lines containing equal angles be drawn cutting the circles, and the points of section joined; the triangles so formed will be reciprocally proportional.*

Let two equal lines AB, CD touch two unequal



circles EBF, GDH ; and from A and C let lines AIK, AEF, CLM, CGH be drawn containing the equal angles KAF, MCH . Join IE, KF, GL, MH ; then will the triangle $AKF : CHM :: CGL : AIE$.

Since AB is equal to CD , the rectangles EA , AF , and GC , CH are equal;

$$\therefore AF : CH :: GC : AE,$$

and for the same reason,

$$AK : CM :: CL : AI,$$

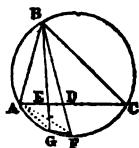
whence $AF \times AK : CH \times CM :: CG \times CL : AE \times AI$,

$$\therefore \text{the triangle } AKF : CMH :: CGL : AIE,$$

since $AK : CM$ in the ratio of the perpendicular from K and M on AF and CH ; and $CL : AI$ in the ratio of the perpendiculars from L and I .

(33.) *If from an angle of a triangle a line be drawn to cut the opposite side, so that the rectangle contained by the sides including the angle, be equal to the rectangle contained by the segments of the side together with the square of the line so drawn; that line bisects the angle.*

From B one of the angles of the triangle ABC , let BD be drawn, so that the rectangle AB , BC may be equal to the rectangle AD , DC together with the square of BD ; BD bisects the angle B .

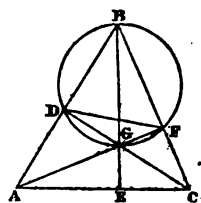


For if not, let BE bisect it; the rectangle AB , BC is equal to the rectangle AE , EC together with the square of BE . About ABC describe a circle, and produce BD , BE to the circumference in F and G ; join FG . The rectangle AD , DC is equal to the rectangle BD , DF ; $\therefore$ the rectangle AB , BC is equal to the rectangle BD , DF together with the square of BD . i. e. to the rectangle BF , BD . In the same way the rectangle AB , BC is equal to the rectangle BG , BE ;

whence the rectangle BG, BE is equal to the rectangle BF, BD ; a circle may therefore be described through D, E, G, F ; whence DEG, DFG are equal to two right angles, i. e. to DEG, DEB ; and $\therefore DFG$ is equal to DEB , or to DAB and ABG ; and $\therefore$ the arc AB equal to the arc BC , which is absurd, unless the triangle be isosceles. Hence $\therefore BG$ does not bisect the angle; and no other but BD can bisect it.

(34.) *In any triangle, if perpendiculars be drawn from the angles to the opposite sides; they will all meet in a point.*

Let ABC be any triangle; and AF, CD perpendiculars drawn upon the opposite sides, intersecting each other in G . Through G draw BGE ; it is perpendicular to AC .



Join FD ; and about the trapezium $BFGD$ describe a circle. The triangles ADG, GFC being equiangular,

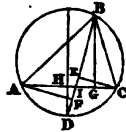
$$AG : GC :: GD : FG,$$

whence also the triangles AGC, FGD are equiangular; and $\therefore$ the angle $ACD = DFG = ABE$; and the angle BAC is common to the two triangles ABE, ACD ; $\therefore$ the angle $AEB = ADC$, i. e. it is a right angle, and BE is perpendicular to AC .

(35.) *If from the extremities of the base of any triangle, two perpendiculars be let fall on the line bisecting the vertical angle; and through the points where they*

meet that line, and the point in the base whereon the perpendicular from the vertical angle falls, a circle be described; that circle will bisect the base of the triangle.

Let ABC be a triangle, whose vertical angle B is bisected by the line BD , on which let fall the perpendiculars AF , CE . From B let fall BG perpendicular to AC ; a circle described passing through E , F , G will also bisect AC .



About the triangle ABC describe the circle ADB ; and from D draw a diameter which will bisect AC in H . Now since the angle AID is common to the triangles AIF , HID , and the angles AFI , IHD are right angles, $\therefore$ the triangles AIF , HID are similar. In the same manner BIG , CEI are similar. Whence

$$HI : ID :: IF : IA,$$

$$\text{and } IG : IB :: IE : IC,$$

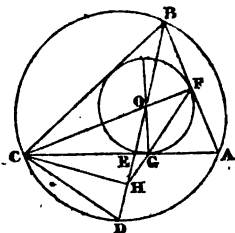
$$\therefore HI \times IG : ID \times IB :: IF \times IE : IA \times IC,$$

and since $ID \times IB = IA \times IC$, $\therefore HI \times IG = IF \times IE$, or a circle passing through E , F , G will pass through H (vi. 13.), and $\therefore$ bisect the base AC .

(36.) If from one of the angles of a triangle a straight line be drawn through the centre of its inscribed circle, and a perpendicular be drawn to this line from one of the other angles; the point of intersection of the perpendicular, and the two points of contact of the inscribed circle, which are adjacent to the remaining angle, are in the same straight line.

Let ABC be a triangle, and O the centre of its inscribed circle. From B draw BOD through the centre;

and from C let fall CH perpendicular to it. Let the circle touch the sides of the triangle in F and G ; join HG , GF . HGF is a straight line.



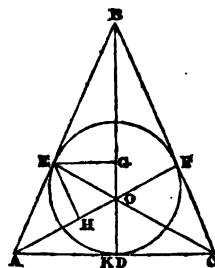
Join OG , OC ; and let a circle be described about the triangle ABC ; join CD . The triangles OGE , CHE , having the vertical angles at E equal, and OGE , CHE right angles, are similar,

$$\therefore CE : HE :: OE : EG,$$

and $\therefore$ the rectangle CE , EG is equal to the rectangle OE , HE ; whence a circle will pass through the points C , O , G , H , $\therefore$ the angle $COH = CGH$. Again (vii. 20.) $CD = DO$, and $AG = AF$, also the angle $CDO = GAF$, $\therefore COE = AGF$; whence $CGH = AGF$; and CG , GA are in the same straight line, $\therefore FG$, GH are in the same straight line.

(37.) *If from the three angles of any triangle three straight lines be drawn to the points where the inscribed circle touches the sides; these lines shall intersect each other in the same point.*

Let ABC be a triangle, in which a circle is inscribed, touching the sides in E , F , D . Join AF , CE , cutting each other in O . Join BO , and produce it; it will pass through D .



For if not let it pass through some other point K ; draw EG , EH respectively parallel to AC , BC . Then the triangles OEH , OFC being similar,

K K

$$OE : EH :: OC : (CF =) CD;$$

in the same manner,

$$EG : EO :: KC : CO,$$

$$\therefore EG : EH :: KC : CD.$$

Again, since EH is parallel to BF ,

$$AE : EH :: AB : (BF =) BE :: AK : EG,$$

$$\therefore AK : AE :: EG : EH :: KC : CD,$$

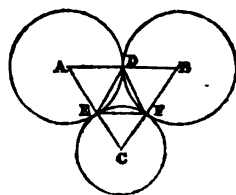
$$\text{or } AK : AD :: KC : CD;$$

$$\therefore AK + KC : AD + DC :: AK : AD;$$

whence $AK = AD$, and K coincides with D .

(38.) *If three circles touch each other, two of which are equal; the vertical angle of the triangle formed by joining the points of contact, is equal to either of the angles at the base of the triangle, which is formed by joining the centres.*

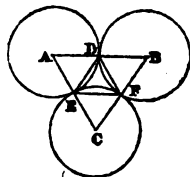
Let three circles, whose centres are A, B, C , touch each other in the points D, E, F ; and let the two circles, whose centres are A and B , be equal. Join AB, BC, CA, ED, DF, FE ; the angle EDF is equal to either of the angles at A or B .



Since AE is equal to BF , the sides of the triangle ACB are cut proportionally, $\therefore EF$ is parallel to AB , and the angle FED is equal to EDA . Now since CA is equal to CB , the angle at A is equal to the angle at B ; but AD, AE are each equal to BF, BD , $\therefore DE$ is equal to DF , and the angle $DFE = FED = EDA = AED$; whence the angle $EDF = DAE = DBF$.

(39.) *If three equal circles touch each other; to compare the area of the triangle formed by joining their centres with the area of the triangle formed by joining the points of contact.*

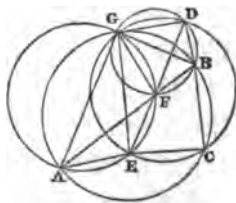
Let three equal circles, whose centres are A, B, C , touch each other in D, F, E . Join AB, BC, CA, ED, DF, FE .



Since the circles are equal, their radii are equal, $\therefore$ the sides of the triangle ACB are cut proportionally, and DF is parallel to AC , and DE to BC ; $\therefore AEFD$ is a parallelogram, and the triangle DEF is equal to ADE . In the same manner it may be proved to be equal to each of the triangles DFB, FCE ; and $\therefore$ it is equal to one fourth of ABC .

(40.) *If four straight lines intersect each other, and form four triangles; the circles which circumscribe them will pass through one and the same point.*

Let the lines AB, AC, DE, DC form the four triangles ABC, AEF, DCE, DBF ; and let the circles circumscribing AEF, DBF , intersect each other in G ; the circles circumscribing the triangles ABC, DEC will also pass through G .

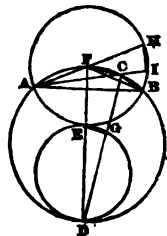


Join GA, GE, GF, GB, GD . Because the points G, F, B, D are in the circumference of a circle, the angle $GDB = GFA = GEA$, i. e. $GDC = GEA$, and

$\therefore$ the points G, E, C, D are in the circumference of the same circle, i. e. the circle circumscribing ECD passes through G . Also since the angle $GAE = GFD = GBD$, i. e. $GAC = GBD$, $\therefore$ the points G, A, C, B are in the circumference of the same circle; or the circle circumscribing ACB also passes through G .

(41.) *Having given the base and vertical angle of a triangle; to determine the locus of the extremity of the line which always bisects the vertical angle, and is equal to half the sum of the sides containing the angle.*

Let AB be the given base; and on it describe a segment of a circle ACB , containing an angle equal to the given vertical angle. Complete the circle; and draw the diameter FD bisecting AB . Join AF, FB ; and with the centre F , and radius FA , describe a circle ABE , cutting FD in E . On DE as a diameter describe a circle; it will be the locus required.



Let ACB be any position of the triangle, and draw CGD ; it bisects the angle at C , since ACD is equal to AFD , i. e. to the half of AFB or to the half of ACB . Produce AC, AF to I and H . Join HI, CF, EG . The angle CAF is equal to CDF , and the angles AIH, FCD, DGE are right angles; $\therefore$ the triangles AIH, CDF, EDG are equiangular,

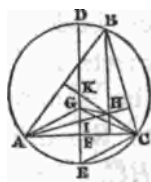
and $AH : AI :: FD : CD :: FE : CG$.

But AH is equal to AF and FB together, i. e. to $2FE$, and AI is equal to AC and CB together (ii. 60.), $\therefore AC$

and CB together are equal to $2CG$, i. e. CG is half the sum of the sides AC and CB , and its extremity is in the circumference of the circle EGD .

(42.) *If from the extremities of the base of a triangle inscribed in a circle, perpendiculars be drawn to the opposite sides, intersecting a diameter which is perpendicular to the base; the segments of the diameter intercepted between these points and a point in it, whose distance from the base is equal to the lesser segment of the diameter made by the base, will be to one another in the ratio of the sides of the triangle.*

Let ABC be a triangle inscribed in a circle, whose diameter DE is perpendicular to the base AC . Make $FG = FE$; and let the perpendiculars be drawn from A and C to the opposite sides, intersecting in H , and meeting the diameter in I and K ;



$$KG : IG :: AB : BC.$$

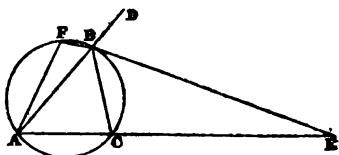
Join AG , GC . Because $GF = FE$, the angles GAF , GCF are each equal to FCE , i. e. to half the vertical angle of the triangle; $\therefore AGC$, ABC are together equal to two right angles (Eucl. i. 32.); and since AHC is equal to its vertically opposite angle, AHC , ABC are equal to two right angles; whence $AGC = AHC$; and A , G , H , C are in the circumference of a circle; $\therefore$ the angle $GHA = GCA =$ half the angle ABC . Now the angle KHI , contained by the perpendiculars, is equal to ABC , $\therefore GH$ bisects the angle KHI . Also the angle $GKH = KHB = BAC$; and $KIH = AIF = ACB$;

$\therefore$ the triangle KHI is equiangular to ABC ; and it has been shewn that GH bisects the angle KHI .

$$\therefore KG : GI :: KH : HI :: AB : BC.$$

(43.) *If the exterior angle of a triangle be bisected by a straight line which cuts the base produced; the square of the bisecting line is equal to the difference of the rectangles of the segments of the base and of the sides of the triangle.*

Let CBD the exterior angle of the triangle ABC be bisected by BE which meets AC produced in E ; the square of BE is equal to the difference of the rectangles AE , EC and AB , BC .



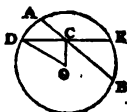
About the given triangle describe the circle ABC ; and produce EB to F ; and join AF . Then because the angle $EBC = EBD = FBA$, and $AFB = BCE$, since each of them together with ACB is equal to two right angles; $\therefore$ the triangles EBC , FBA are equiangular, and $AB : BF :: EB : BC$,

$\therefore$ the rectangle AB , BC is equal to the rectangle EB , BF ; to each of these equals add the square of BE , and the rectangle AB , BC together with the square of BE is equal to the rectangle EB , BF together with the square of BE , i. e. to the rectangle FE , EB , or its equal AE , EC ; and $\therefore$ the square of BE is equal to the difference between the rectangles AE , EC and AB , BC .

SECT. VIII.

(1.) *If from the centre of a circle a line be drawn to any point in the chord of an arc; the square of that line together with the rectangle contained by the segments of the chord will be equal to the square described on the radius.*

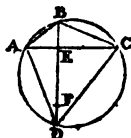
From the centre O of the circle ABD , let OC be drawn to a point C in any chord AB ; the square of OC together with the rectangle AC, CB is equal to the square described on the radius.



Through C draw DE perpendicular to OC . Join OD . Then $DC = CE$, and the rectangle DC, CE is equal to the square of DC ; but the rectangle DC, CE is equal to the rectangle AC, CB , $\therefore$ the rectangle AC, CB together with the square of CO is equal to the squares of DC, CO , i. e. to the square of DO .

(2.) *If two straight lines in a circle cut each other at right angles; the sums of the squares of the two lines joining their extremities will be equal.*

Let the two straight lines AC, BD cut each other at right angles in E ; join AB, BC, CD, DA ; the squares of AB, CD are equal to the squares of AD and CB .

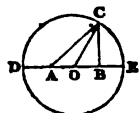


For the squares of AB and CD are equal

to the squares of AE , EB , DE , EC . But the squares of AE and DE are equal to the square of AD , and the squares of EC and EB are equal to the square of BC ; $\therefore$ the squares of AB and CD are equal to the squares of AD and BC .

(3.) *If two points be taken in the diameter of a circle, equidistant from the centre; the sum of the squares of two lines drawn from these points to any point in the circumference will always be the same.*

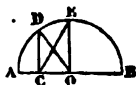
Let A and B be two points in the diameter of the circle CDE , equally distant from the centre O ; if lines AC , BC be drawn to a point in the circumference, the sum of the squares of AC , CB will be the same, in whatever point of the circumference C is taken.



Join CO ; then (iv. 30.) the sum of the squares of AC , CB is double of the sum of the squares of AO and OC , which is an invariable quantity.

(4.) *If from any point in the diameter of a semicircle there be drawn two straight lines to the circumference, one to its point of bisection, and the other at right angles to the diameter; the squares of these two lines are together double of the square of the semi-diameter.*

From any point C in the diameter AB , let CD , CE be drawn; of which CD is perpendicular to AB , and CE is drawn to the

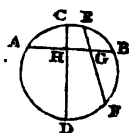


middle point E of the semi-circumference AEB ; the squares of CD and CE together will be double of the square of the semi-diameter.

Join DO , OE . The angle EOC is a right angle, and $\therefore$ the square of EC is equal to the sum of the squares of EO and OC ; but the square of DC is equal to the difference of the squares of DO and OC ; $\therefore$ the squares of EC and CD together are equal to the squares of EO and DO together, i. e. are double of the square of EO .

(5.) *If a straight line be drawn at right angles to the diameter of a circle, and be cut by any other line; the rectangle contained by the segments of this cutting line, together with the square of that part of the perpendicular line which is intercepted between it and the diameter, is always of the same magnitude.*

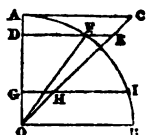
Let AB be drawn at right angles to CD the diameter of the circle ABC ; and let it be cut in G , by any other line EF ; the rectangle EG , GF , together with the square of HG is of invariable magnitude.



For the rectangle EG , GF is equal to the rectangle AG , GB , and the rectangle AG , GB together with the square of HG is equal to the square of AH , $\therefore$ the rectangle EG , GF together with the square of HG is equal to the square of half AB , which is always the same.

(6.) *A straight line being drawn from the centre of a quadrant bisecting the arc and meeting a tangent drawn from one extremity; if from any point in the bounding radius a line be drawn parallel to the tangent, the sum of the squares of the segments of it, cut off by the aforesaid line and by the circumference will be equal to the square of the radius.*

From the centre O let OC be drawn bisecting the quadrantal arc AB , and meeting a tangent to the point A in C . From any point D in AO draw a perpendicular DE ; the squares of DF and DE are together equal to the square of OB .



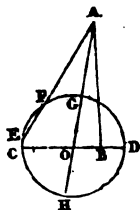
Join FO . Since the angle DOE is half a right angle, and the angle at D a right angle, $\therefore DEO$ is half a right angle, and equal to DOE ; whence $DE = DO$. Now the squares of DO and DF are together equal to the square of OF ; $\therefore$ also the squares of DE and DF are together equal to the square of OF , or OB . In the same manner it may be shewn that the squares of GH and GI are together equal to the square of OB .



(7.) *If from a point without a circle there be drawn two straight lines, one of which is perpendicular to a diameter, and the other cuts the circle; the square of the perpendicular is equal to the rectangle contained by the whole cutting line and the part without the circle, together with the rectangle contained by the segments of the diameter.*

From the point A let AB be drawn perpendicular

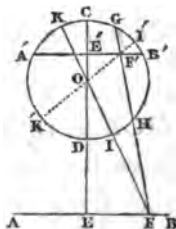
to CD the diameter of the circle DEC , and AFE cutting the circle; the square of AB is equal to the rectangles EA , AF , and CB , BD together.



Through the centre O draw AGH . The squares of AB , BO are equal to the square of AO , i. e. to the rectangle HA , AG together with the square of GO (Eucl. ii. 6.), i. e. to the rectangle HA , AG together with the rectangle DB , BC and the square of OB ; and $\therefore$ the square of AB is equal to the rectangles HA , AG and DB , BC together.

(8.) *If any straight line be drawn perpendicular to the diameter of a given circle, and produced to cut any chord; the rectangle contained by the segments of the diameter will be less or greater than the rectangle contained by the segments of the chord, by the square of the line intercepted between them, according as it is drawn without or within the circle.*

Let AB meet the diameter CD of the circle CGD at right angles in the point E , and any other chord GH in F ; the rectangle CE , ED is less or greater than the rectangle GF , FH , by the square of EF , according as AB is without or within the circle.

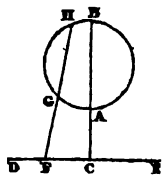


Take O the centre of the circle, and through it draw FOK cutting the circle in I and K . Then because KI is bisected in O , and produced to F , the rectangle KF , FI together with the square of OI is equal to the square of OF , i. e. to the squares of OE and EF . But when E

is without the circle, the rectangle CE , ED together with the square of OD is equal to the square of OE ; $\therefore$ the rectangle KF , FI together with the square of OI is equal to the rectangle CE , ED , together with the squares of OD , and EF . And since OI is equal to OD , and the rectangle KF , FI is equal to the rectangle GF , FH , (Eucl. iii. 36. Cor.); $\therefore$ the rectangle GF , FH is equal to the rectangle CE , ED together with the square of EF . In nearly the same manner it is demonstrated if AB be within the circle.

(9.) *If a diameter of a circle be produced to bisect a line at right angles, the length of which is the double of a mean proportional between the whole line through the centre and the part without the circle; and from any point in the double of the mean proportional a line be drawn cutting the circle; the sum of the squares of the segments of the double mean proportional will be equal to twice the rectangle contained by this cutting line and the part without the circle.*

Let the diameter BA produced bisect DCE at right angles, and let CD and CE be each a mean proportional between AC and CB ; and through any point F let FGH be drawn cutting the circle in G and H ; the squares of DF , and FE are together equal to twice the rectangle GF , FH .



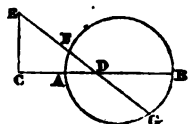
Since the rectangle AC , CB is equal to the square of CD , the rectangle AC , CB together with the square of CF is equal to the squares of CD and CF . But (viii. 8.)

the rectangle AC, CB together with the square of CF is equal to the rectangle GF, FH ; whence the rectangle GF, FH is equal to the squares of CD and CF together; and the doubles of equals are equal, $\therefore$ twice the rectangle GF, FH is equal to twice the squares of CD and CF together, *i. e.* to the squares of DF and FE together (Eucl. ii. 9.).

COR. If from F tangents be drawn to the circle, the sum of their squares will together be equal to the sum of the squares of DF and FE .

(10.) *If from a point without a circle two straight lines be drawn, one through the centre to the circumference, and the other perpendicular to it, and on the former a mean proportional be taken between the whole line and the part without the circle; any other line passing through that extremity of the mean proportional which is within the circle, and terminated by the circumference and perpendicular, will be similarly divided.*

From a point C without the circle ABG , let CAB be drawn through the centre; take a point D such that $AC : CD :: CD : CB$; and from C let CE



be drawn perpendicular to CB ; if through D , any line EFG be drawn terminated by the circumference and the perpendicular CE , $EF : ED :: ED : EG$.

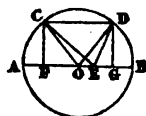
For the rectangle AC, CB together with the square of CE is equal, by construction, to the squares of DC, CE , *i. e.* to the square of DE . But (viii. 8.) the rectangle AC, CB together with the square of CE is equal

to the rectangle FE, EG ; $\therefore$ the rectangle FE, EG is equal to the square of ED ;

and $EF : ED :: ED : EG$.

(11.) *If a chord be drawn parallel to the diameter of a circle, and from any point in the diameter lines be drawn to its extremities; the sum of their squares will be equal to the sum of the squares of the segments of the diameter.*

Let CD be drawn parallel to AB the diameter of the circle ACD ; and from any point E in AB , let EC, ED be drawn; the squares of EC and ED are together equal to the squares of EA and EB .

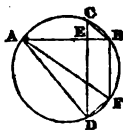


Take O the centre, and join CO, DO ; and let fall the perpendiculars CF, DG . Then since CD is parallel to AB , the angles AOC, BOD are equal, and $OF = OG$.

Now (Eucl. ii. 12.) $CE^2 = CO^2 + OE^2 + 2 OF \times OE$,
 and (Eucl. ii. 13.) $ED^2 = DO^2 + OE^2 - 2 OG \times OE$,
 whence the squares of CE, ED are equal to twice the squares of CO, OE , or twice the squares of AO, OE ,
i. e. to the squares of AE, EB (Eucl. ii. 9.).

(12.) *If through a point within or without a circle, two straight lines be drawn at right angles to each other, and meeting the circumference; the squares of the segments of them are together equal to the square of the diameter.*

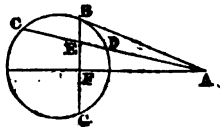
Let AB , CD cut one another at right angles in E ; the sum of the squares of AE , EB , CE , ED will be equal to the square of the diameter.



Draw the diameter AF . Join FB , BC , FD , DA ; then ABF being a right angle is equal to AED , and $\therefore BF$ is parallel to CD , and (ii. 1. Cor. 2.) $BC = FD$. And since the angles at E are right angles, the squares of CE , EB are equal to the square of CB , i. e. to the square of DF ; but the squares of AE , ED are equal to the square of AD ; $\therefore$ the squares of CE , EB , AE , ED are equal to the squares AD , DF , i. e. to the square of AF , ADF being a right angle.

(13.) *If from a point without a circle there be drawn two straight lines, one of which touches the circle and the other cuts it; and from the point of contact a perpendicular be drawn to the diameter; the square of the line which touches the circle is equal to the square of that part of the cutting line which is intercepted by the perpendicular, together with the rectangle contained by the segments of that part of it which is within the circle.*

From the point A without the circle BCD let two lines AB , AC be drawn; of which AB touches the circle, and AC cuts it; and from B let



BFG be drawn perpendicular to the diameter; the square of AB is equal to the square of AE together with the rectangle CE , ED .

For the square of AB is equal to the squares of AF , FB . But (Eucl. ii. 5.) the square of FB is equal to the rectangle BE , EG together with the square of EF , i. e. to the rectangle CE , ED together with the square of EF ; $\therefore$ the square of AB is equal to the squares of AF , FE together with the rectangle CE , ED , i. e. to the square of AE together with the rectangle CE , ED .

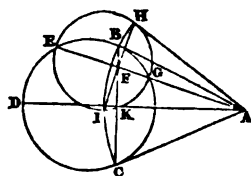
(14.) *A straight line drawn from the concourse of two tangents to the concave circumference of a circle is divided harmonically by the convex circumference and the chord which joins the points of contact.*

Let AB , AC touch the circle ADC , and AGE cut it. Join BC ; then will

$$AE : AG :: EF : FG.$$

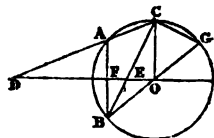
On EG as diameter describe a circle EHG , and through F draw HFI perpendicular to EG . Join AH . Then the rectangle BF , FC is equal to the rectangle EF , FG , i. e. to the square of HF , or the rectangle HF , FI ; and $\therefore$ the points H , B , I , C are in the circumference of a circle. And since the square of AH is equal to the squares of AF , FH , or to the square of AF and the rectangle BF , FC , i. e. to the squares of AK , KF , together with the rectangle BF , FC , i. e. to the squares of AK , KB , or to the square of AB ; $\therefore AH = AB$; and since the square of AH is equal to the rectangle EA , AG , AH is a tangent at H . And since EG is a diameter, (ii. 42.)

$$AE : AG :: EF : GF.$$



(1f.) *If from the extremities of any chord in a circle straight lines be drawn to any point in the circumference meeting a diameter perpendicular to the chord; the rectangle contained by the distances of their points of intersection from the centre is equal to the square described upon the radius.*

From A and B , the extremities of the chord AB , let AC , BC be drawn to any point C in the circumference; and let them meet a diameter perpendicular to AB in D and E . Take O the centre; the rectangle DO , OE is equal to the square described on the radius.



Draw the diameter BOG . Join CG , CO . Since the angle OCB is equal to OBC , and BGC to FAD , and that CBG , and BGC together are equal to a right angle; $\therefore$ OCE and FAD together are equal to a right angle, and $\therefore$ to FAD and ADF together; hence OCE is equal to ADO , $\therefore$ the triangles COD , COE are equiangular,

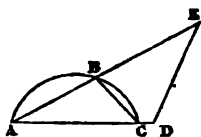
and $DO : OC :: OC : OE$.

$\therefore$ the rectangle DO , OE is equal to the square of OC .



(16.) *If from any point in the base or base produced, of the segment of a circle, a line be drawn making therewith an angle equal to the angle in the segment, and from the extremity of the base any line be drawn to the former, and cutting the circumference; the rectangle contained by this line and the part of it within the segment is always of the same magnitude.*

Let ABC be a segment of a circle on the base AC ; and from any point D , let DE be drawn, making with AC an angle equal to the angle in the segment, and meeting any line AB drawn from the extremity A ; the rectangle EA, AB is of invariable magnitude.



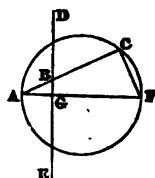
Since the angle ADE is equal to ABC , and the angle at A common to the triangles ADE, ABC , the triangles are $\therefore$ similar; whence

$$AD : AE :: AB : AC,$$

and the rectangle AE, AB is equal to the rectangle AD, AC , which is invariable.

(17.) To determine the locus of the extremities of any number of straight lines drawn from a given point, so that the rectangle contained by each and a segment cut off from each by a line given in position may be equal to a given rectangle.

Let A be the given point, and DE the line given in position. Draw AGF perpendicular to DE ; and take AF such that the rectangle AG, AF may be equal to the given rectangle; and on AF as diameter describe a circle; it will be the locus required.



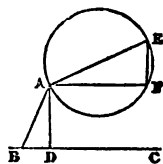
Draw any line AC ; and join FC . The triangles ABG, AFC being similar,

$$AF : AC :: AB : AG,$$

$\therefore$ the rectangle AC, AB is equal to the rectangle AF, AG , i. e. to the given rectangle. And the same may be proved of any other line drawn from A to the circumference, which $\therefore$ is the locus.

(18.) *If from a given point two straight lines be drawn, containing a given angle, and such that their rectangle may be equal to a given rectilineal figure, and one of them be terminated by a straight line given in position; to determine the locus of the extremity of the other.*

Let A be the given point, and BC the line given in position. From A draw AD perpendicular to BC ; and draw AE , making with it the angle DAE equal to the given angle; and make AE of such a magnitude that the rectangle AD, AE may be equal to the given figure. On AE as diameter describe a circle AFE : it will be the locus required.



Draw any other line AB , and AF making with it the angle FAB equal to the given angle; join FE . Then the triangles ABD, AFE , being equiangular,

$$AB : AD :: AE : AF,$$

whence the rectangle AB, AF is equal to the rectangle AD, AE , i. e. to the given figure; and the same may be proved, of any other two lines, similarly drawn from A .

(19.) *If from the vertical angle of a triangle two lines be drawn to the base making equal angles with the adjacent sides; the squares of those sides will be proportional to the rectangles contained by the adjacent segments of the base.*

Let AD, AE be drawn from the vertical angle A making equal angles BAD, EAC with the adjacent sides; then will $AB^2 : AC^2 :: BD \times BE : CD \times CE$.

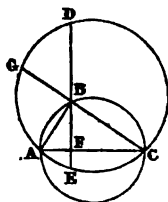


About the triangle ADE describe a circle, cutting AB , AC (produced if necessary) in G and F . Join FG . Then (Eucl. iii. 26.) the arcs GD , FE are equal, $\therefore$ (ii. 1. Cor.) FG is parallel to BC ;

$$\begin{aligned} \therefore AB : AC &:: BG : CF, \\ \text{and } AB^2 : AC^2 &:: AB \times BG : AC \times CF, \\ &:: BD \times BE : CD \times CE \text{ (Eucl. iii. 36.)}. \end{aligned}$$

(20.) *If a line placed in one circle be made the diameter of a second, the circumference of the latter passing through the centre of the former; and any chord in the former circle be drawn through this diameter perpendicularly; the rectangle contained by the segments made by the circumference of the latter circle will be equal to that contained by the whole diameter and a mean proportional between its segments.*

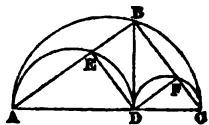
Let a line AC , placed in the circle ADC , be the diameter of the circle ABC , whose circumference passes through the centre of ADC . Through any point B let a line DBE be drawn perpendicular to AC ; the rectangle DB , BE is equal to the rectangle AC , BF .



Draw CBG . And since the circumference ABC passes through the centre of AGD , $\therefore$ (ii. 60.) AB is equal to BG , and the rectangle AB , BC is equal to the rectangle GB , BC , i.e. to the rectangle DB , BE . Also the rectangle AB , BE is equal to the rectangle AC , BF , (Eucl. vi. C.), $\therefore$ the rectangle DB , BE is equal to the rectangle AC , BF .

(21.) *If semicircles be described on the segments of the base made by a perpendicular drawn from the right angle of a triangle; they will cut off from the sides, segments which will be in the triplicate ratio of the sides.*

From the right angle B let BD be drawn perpendicular to AC ; and on AD , DC let semicircles be described, cutting AB , CB in E and F ; $AE : CF$ in the triplicate ratio of $AB : CB$.



Join DE , DF ; they are perpendicular to AB , BC respectively; $\therefore$ (Eucl. vi. 8. Cor.)

$$AC : AB :: AB : AD$$

$$AB : AD :: AD : AE;$$

hence $AC : AE$ in the triplicate ratio of $AC : AB$.

In the same manner it may be shewn that

$$AC : CF \text{ in the triplicate ratio of } AC : CB,$$

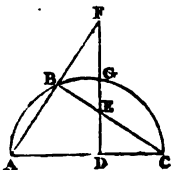
$\therefore$ *inv.* and *ex aequo*,

$$AE : CF \text{ in the triplicate ratio of } AB : CB.$$



(22.) *If from any point in the diameter of a semicircle a perpendicular be drawn, and from the extremities of the diameter lines be drawn to any point in the circumference, and meeting the perpendicular; the rectangle contained by the segments which they cut off from the perpendicular, will be equal to the rectangle contained by the segments of the diameter.*

From any point D in the diameter AC of the semicircle ABC , let a perpendicular DF be drawn; and to any point B in the circumference let the lines AB , CB be drawn, meeting the perpendicular



in E and F ; the rectangle FD , DE is equal to the rectangle AD , DC .

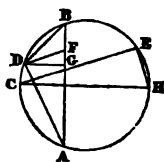
Since the angle ABC in a semicircle is a right angle, FBE is also a right angle, and $\therefore$ equal to FDC ; and the vertically opposite angles at E are equal, $\therefore$ the angle BFE is equal to ECD , and the triangles FDA , DEC are equiangular,

and $FD : DA :: DC : DE$,

$\therefore$ the rectangle FD , DE is equal to the rectangle AD , DC .

(23.) *If from the point of bisection and any other point in a given arc of a circle, two parallel lines be drawn, the former terminated by the circumference, the latter by the chord of the arc; the rectangle contained by these two lines will be equal to that contained by the lines which join the latter point with each extremity of the given arc.*

From C the middle point of the arc ACB , and D any other point, let any two parallel lines CE , DF be drawn, of which CE is terminated by the circumference of the circle, and DF by the chord AB .



Join AD , DB ; the rectangle CE , DF is equal to the rectangle AD , DB .

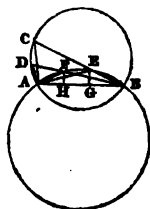
Draw DG perpendicular to AB ; draw the diameter CH ; and join EH . The angle FDG being equal to ECH , and the angles at G and E right angles, the triangles FDG , ECH are equiangular,

$\therefore DG : DF :: EC : CH$,

whence the rectangle EC , DF is equal to the rectangle DG , CH , i. e. (Eucl. vi. C.) to the rectangle AD , DB .

(24.) *If two circles cut each other, and from either point of intersection lines be drawn meeting both circumferences; the rectangles contained by the segments of these lines are to one another in the ratio of the perpendiculars drawn from their intersection with the inner circumferences upon the line joining the intersections of the circles.*

Let the two circles ABC , ABE cut each other in A and B ; join AB ; and from B draw any two lines BC , BD cutting the circles in E , F , C , D ; let fall the perpendiculars EG , FH ; the rectangles BE , EC and BF , FD are to one another as EG to FH .



Join AD , AC . Since the angle $AFB = AEB$, $\therefore AFD = AEC$; but $ADF = ACE$ in the same segment, $\therefore$ the triangles AFD , AEC are similar;

and $EC : EA :: FD : FA$.

But $EC : EA :: EC \times EB : EA \times EB$

$= EG \times \text{diameter of the circle } ABE,$

and $FD : FA :: BF \times FD : BF \times FA$

$= FH \times \text{diameter of the circle } ABE,$

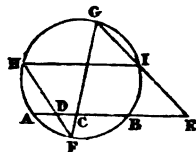
whence $EC \times EB : BF \times FD :: EG \times D : FH \times D ::$

$[EG : FH.$

(25.) *If on opposite sides of any point in the chord of a circle two lines be taken, one terminating in the chord, the other in the chord produced, whose rectangle is equal to that contained by the segments of the chord; and the extremities of the lines so taken be joined to those*

of any other chord passing through the same point; the line joining their intersections of the circle will be parallel to the first chord.

On opposite sides of any point C of the chord AB of the circle ABG let two lines CD and CE be taken, such that the rectangle DC, CE may be equal to the rectangle AC, CB ; and through C let any chord GCF be drawn, and DF, GE joined, meeting the circumference in H and I . Join HI ; it will be parallel to AB .



Since the rectangle DC, CE is equal to the rectangle AC, CB , i. e. to the rectangle GC, CF ,

$$\therefore DC : CF :: GC : CE,$$

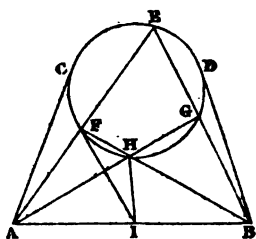
$\therefore$ the triangles DCF, GCE are equiangular, and the angle FDC is equal to CGE or FHI in the same segment; $\therefore HI$ is parallel to AB .

(26.) *If from two points without a circle two tangents be drawn, the sum of the squares of which is equal to the square of the line joining those points; and from one of them a line be drawn cutting the circle, and two lines from the other point to the intersections with the circumference; the points in which these two lines cut the circle, are in the same straight line with the former point.*

From A and B two points without the circle CDE let tangents AC, BD be drawn, such that the sum of their squares may be equal to the square of AB . If

from A any line AFE be drawn, and BF , BE joined; the points A , H , G will be in a straight line.

In AB take a point I , so that the rectangle AB , BI may be equal to the square of BD . Join IF , IH , AH , HG . Then the rectangles AB , AI and AB , BI are together equal to the square of AB , i.e. to the squares of AC , BD ; $\therefore$ the rectangle AB , AI is equal to the square of AC or to the rectangle AF , AE ;



$$\therefore AF : AB :: AI : AE,$$

and $\therefore$ (Eucl. vi. 6.) the angle AIF is equal to AEB , whence also $FIB = FHG$. Now the rectangle BI , BA being equal to the square of BD , or to the rectangle BH , BF ,

$$\therefore BF : BA :: BI : BH,$$

and $\therefore$ the angle AHB is equal to FIB , or FHG ; and BHF is a straight line, $\therefore AHG$ is a straight line.

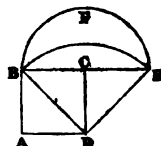
(27.) *If from the vertex of a triangle there be drawn a line to any point in the base, from which point lines are drawn parallel to the sides; the sum of the rectangles of each side and its segment adjacent to the vertex will be equal to the square of the line drawn from the vertex together with the rectangle contained by the segments of the base.*

From the vertex A of the triangle ABC let a line AD be drawn to any point D in the base; from which let DF , DE be drawn parallel respectively to AB , AC ; the

Since circles are as the squares of their radii, the quadrant $AEBO : ADC :: AO^2 : AC^2 :: 2 : 1$, $\therefore$ the quadrant $AEBO$ is equal to the semicircle ADB ; and taking away the part $AEBC$, the lune $ADBE$ is equal to the triangle ABO .

(29.) *If from the extremities of the side of a square circles be described with radii equal the former to the side, and the latter to the diagonal of the square; the area of the lune so formed will be equal to the area of the square.*

From D and C the extremities of DC the side of a square, with radii DB and CB , let circles be described, cutting each other again in E ; the area of the lune BFE is equal to the square AC .

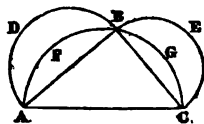


Join CE , ED . Since BC is equal to CE , and CD is common, and BD is equal to DE ; $\therefore$ the angles BCE , ECD are equal; whence BE is a straight line; also the angles BDC , EDC are equal; and BDC being half a right angle, BDE is a right angle; $\therefore$ the arc BE is a quadrant; $\therefore$ the lune BFE is equal to the triangle BDE (viii. 28.) i. e. to the square AC .

(30.) *If on the sides of a triangle inscribed in a semicircle, semicircles be described; the two lunes formed thereby will together be equal to the area of the triangle.*

Let ABC be a triangle inscribed in a semicircle. On AB , BC let semicircles ADB , BEC be described;

the lunes $ADBF$, $BGCE$ are together equal to the triangle ABC .



Since the areas of circles are as the squares of their diameters, the semicircle $ABC : ADB :: AC^2 : AB^2$,

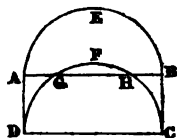
and $ABC : BEC :: AC^2 : BC^2$,

$\therefore ABC : ADB + BEC :: AC^2 : AB^2 + BC^2$,

i. e. in a ratio of equality, $\therefore ABC = ADB + BEC$;
from these equals take away the segments AFB , BGC ,
and the triangle $ABC = AFBD + BGCE$.

(31.) *If on the two longer sides of a rectangular parallelogram as diameters, two semicircles be described towards the same parts; the figure contained by the two remaining sides of the parallelogram and the two circumferences shall be equal to the parallelogram.*

Let $ABCD$ be a rectangular parallelogram, on the sides AB , DC of which let semicircles AEB , DFC be described; the figure $DAEBCHFG$ is equal to $ABCD$.

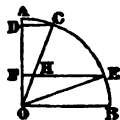


Since $AB = DC$, the semicircles are equal; from each of which take away FGH , and $AGFHBE = DGHC$; if to these equals be added ADG and BHC , the whole $ADGFHCBE$ will be equal to the whole $ABCD$.

(32.) *If two points be taken at equal distances from the extremities of a quadrant, and perpendiculars be*

drawn from these points to the radius; the mixtilinear space cut off, shall be equal to the sector which stands on the arc between them.

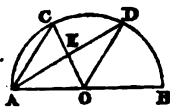
Let two points C, E be taken at equal distances from A and B , the extremities of the quadrant AB ; and let fall the perpendiculars CD, EF on AO . Join CO, EO ; the figure $CDFE$ is equal to the sector COE .



Since the arc $AC = EB$, the angle $AOC = EOB = OEF$, and the angles at D and F are right angles, and $CO = OE$, $\therefore$ the triangle $COD = EOF$; from each of these take away OFH , $\therefore DFHC = OHE$; to each of these add CHE , and $CDFE = COE$.

(33.) *If the arc of a semicircle be trisected, and from the points of section lines be drawn to either extremity of the diameter; the difference of the two segments thus made, will be equal to the sector which stands on either of the arcs.*

Let the arc of the semicircle ACB be divided into three equal parts in the points C , and D . From A the extremity of the diameter draw AC, AD ;



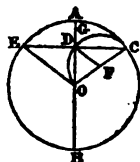
take O the centre, and join OC, OD ; the difference of the segments AC and ACD is equal to the sector COD .

Since the angles CAD, DAB stand on equal circumferences, they are equal; but the angle $DAO = ODA$, $\therefore CAE = EDO$; and $CEA = OED$, $\therefore$ the triangles CAE, EOD are equiangular; and since OE is drawn bisecting the vertical angle O of the isosceles triangle

AOD , it bisects the base, $\therefore AE = ED$; and consequently the triangle AEC is equal to the triangle DOE ; add to each CED , and $CAD = COD$.

(34.) *If a straight line be placed in a circle, and on the radius passing through one extremity, as a diameter, another circle be described; the segments of the two circles cut off by the above straight line will be similar, and in the ratio of four to one.*

Let EC be a straight line placed in the circle ABC . Take O the centre, and join OC ; and upon it describe a semicircle ODC . The segments EAC , DGC are similar, and in the ratio of 4 : 1.



Join OD , and produce it both ways to the circumference. Take F the centre of the semicircle ODC . Join OE , FD . Then ODC being a right angle, $ED = DC$, and $OF = FC$, $\therefore$ (Eucl. vi. 2.) DF is parallel to EO ;

and $CE : EO :: CD : DF$,

and the segments EAC , DGC are similar;

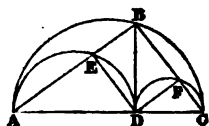
whence $EAC : DGC :: EC^2 : CD^2 :: 4 : 1$.

Cor. The segment ADC is bisected by the circumference DGC .

(35.) *If on any two segments of the diameter of a semicircle semicircles be described; the area included between the three circumferences will be equal to the*

area of a circle, whose diameter is the mean proportional between the segments.

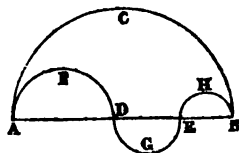
On AD , and DC , segments of AC the diameter of the semicircle ABC let semicircles AED , CFD be described; from D draw DB perpendicular to AC , and $\therefore$ a mean proportional between AD and CD ; the figure $AEDFCB$ is equal to the circle described upon DB .



Join AB , BC . Since ADB is a right angle, the semicircle on AB is equal to those on AD and DB together; and the semicircle on BC is equal to those on BD and DC ; $\therefore$ the semicircles on AB and BC , or the semicircle ABC which is equal to them, will be equal to the semicircles AED , DFC and the circle described upon DB . From these equals take away the semicircles AED , DFC , and the figure $AEDFCB$ is equal to the circle described upon DB .

(36.) *If the diameter of a semicircle be divided into any number of parts, and on them semicircles be described; their circumferences will together be equal to the circumference of the given semicircle.*

Let AB the diameter of the semicircle ACB be divided into any number of parts in the points D , E ; and on AD , DE , EB , let semicircles be described; their circumferences are together equal to ACB .



For since the circumferences of circles are as their diameters,

$$ACB : AFD :: AB : AD$$

$$ACB : DGE :: AB : DE$$

$$ACB : EHB :: AB : EB,$$

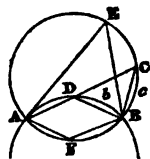
whence $ACB : AFD + DGE + EHB :: AB : AD +$
 $[DE + EB,$

in which proportion the third term being equal to the fourth,

$$ACB = AFD + DGE + EHB.$$

(37.) *If two equal circles cut each other, and from either point of section a line be drawn meeting the two circumferences; the area cut off by the part of this line between the two circumferences will be equal to the area of the triangle contained by that part and lines drawn to its extremities from the other point of section.*

Let the two equal circles ADB , ACB cut each other in A and B ; and from A draw any line AC , cutting the circles in D and C ; join DB , BC ; the figure $DbbC$ CD is equal to the triangle DBC .

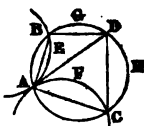


Take any points E , F in the circumferences AEB , AFB ; join AE , EB , AF , FB . Since the arcs ADB , AFB are equal, the angles ADB , AFB are equal. But the angles AFB , AEB are equal to two right angles, and $\therefore$ to ADB , BDC ; whence the angle $BDC = AEB = ACB$, and $BD = BC$; $\therefore$ the segment Dbb is equal to the segment BcC ; to each of these add $DbBC$, and the triangle DBC is equal to $DbbC$ CD .

COR. If AE is a tangent to ADB at A ; the area $ADbbCCEA$ will be equal to the triangle ABE .

(38.) *If two equal circles touch each other externally, and through the point of contact another be described with the same radius; the area contained by the convex circumferences cut off from the touching circles, and the part of the third without them, is equal to the area of the quadrilateral figure formed by lines drawn from the points of intersection to the point of contact, and to the point where the third circle is cut by a tangent drawn to the point of contact of the two circles.*

Let two equal circles touch each other in A ; and through the point of contact let an equal circle ABC be described, cutting the former in B and C . Join AB , AC ; and to the point A let a tangent AD be drawn; join BD , DC . The area contained by AEB , AFC and the intercepted arc BDC is equal to the quadrilateral figure $ABDC$.

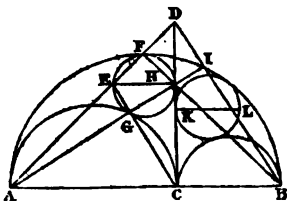


Since DA touches the circle AEB , the angle DAB is equal to the angle in the alternate segment, and $\therefore$ equal to the angle in the segment BCA , i. e. equal to the angle BDA , whence $BA=BD$; $\therefore$ the segment BEA is equal to the segment BGD ; and $AEBGDA$ is equal to the triangle ABD . In the same manner it may be shewn that $AFCHDA$ is equal to the triangle ACD ; $\therefore$ $AEBGHCFDA$ is equal to the quadrilateral figure $ABDC$.

(39.) *If a straight line be divided into any two parts, and upon the whole and the two parts semicircles be described; and from the point of section a perpendicular*

be drawn, on each side of which circles are described touching it and the semicircles; these circles will be equal.

Let AB be divided into any two parts in C ; and on AB , AC , CB let semicircles be described; from C draw the perpendicular CD , on each side of which let a circle be described touching the perpendicular and each of the semicircles. These circles are equal.



Let $EFGH$ touch the perpendicular in H , and the semicircles in F and G . Draw the diameter EH parallel to AB . Join FE , EA ; AF will be a straight line (ii. 35.). Produce it to meet the perpendicular in D . Join FH , HB ; FB will also be a straight line, and perpendicular to AD at the point F . Join EG , GC , HG , GA ; EC and HA will also be straight lines. Produce AH to I . Join BI ; it will be perpendicular to AI , and pass through D , since the perpendiculars to the three sides of the triangle AHB meet in a point. And since the angles AGC , AIB are equal, EC is parallel to DB ,

$$\therefore AD : DE :: AB : BC;$$

and AC , HE are parallel,

$$\therefore AD : DE :: AC : EH,$$

$$\therefore AB : BC :: AC : EH,$$

In the same manner it may be proved that

$$AB : AC :: BC : KL$$

KL being the diameter of the other circle drawn parallel to AB . Hence $EH = KL$, and the circles are equal.

SECT. IX.

(1.) *GIVEN one angle, a side adjacent to it, and the difference of the other two sides ; to construct the triangle.*

Let CB be equal to the given side ; draw the indefinite line CA , making with it an angle equal to the given angle ; and cut off CD equal to the given difference. Join BD ; and make the angle DBA equal to BDA ; ABC is the triangle required.



The angle DBA being equal to ADB , the side AD is equal to AB ; and the differences between CA and AB is equal to CD , i. e. to the given difference.

(2.) *Given one angle, a side opposite to it, and the difference of the other two sides ; to construct the triangle.*

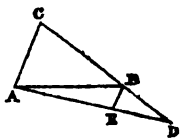
In any line CA , (see last Fig.) take CD equal to the given difference ; make the angle CDB greater than a right angle by half the given angle ; from C draw CB equal to the given side, and meeting DB in B ; and make the angle DBA equal to BDA ; ABC is the triangle required.

Since the angles ABD , ADB are equal, AB is equal to AD ; $\therefore$ the difference between CA and AB is equal.

to CD , *i. e.* to the given difference. Also the angle at A is equal to the difference between the angles CDB , DBA , or CDB , BDA , *i. e.* to the given angle. And CB was made equal to the given side.

(3.) *Given the base and one of the angles at the base; to construct the triangle, when the side opposite the given angle is equal to half the sum of the other side and a given line.*

Let AB be the given base, and ABC the given angle; produce CB to D , making BD equal to the given line. Join AD ; and from B to AD draw BE , equal to half BD . From A draw AC parallel to BE ; ABC is the triangle required.

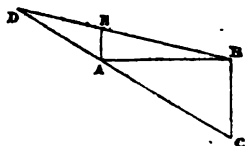


For AB and ABC are made equal to the given base and angle; and since BE is parallel to AC ,

$$AC : CD :: BE : BD :: 1 : 2.$$

(4.) *Given the base of a right-angled triangle, and the sum of the hypotenuse and a straight line, to which the perpendicular has a given ratio; to construct the triangle.*

Let AB be equal to the given base. From B draw BC perpendicular to it, and such that it may have to the given sum, the given



ratio. Join CA , and produce it; and from B to CD , draw BD equal to the given sum. From A draw AE perpendicular to AB ; ABE is the triangle required.

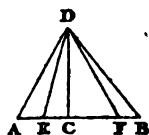
For AE being parallel to BC ,

$AE : ED :: BC : BD$, i. e. in the given ratio;

$\therefore AE$ is equal to the perpendicular; and AB was made equal to the given base.

(5.) *Given the perpendicular drawn from the vertical angle to the base, and the difference between each side and the adjacent segment of the base made by the perpendicular; to construct the triangle.*

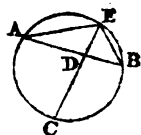
From any point C in an indefinite line AB , erect a perpendicular CD equal to the given perpendicular; and take CE equal to the given difference between the side and adjacent segment on the opposite side of the perpendicular. Also take CF equal to the other given difference. Join ED , FD ; and make the angle $FDA = DFA$, and $EDB = DEB$; ADB is the triangle required.



Since the angles ADF , AFD are equal, $AD = AF$, $\therefore$ the difference between AD , AC is equal to CF , i. e. to the given difference. In the same manner the difference between BD and BC is equal to CE , i. e. to the given difference.

(6.) *Given the vertical angle, and the base; to construct the triangle, when the line drawn from the vertex cutting the base in any given ratio, bisects the vertical angle.*

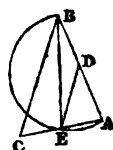
Let AB be equal to the given base; and upon it describe a segment of a circle containing an angle equal to the given angle; and let the base be divided in the given ratio in D . Complete the circle; and bisect the arc ACB in C ; join CD , and produce it to E ; join AE , EB ; AEB is the triangle required.



For AEB is equal to the given angle; and since the arc $AC = CB$, the line ED , which divides AB in the given ratio in D , makes the angle $AED = DEB$.

(7.) *Given the vertical angle, and one of the sides containing it; to construct the triangle, when the line drawn from the vertex making a given angle with the base, bisects the triangle.*

Let AB be equal to the given side; and on it describe a segment of a circle containing an angle equal to the given angle made by the bisecting line with the base; and make the angle ABC equal to the given vertical angle.

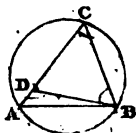


Bisect AB in D ; and draw DE parallel to BC , meeting the circle in E ; join AE , and produce it to C ; ABC is the triangle required.

Join BE . Since DE is parallel to BC , (Eucl. vi. 2.) AE is equal to EC , and $\therefore BE$ bisects the triangle; and it makes with the base AE an angle equal to the given angle. Also AB is equal to the given side, and ABC to the given angle at the vertex.

(8.) *Given one angle, a side opposite to it, and the sum of the other two sides ; to construct the triangle.*

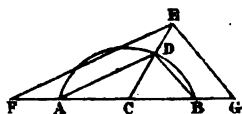
Let AB be the given side. Upon it describe a segment of a circle containing an angle equal to half the given angle ; and from A draw AC equal to the given sum of the two sides ; join BC ; and make the angle CBD equal to BCD ; ABD is the triangle required.



Since the angle DCB is equal to DBC , DB is equal to DC ; $\therefore AD, DB$ together are equal to the given sum. And the angle ADB is equal to DBC, DCB , i.e. to twice DCB , and $\therefore$ is equal to the given angle.

(9.) *Given the vertical angle, the line bisecting the base, and the angle which the bisecting line makes with the base ; to construct the triangle.*

On any line AB describe a segment of a circle containing an angle equal to the given angle. Bisect AB in C ; and at C make the angle BCD equal to the given angle which the bisecting line makes with the base ; produce it, if necessary, till CE is equal to the given line ; join DA, DB ; and draw EF, EG respectively parallel to them ; EFG is the triangle required.



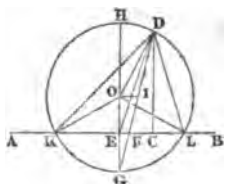
For FE and EG being respectively parallel to DA, DB ,

$FC : CE :: (AC =) CB : CD :: CG : CE$,
 $\therefore FC = CG$, and EC , which is equal to the given line, bisects the base ; and the angles FEC, CEG being equal

For AB is the given base; BE is the given line; and $AE : EC :: AD : DB$, i. e. in the given ratio; and the angle at C is equal to AED , i. e. to the given angle.

(12.) *Given the perpendicular, the line bisecting the vertical angle, and the line bisecting the base; to construct the triangle.*

From any point C in the indefinite line AB , draw a perpendicular CD equal to the given perpendicular; and with D as centre, and radii equal to the two given lines describe circles, cutting AB in E and F . Through E draw GEH perpendicular to AB ; join DE , DF ; and produce DF to meet HE in G . Bisect DG in I ; and draw IO at right angles to DG , meeting GH in O . With the centre O , and radius OG describe a circle cutting AB in K and L ; join DK , DL ; DKL is the triangle required.



Join OD , OK , OL . Since OI bisects DG at right angles, $OD = OG$, and the circle passes through D . And since OE is perpendicular to KL , $KE = EL$, or KL is bisected by DE , which is equal to the given bisecting line; and the arc $KG = GL$, and the angle KDF is equal to FDL , or the angle KDL is bisected by DF , which is equal to the given line; and DC was made equal to the given perpendicular.

(13.) *Given the line bisecting the vertical angle,*

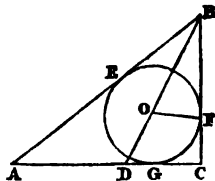
the line bisecting the base, and the difference of the angles at the base; to construct the triangle.

Construct a right-angled triangle FDC , (see last Fig.) having its hypotenuse FD equal to the given line which bisects the vertical angle, and the angle FDC equal to half the given difference of the angles at the base. Produce FC both ways; and to it from D draw DE equal to the given line which bisects the base. Draw HEG parallel to DC , meeting DF produced in G . Bisect GD in I ; and from I draw IO at right angles to DG , meeting GH in O . With the centre O , and radius OG , describe a circle, cutting FC produced in K and L ; join DK, DL ; DKL is the triangle required.

For $KE = EL$, *i. e.* the base KL is bisected by DE , which is equal the given line; and the angle KDF is equal to FDL , being on equal circumferences KD, DL ; *i. e.* the vertical angle KDL is bisected by DF , which was made equal to the given bisecting line. Also (iii. 5.) the difference between the angles DLK and DKL is equal to twice the angle FDC , *i. e.* to the given angle.

(14.) *Given the vertical angle, and the line drawn to the base bisecting the angle, and the difference between the base and the sum of the sides; to construct the triangle.*

Let ABC be equal to the given angle, and BD the line bisecting it. Make BE and BF , each equal to half the given difference. From F draw FO perpendicular to BC , meet-

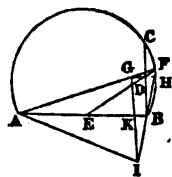


ing BD in O . With the centre O , and radius OF describe a circle, it will touch AB in E (Eucl. iv. 4.). Through D draw a line AC touching the circle in G ; ABC will be the triangle required.

• For (Eucl. iii. 36. Cor.) $AE = AG$, and $GC = CF$, $\therefore AC$ is equal to AE and CF together; whence the difference between AC and the sum of the sides AB , BC is equal to BE , BF together, i. e. to the given difference. Also BD is equal to the given line, and it bisects the angle ABC , which is equal to the given vertical angle.

(15.) *Given the line bisecting the vertical angle, the perpendicular drawn to it from one of the angles at the base, and the other angle at the base; to construct the triangle.*

Let AB be equal to the given bisecting line; and upon it describe a segment of a circle containing an angle equal to the given angle. Draw BC perpendicular to AB , and make BD equal to the given perpendicular. Bisect AB in E ; join ED , and produce it to F ; join FA , FB ; and through D , draw GDH parallel to AB . In FB produced take BI equal to BH . Join AI ; AFI is the triangle required.

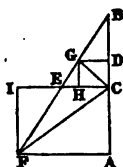


Join IG , cutting AB in K . Because GH is parallel to AB , and FE bisects AB , it also bisects GH , i. e. $GD = DH$; but HB also is equal to BI ; $\therefore BD$ is parallel to GI , and IK is half of IG , and $\therefore$ equal to BD the

given perpendicular. Also since AB bisects GI at right angles, it also bisects the angle IAG ; and it is equal to the given bisecting line. And AFI is equal to the other given angle.

(16.) *Given the straight line bisecting the vertical angle, and the perpendiculars drawn to that line from the extremities of the base; to construct the triangle.*

On the indefinite straight line AB , take AC , CD respectively equal to the greater and less perpendiculars; and from C draw CE at right angles to AB , and equal to the given bisecting line. Take $AB : BD :: AC : CD$. Join BE ; and produce it to meet AF , DG , drawn from A and D , perpendicular to AB . Join GC , CF ; GCF is the triangle required.



Draw FI , GH perpendicular to CE . Then the triangles BGD , ABF being similar,

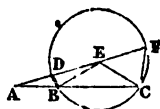
$$AF : GD :: AB : BD :: AC : CD,$$

and the angle at A is equal to GDC ; whence (Eucl. vi. 6.) AFC , GDC are similar, and the angle ACF is equal to GCD ; and $\therefore$ the angle FCE is equal to ECG , i.e. the angle FCG is bisected by EC , which was made equal to the given bisecting line. Also FI , GH are respectively equal to AC , CD , which were made equal to the given perpendiculars.

(17.) *Given the vertical angle, the difference of the two sides containing it, and the difference of the*

segments of the base made by a perpendicular from the vertex ; to construct the triangle.

Let AB be equal to the given difference of the segments of the base. Draw BD , making the angle ABD equal to half the given angle ; and from A draw AD meeting it in D , and equal to the given difference of the sides ; produce it, and make the angle $DBE = EDB$, and with the centre E and radius EB describe the circle DBC meeting AB , AD produced in C and F ; join EC ; AEC is the triangle required.

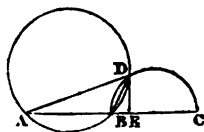


Join FC . Since $BDFC$ is a quadrilateral figure inscribed in a circle (Eucl. iii. 22.) the angles ABD , DFC are equal ; but AEC is double of DFC (Eucl. iii. 20.), and $\therefore$ also of ABD , *i. e.* it is equal to the given angle. Also since the angles EDB , EBD are equal, $ED = EB = EC$, $\therefore$ the difference of the sides AE , EC is equal to AD , *i. e.* to the given difference ; and AB is evidently equal to the difference of the segments of the base.



(18.) *Given the base and vertical angle ; to construct the triangle, when the square of one side is equal to the square of the base, and three times the square of the other side.*

Let AB be equal to the given base. Upon it describe a segment of a circle containing an angle equal to the given angle. Produce AB to C , and make BC equal to BA ; and upon BC describe a semicircle

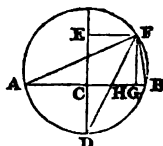


BDC cutting the segment in D . Join AD , BD ; ABD is the triangle required.

Let fall the perpendicular DE ; then (Eucl. vi. 8.) the square of DB is equal to the rectangle CB , BE or to the rectangle AB , BE ; and (Eucl. ii. 12.) the square of AD is equal to the squares of AB , BD , and twice the rectangle AB , BE , i. e. to the square of AB , and three times the square of BD . Also, by construction, ADB is equal to the given angle.

(19.) *Given the base and perpendicular; to construct the triangle, when the rectangle contained by the sides is equal to twice the rectangle contained by the segments of the base made by the line bisecting the vertical angle.*

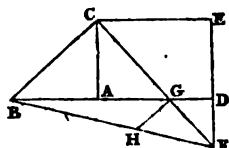
Let AB be equal to the given base; and draw the indefinite line ED bisecting it at right angles. Take CE , CD each equal to the given perpendicular; and through the points A , D , B describe a circle. Draw EF parallel to AB meeting the circle in F . Join AF , FB ; AFB is the triangle required.



Draw FG perpendicular to AB , it is equal to the given perpendicular. Join FD . Since DC is equal to CE , DF is equal to twice DH ; and $\therefore$ the rectangle DF , FH is double of the rectangle DH , HF , i. e. it is double of the rectangle AH , HB , contained by the segments of the base, made by DF which bisects the angle AFB . And AB was made equal to the given base.

(20.) *In a right-angled triangle, having given the sum of the base and hypotenuse, and the sum of the base and perpendicular ; to construct the triangle.*

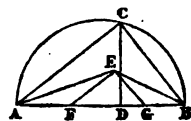
Let BD be taken equal to the sum of the base and hypotenuse, and BG equal to the sum of the base and perpendicular. At the point G in the straight line GB , make the angle BGC equal to half a right angle ; and let CG be produced to meet a perpendicular to BD drawn through D , in F . Join BF ; and from G to BF draw $GH = GD$. From B draw BC parallel to GH , meeting GC in C ; and let fall the perpendicular CA ; ACB is the triangle required.



Draw CE parallel to AD . Because GH is equal to GD , BC is equal to CE , i. e. to AD ; and $\therefore BC$ and BA together are equal to BD , the given sum of the hypotenuse and base. And since AGC is half a right angle, and the angles at A right angles, AG is equal to AC , $\therefore BA$ and AC together are equal to BG , the given sum of the base and perpendicular.

(21.) *Given the perimeter of a right-angled triangle whose sides are in geometrical progression ; to construct the triangle.*

Let AB be equal to the given perimeter ; and on it describe a semicircle ACB . Divide AB in extreme and mean ratio in D ; and from D draw DC at



right angles to AB . Join AC , CB . Bisect the angles CAB , CBA by the lines AE , BE , meeting in E ; and draw EF , EG respectively parallel to AC , CB ; FEG is the triangle required.

Since FE is parallel to AC , the angle $FEA = EAC = EAF$, and $\therefore AF = FE$. And in the same manner it may be shewn that $EG = GB$. Hence GF , FE , EG are together equal to AB the given perimeter. And the angles at F and G being equal to the angles BAC , BCA , the angle FEG is equal to ACB , and is $\therefore$ a right angle.

Also since $AB : AD :: AD : DB$

and (Eucl. vi. 8. Cor.) $AB : BC :: BC : DB$,

$\therefore AD = BC$;

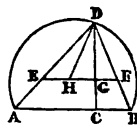
whence also $AB : AC :: AC : (AD =) BC$;
and since the triangles ABC , FGE are equiangular,

$FG : FE :: FE : EG$,

i. e. the sides are in geometrical progression.

(22.) *Given the difference of the angles at the base, the ratio of the segments of the base made by the perpendicular, and the sum of the sides; to construct the triangle.*

Take any line AB , and divide it in C , in the given ratio of the segments. On AB describe a segment of a circle containing an angle equal to the given difference. From C draw the perpendicular CD . Join AD , DB ; and take $DE : DA$ in the ratio of the given sum, to the sum of AD , DB ; and through E draw EF parallel to AB ; and make the angle GDH equal to GDF , and



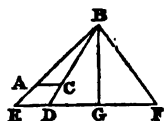
$\therefore DH$ equal to DF , and GH equal to GF ; DEH is the triangle required.

For $DE : DA :: DF : DB :: DE + DH : DA$
 $[+ DB,$
 $\therefore DE + DH$ is equal to the given sum. Also the angle EHD is equal to the two HDF, DFH , i. e. to HDF, DHF , and DEH is equal to the difference between DHF and EDH ; $\therefore$ the difference between EHD and DEH is equal to the sum of HDF and EDH , i. e. to EDF , or the given difference.

Also $EG : GH :: EG : GF :: AC : CB$,
 i. e. in the given ratio.

(23.) *Given the difference of the angles at the base, the ratio of the sides, and the length of a third proportional to the difference of the segments of the base made by a perpendicular from the vertex and the shorter side; to construct the triangle.*

Let ABC be equal to the given difference of the angles at the base; and take $AB : BC$ in the given ratio of the sides. Join AC ; and produce BC to D , so that BD may be to the given third proportional in the ratio $CA : CB$. Through D draw EDF parallel to AC ; and from B draw BG perpendicular to it; make $GF = GD$; join BF ; EBF is the triangle required.



Since $DG = GF$, and the angles at G are right angles; $\therefore BF = BD$, and the angle $BFD = BDF$. Hence the difference between BFE and BEF is equal to the difference between BDF and BEF , i. e. to EBD or the given difference.

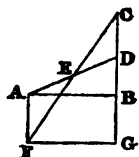
Also $EB : BF :: EB : BD :: AB : BC$,

i. e. in the given ratio. And

$DE : BF :: DE : BD :: CA : CB :: (BD =) BF :$
the given third proportional, whence DE is equal to the
given difference of the segments of the base.

(2A.) *Given the base of a right-angled triangle; to construct it, when parts, equal to given lines, being cut off from the hypotenuse and perpendicular, the remainders have a given ratio.*

Let AB be equal to the given base. From B draw BC at right angles to it, and equal to the part to be cut off from the perpendicular. Take CD to the given part to be cut off from the hypotenuse, in the given ratio of the remainders; and from C to DA , draw CE equal to that given part, and produce it. From A draw AF perpendicular to AB , meeting CE produced in F . Draw FG parallel to AB ; CFG is the triangle required.



Since AF is parallel to CD , the angles AFE , ECD are equal, and the angles at E are equal, $\therefore$ the triangles AEF , CED are equiangular,

and $\therefore FE : AF :: CE : CD$.

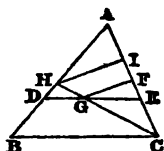
And since AG is a parallelogram, $AF = BG$,

$\therefore FE : BG :: CE : CD$,

i. e. in the given ratio of the remainders; and FG is equal to the given base; and CB , CE are equal to the given parts to be cut off.

(25.) *Given one angle of a triangle, and the sums of each of the sides containing it and the third side; to construct the triangle.*

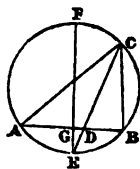
Let BAC be equal to the given angle; and AB , AC equal to the given sums. Join BC ; and draw any line DE parallel to it. Make CF , FG , each equal to BD ; join CG , and produce it to H ; and draw HI parallel to GF . AHI is the triangle required.



Since DG is parallel to BC , and GF to HI ,
 $BD : BH (:: CG : CH) :: GF : HI :: CF : CI$,
 and since BD , GF and FC are equal, BH , HI and IC
 are also equal; whence AH and HI together are equal
 to AB ; and AI , IH are together equal to AC ; and
 HAI is equal to the given angle.

(26.) *Given the vertical angle, and the ratio of the sides containing it, as also the diameter of the circumscribing circle; to construct the triangle.*

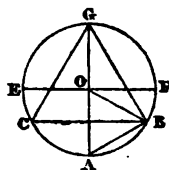
On the given diameter describe a circle; and from it cut off a segment ACB containing an angle equal to the given vertical angle. Divide the base AB in D , in the given ratio of the sides; and draw the diameter EF at right angles to AB . Join ED , and let it meet the circumference in C ; join AC , CB ; ACB is the triangle required.



Since $AE = EB$, the angle ACB is bisected by CE ,
 $\therefore AC : CB :: AD : DB$, i. e. in the given ratio.

(27.) *Given the vertical angle, and the radii of the inscribed and circumscribing circles ; to construct the triangle.*

With the given radius describe the circumscribing circle; and from any point A in it take AB , AC each equal to the arc subtending at the centre an angle equal to the given angle. Join BC ; and parallel to it, at a distance equal to the radius of the inscribed circle draw EF . Join AB ; and from A to EF draw $AO=AB$, and produce AO to G . Join CG , BG ; GCB is the triangle required.

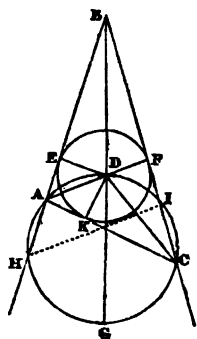


For the angle CGB (Eucl. iii. 20.) is equal to the angle at the centre, which stands on AC , and $\therefore$ is equal to the given angle. Also the angle $CGA=AGB$. Join BO ; and since $AB=AO$, the angle $AOB=ABO$; but AOB is equal to the two OBG , OGB ; and $AGB=ABC$, $\therefore CBO=OBG$, and BO bisects the angle CBG ; whence O , the point of intersection of the bisecting lines GA and BO , is the centre of the inscribed circle; and its distance from BC was made equal the given radius.

(28.) *Given the vertical angle, the radius of the inscribed circle, and the rectangle contained by the straight lines drawn from the centre of that circle to the angles at the base ; to construct the triangle.*

Let the angle ABC be equal to the vertical angle, which bisect by the straight line BD . Let D (ii. 14.) be the centre of a circle which would touch BA and BC ;

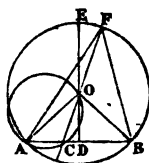
and let E and F be the points of contact; join DE , DF , and produce BD to G , so that the rectangle DE , DG may be equal to the given rectangle. On DG describe a circle cutting BA , BC in A and C ; and in H and I . Join AC (or HI). ABC or AHI is the triangle required.



Join AD , DC ; and draw DK perpendicular to AC . Then since BA and BC make equal angles with BG which passes through the centre, the arcs AD , DI , which they cut off, are equal; $\therefore$ the angle ACD is equal to DCI , *i. e.* ACB is bisected by CD , or D is the centre of the circle inscribed in the triangle ABC ; $\therefore K$ is the point of contact, and $DK = DE$. But (Eucl. vi. C.) the rectangle AD , DC is equal to the rectangle DK , DG , *i. e.* to the rectangle DE , DG , which is equal to the given rectangle.

(29.) *Given the base, one of the angles at the base, and the point, in which the diameter of the circumscribing circle drawn from the vertex meets the base; to construct the triangle.*

Let AB be equal to the given base, and C the given point. Bisect AB in D , and draw DE at right angles to AB . Upon AC describe a segment of a circle containing an angle equal to twice the difference between the given angle and a right angle; and let it meet DE in O . Join AO , OC ; and produce

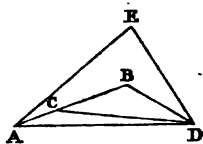


CO to F , making $OF = OA$; join AF, FB ; AFB is the triangle required.

Join OB . Because $AD = DB$, and DO is common, and the angles at D right angles, $\therefore OB = OA = OF$; and a circle described from O as a centre, and radius OA , will pass through F and B , and circumscribe the triangle ABF . And FOC produced is a diameter. Also the angles AOC, AOF are equal to two right angles, i. e. to AOC and twice the given angle; $\therefore AOF$ is equal to twice the given angle; and since it is double of ABF , ABF will be equal to the given angle.

(30.) *Given the vertical angle, the base, and the difference between two lines drawn from the centre of the inscribed circle to the angles at the base; to construct the triangle.*

Take any line AB , unlimited towards B , and cut off AC equal to the given difference; and at the point C make the angle BCD such that its quadruple together with the given angle at the vertex may be equal to two right angles. From A to CD , draw AD equal to the given base. At the point A make the angle $BAE = BAD$, and at the point D make the angle $CDB = DCB$, and $BDE = BDA$; EDA is the triangle required.

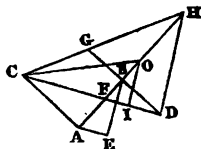


For the angles EAD, EDA being bisected by AB, BD , B is the centre of the inscribed circle. And the angle BCD being equal BDC , BC is equal to BD , and

$\therefore$ the difference between BA and BD is equal to AC ,
i. e. to the given difference. Also the angles EAD ,
 EDA are together double of the angles BAD , BDA ,
i. e. of BCD and BCD , and $\therefore$ are equal to $4BCD$;
 whence the angle AED together with $4BCD$ will be
 equal to two right angles; and $\therefore$ the angle AED is equal
 to the given vertical angle. Also AD is equal to the
 given base.

(31.) *Given that segment of the line bisecting the vertical angle which is intercepted by perpendiculars let fall upon it from the angles at the base ; the ratio of the sides ; and the ratio of the radius of the inscribed circle to the segment of the base which is intercepted between the line bisecting the vertical angle and the point of contact of the inscribed circle ; to construct the triangle.*

Let AB be equal to the given segment of the bisecting line; from A and B , on opposite sides, draw AC , BD at right angles to it. On AB , as hypotenuse, describe a right-angled triangle AEB , whose sides are in the given ratio of the radius of the inscribed circle to the segment of the base intercepted between the bisecting line and the point of contact of the inscribed circle. Divide AB in F in the ratio of the sides; and through F , draw CFD parallel to AE , and meeting the perpendiculars in C and D . Make $BG = BD$. Join CG , and produce it to meet AB produced in H ; join HD ; HCD is the triangle required.



Draw CO bisecting the angle HCD . Since $BG = BD$, and BH is perpendicular to GD , it bisects the

angle GHD ; whence O is the centre of the inscribed circle. From O draw OI perpendicular to CD , and $\therefore$ parallel to BE ; whence

$$OI : IF :: BE : EA,$$

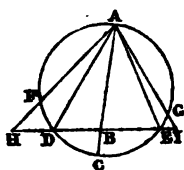
i. e. in the given ratio of the radius of the inscribed circle to the segment of the base intercepted between the bisecting line and the point of contact of the inscribed circle. Also from the similar triangles AFC , BFD ,

$$AF : FB :: CF : FD :: CH : HD,$$

$\therefore CH : HD$ in the given ratio of the sides.

(32.) *Given the line bisecting the vertical angle, and the differences between each side and the adjacent segment of the base made by the bisecting line; to construct the triangle.*

Let AB be equal to the given bisecting line. Produce it to C , so that BC may be a fourth proportional to AB and the given differences. On AC , as a diameter, describe a circle ADE ; in which



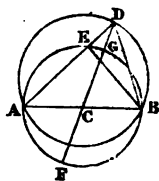
place a line DBE , passing through B , equal to the sum of the given differences; and from A draw AF , AG to the circumference, each equal to DE ; and produce them to meet DE produced in H and I ; AHI is the triangle required.

Since DE is equal to the sum of the given differences, and the rectangle DB , BE , is equal to the rectangle AB , BC , i. e. to the rectangle contained by the given differences, DB and BE will be the given differences. And since $AF = AG$, the arcs AF , AG are

equal and AC being a diameter, FC and CG will also be equal, $\therefore$ the angles FAC , CAG are equal, or HAI is bisected by AC . Also since $FA=DE$, AH and HE will be equal; $\therefore$ the difference between AH and HB is equal to BE one of the given differences. In the same manner $AI=ID$; and $\therefore$ the difference between AI and IB is equal to BD , the other given difference.

(33.) *Given one of the angles at the base, the side opposite to it, and the rectangle contained by the base and that segment of it made by the perpendicular which is adjacent to the given angle; to construct the triangle.*

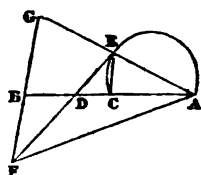
Let AB be equal to the given side. Upon it describe a segment of a circle containing an angle equal to the given angle. Bisect AB in C ; and from C draw to the circumference, a line CD such that the difference between the squares of CD and CB may be equal to the given rectangle. Join AD , DB ; ADB is the triangle required.



On AB describe a circle ABE . Join BE . Then the rectangle AD , DE is equal to the rectangle GD , DF , i. e. to the difference of the squares of CD and CG or to the given rectangle. And BE is perpendicular to AD ; AB is equal to the given side, and ADB to the given angle.

(34.) *Given the vertical angle, and the lengths of two lines drawn from the extremities of the base to the points of bisection of the sides; to construct the triangle.*

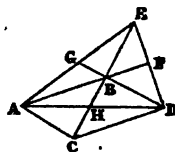
Let AB be equal to one of the given lines; bisect it in C ; and on AC describe a segment of a circle containing an angle equal to the given angle. From BA cut off BD equal to one third of BA ; and from D to the circle, draw DE equal to one third of the other given line. Produce ED , and make DF equal to twice DE . Join AE , FB ; and produce them to meet in G . Join AF ; AFG is the triangle required.



Join CE . Since BD is double of DC , and DF double of DE ; EC is parallel to FG ; and $\therefore$ the angle AGF is equal to AEC , i. e. to the given angle. Hence also AE is equal to EG ; and BF being double of EC , is equal to BG ; $\therefore AB$ and FE , equal to the given lines, are drawn to the points of bisection of the sides.

(35.) *Given the lengths of three lines drawn from the angles to the points of bisection of the opposite sides; to construct the triangle.*

Describe a triangle ABC whose sides are respectively equal to two thirds of the given lines; and complete the parallelogram $ABDC$. Join AD ; and produce CB to E , making $BE = BC$. Join AE , ED ; AED is the triangle required.

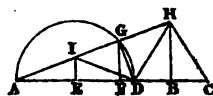


Produce AB , DB to F and G . Since the diagonals of parallelograms bisect each other, AH is equal to HD , and BH to HC ; $\therefore EH$ is equal to EB and BH together, i. e. to BC and BH or to $\frac{1}{2}BC$ and $\therefore$ is equal to

one of the given lines. Again, since GB is parallel to AC , (Eucl. vi. 2.) it bisects AE , and $BG = \frac{1}{2}AC$; but $DB = AC$, $\therefore DG = \frac{1}{2}AC$, and $\therefore$ is equal to another of the given bisecting lines. In the same manner, AF may be shewn to be equal to the other given line, and to bisect DE in F .

(36.) *Given the segments of the base made by the perpendicular, and one of the angles at the base triple the other; to construct the triangle.*

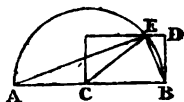
Let AB, BC equal to the given segments, be placed in the same straight line. Make $BD = BC$; bisect AD in E , and BE in F . On AD describe a semicircle; and from F draw FG at right angles to AD . Join AG, GD ; and let AG meet the perpendicular BH in H . Join HC ; AHC is the triangle required.



Draw EI perpendicular to AD ; join DI, DH . Then AE being equal to ED and the angles at E right angles, $AI = ID$, and the angle $IAD = IDA$; whence the angle DIH is double of DAH . But since EF is equal to FB , and GF parallel to IE and BH , $\therefore IG = GH$; and the angles DGI, DGH being right angles, DI is equal to DH , and the angle DHI equal to DIH , and $\therefore$ double of DAH . Also since $DB = BC$, and the angles at B are right angles, $\therefore$ the angle HCB is equal to HDB , i.e. to DHA, DAH together, or to three times the angle DAH . Also AB, BC , by construction, are equal to the given segments made by the perpendicular.

(37.) *The area and hypotenuse of a right-angled triangle being given; to construct the triangle.*

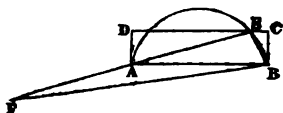
Let AB be equal to the given hypotenuse. Bisect it in C , and on CB describe a rectangular parallelogram CD equal to the given area. On AB describe a semicircle AEB , cutting the side parallel to AB , in E . Join AE , EB ; AEB is the triangle required.



Join CE . Since $AC = CB$, the triangle AEB is double of CEB (Eucl. i. 38.), and $\therefore$ equal to the rect-
angle CD , i. e. to the given area.

(38.) *Given one angle, and a line drawn from one of the others bisecting the side opposite to it; to construct the triangle, when the area is also given.*

Let AB be the given bisecting line; and upon it describe a segment of a circle containing an angle equal to the given angle.



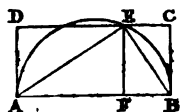
Upon AB also describe a rectangular parallelogram $ABCD$ equal to the given area; and let DC meet the circle in E ; join EA , and produce it, making $AF = AE$; join FB , BE ; FEB is the triangle required.

For BA bisects the side EF ; the angle BEF is equal to the given angle; and the triangle BEF is double of BAE (Eucl. i. 38.) and $\therefore$ equal to $ABCD$, i. e. to the given area.

(39.) *In two similar right-angled triangles, the sum of the base of the one and perpendicular of the other*

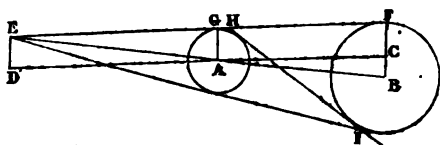
is given ; to determine the triangles such that their hypotenuses may contain the right angle of another triangle similar to them, and the sum of the three areas may be equal to a given area.

Let AB be equal to the given sum ; and upon it describe a rectangular parallelogram equal to the given area. On AB also describe a semicircle, cutting CD in E ; join AE , EB ; the triangles AED , BEC , AEB are the triangles required ; as is evident from the construction.



(40.) Given the vertical angle, the area, and the distance between the centres of the inscribed circle and the circle which touches the base, and the two sides produced ; to construct the triangle.

Let AB be equal to the given distance between the centres ; and make the angle BAC equal to half the given angle at the vertex. On AC let fall the perpen-



dicular BC ; and produce CA , till the rectangle AD , DC is to the given area, in the ratio of $AC : CB$. Complete the parallelogram $DEFC$; and from the centres A and B , describe two circles touching EF in G and F . Draw HI , IE touching the two circles ; HIE is the triangle required.

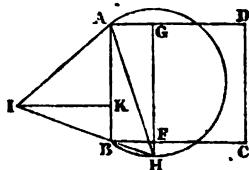
For AB is equal to the given distance between the centres. The angle IEF is double of AEF , i. e. of BAC , and $\therefore$ is equal to the given angle. And from the similar triangles ABC , AED ,

$$AC : CB :: AD : (DE =) AG :: AD \times DC : AG \quad [\times EF.]$$

But since EF is equal to half the perimeter of the triangle EHI , the rectangle AG , EF is equal to the triangle EHI ; whence EHI is equal to the given area.

(41.) *Given the area, the line from the vertex dividing the base into segments which have a given ratio, and either of the angles at the base; to construct the triangle.*

On AB the given line, describe a segment of a circle containing an angle equal to the given angle. Describe a rectangular parallelogram $ABCD$ equal to twice the given



area; and divide BC in the given ratio at F . Through F draw GH parallel to AB . Join BH , HA ; and produce HB so that IB may be to BH in the given ratio. Join IA ; IAH will be the triangle required.

Draw IK perpendicular to AB . The triangles IKB , BFH are similar,

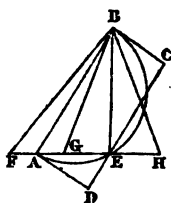
$$\therefore IK : BF :: IB : BH :: FC : FB,$$

$\therefore IK = FC$; and the triangle IAB is equal to half of the parallelogram GC ; and the triangle ABH is equal to half of BG ; $\therefore IAH$ is equal to half the parallelogram

AC , and $\therefore$ equal to the given area. And AB is equal to the given dividing line, and $IB : BH$ in the given ratio.

(42.) *Given the difference between the segments of the base made by the perpendicular, the sum of the squares of the sides, and the area ; to construct the triangle.*

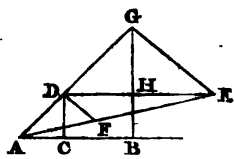
Take a line AB such that its square may be equal to the difference between half the given sum of the squares, and the square of half the given difference of the segments of the base. Upon AB describe a rectangular parallelogram $ABCD$ equal to the given area ; and also on AB describe a semicircle cutting CD in E . Join AE , and produce it both ways ; and make AF, AG , each equal to half the given difference of the segments, and make $EH=EG$. Join BF, BH ; BFH is the triangle required.



Join BG, BE . Since $GE=EH$ and the angles at E are right angles, $\therefore GB=BH$; and the sum of the squares of FB, BH is equal to the sum of the squares of FB, BG , i. e. (iv. 30.) is equal to twice the sum of the squares of FA and AB , i. e. by construction, is equal to the given sum. Also the difference between FE and EH is equal to the difference between FE and EG , i. e. to FG , which is equal to the given difference. And the area of the triangle FBH is double of the area of the triangle ABE , and $\therefore$ (Eucl. i. 41.) equal to $ABCD$, or the given area.

(43.) *Given the base, one of the angles at the base, and the difference between the side opposite to it and the perpendicular; to construct the triangle.*

Take an indefinite line AB ; and from any point C in it draw CD perpendicular to it, and equal to the given difference. From D draw DA making the angle DAC equal to the given angle. Draw DE parallel to AB , and equal to the given base. Join AE ; to which from D draw $DF = DC$. Draw EG parallel to FD , and meeting AD produced; EDG will be the triangle required.



From G draw GB perpendicular to AB . Since DF is parallel to GE , and DC to GB ,

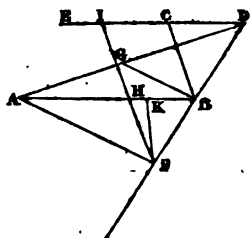
$$DC : GB :: AD : AG :: DF : GE.$$

But DC being equal to DF , GB is equal to GE ; whence the difference between GE and GH is equal to HB , or DC , i. e. to the given difference. And the angle GDE is equal to DAC , i. e. to the given angle; and DE is equal to the given base.

(44.) *Given the vertical angle, the difference of the base and one side, and the sum of the perpendicular drawn from the angle at the base contiguous to that side upon the opposite side and the segment cut off by it from that opposite side contiguous to the other angle at the base; to construct the triangle.*

Let AB be equal to the given sum, and BC to the given difference; and let them be placed so as to contain

an angle ABC , which with the given vertical angle will be equal to two right angles. Through C draw DCE parallel to AB , and through B draw DBF making half a right angle with it. Join AD ; and to it from B draw BG equal to BC ; and parallel to these respectively draw AF, FH ; AFH will be the triangle required.



Produce FH to I ; and let fall the perpendicular FK . Since BC is parallel to FI , and BG to AF ,

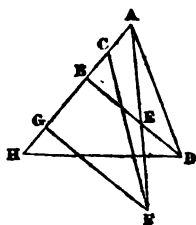
$$BC : FI :: BD : FD :: BG : AF,$$

but $BC=BG$, $\therefore FI=AF$; and $HI=BC$ will be the difference between the base AF and the side FH . Also since the angle BFK is half a right angle, $FK=KB$; and AB is the sum of the perpendicular FK and the segment KA . Also $BCIH$ being a parallelogram, the angle AHF , or its vertically opposite IHB , together with ABC will be equal to two right angles; and $\therefore AHF$ is equal to the given vertical angle.



(45.) *Given the base, the difference of the sides, and the segment intercepted between the vertex and a perpendicular from one of the angles at the base upon the opposite side; to construct the triangle.*

Let AB be equal to the given segment intercepted between the perpendicular and the vertical angle; BC equal to the given difference of the sides; draw BD perpendicular to AB , and make $BE = BA$; join AE , and



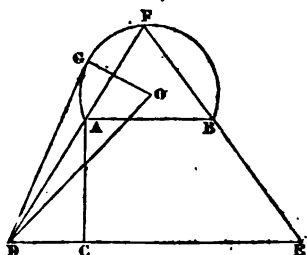
produce it; and from C to AE draw CF such that its square may be equal to the given squares of the base and segment AB ; draw FG perpendicular to AB ; and make $GH=BC$. From H to BD , draw HD equal to the given base; join AD ; AHD is the triangle required.

For AB is made equal to the given segment, and HD to the given base. Also since $AB=BE$, $AG=GF$, and $HB=GC$; whence the squares of GA and HB are equal to the squares of GA and GC , i. e. to the square of CF , or the squares of HD and AB , by construction. But (iv. 29.) the squares of AD and HB are equal to the squares of HD and AB ; $\therefore$ the squares of GA and HB are equal to the squares of AD and HB ; whence $GA=AD$; and $\therefore$ the difference between AH and AD is equal to HG , i. e. to BC , or the given difference.

(46.) *Given the vertical angle, the side of the inscribed square, and the rectangle contained by one side and its segment adjacent to the base made by the angular point of the inscribed square. To construct the triangle.*

Let AB be equal to a side of the inscribed square; and upon it describe a segment of a circle containing an angle equal to the given angle. From A draw AC perpendicular and equal to AB ; and through C draw DCE parallel to AB .

Find O the centre of the circle; and from it to DE , draw OD such that the difference of the squares of OD



and the radius of the circle may be equal to the given rectangle. Join DA , and produce it to F . Join FB , and produce it to E ; DFE is the triangle required.

From D draw the tangent DG . Join GO . Then the squares of DG , GO are together equal to the square of DO , *i. e.* to the square of GO and the given rectangle; and $\therefore$ the square of DG , *i. e.* the rectangle AD , DF is equal to the given rectangle; $\therefore DF$ is a side of the triangle, in which AB is the side of an inscribed square. And DFE is equal to the given vertical angle.





APPENDIX.

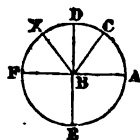
PLANE TRIGONOMETRY.

(1.) **DEF. 1.** *PLANE Trigonometry* is that branch of Mathematical science which treats of the measures of angles and the relation between the respective sides and angles of plane triangles.

LEMMA I.

(2.) *If from the centre of a circle two straight lines be drawn to its circumference, the included angle will be to four right angles, as the intercepted arc is to the whole circumference.*

Let BA , BC be two straight lines drawn from the centre B of the circle ADE , meeting the circumference in A and C ; the angle ABC will be to four right angles as the arc AC is to the whole circumference.

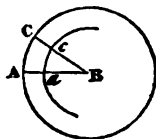


Produce AB till it meets the circumference again in F ; and through B draw DE perpendicular to AB , meeting the circle in D and E . Then (Eucl. vi. 33.) the angle ABC : the right angle ABD :: the arc AC : AD ; and quadrupling the consequents, the angle ABC : four right angles :: the arc AC : $4AD$, or the whole circumference.

LEMMA II.

(3.) *If from the common centre of two circles two straight lines be drawn intersecting the circumferences; the intercepted arc of either circle will be to the intercepted arc of the other circle, as the circumference of the first circle to the circumference of the second.*

From B the common centre of two circles are drawn the lines BA, BC , intersecting the circumferences in A, C, a, c ; the arc $AC : ac ::$ the whole circumference of the first circle : the whole circumference of the second.



For (2) the arc AC : the whole circumference (C) of which it is an arc :: the angle ABC : four right angles; which is the same ratio with that of ac : the whole circumference (C') of which it is an arc,

$$\therefore (\text{Eucl. v. 15.}) \quad AC : C :: ac : C',$$

and *alt.* $AC : ac :: C : C'.$

(4.) COR. 1. If the circumferences of any two circles be severally divided into the same number of equal parts, whatever number of such parts is contained in any arc (AC) of one circle, the same number will be contained in that arc (ac) of the other circle, which subtends the same angle.

(5.) COR. 2. Any arc of a circle is the measure of the angle which it subtends at the centre.

For (See Fig. 1.) draw any straight line Bx to the circumference; then (Eucl. vi. 33.) the arc AC : the arc $Ax ::$ the angle ABC : the angle ABx , i. e. in whatever proportion the angle at the centre is increased, the subtending arc is increased in the same proportion*.

* If the angle be not at the centre, but between it and the circumference;

(6.) Angles at the centres of different circles vary as the arcs which subtend them, directly, and the radii of the circles inversely.

For (2) the arc AC : the circumference $ADEA$:: the angle ABC : four right angles. Hence the angle $ABC = \frac{\text{the arc } AC}{ADEA} \times \text{four right angles}$; and since the circumference $ADEA \propto AB$, and four right angles is an invariable quantity, the angle $ABC \propto \frac{AC}{AB}$.

(7.) Till lately, Geometers *almost unanimously* agreed to divide the circumference of the circle into 360 equal parts called Degrees; each degree into 60 equal parts called Minutes; and each minute into 60 equal parts called Seconds, &c. This method appeared to the Greek Geometers to afford some facilities for calculations, in consequence of the great number of divisors of 360 and 60; but it is in reality subject to inconvenience from complicated numbers, and tediousness in operations. Upon the introduction therefore of a new system of weights and measures into France, in which they were decimally divided and subdivided, it was thought proper to make a similar division of the quadrant. The quadrant is accordingly divided into 100 minutes, each minute into 100 seconds, &c. One great advantage of which method is its identity with the common decimal scale of

ference; it will be measured by half the sum of the arcs intercepted between the sides, and the sides produced. If it be without the circle, it will be measured by half their difference (ii. 24.).

If the angle be formed by a tangent and chord, it will be measured by half the arc subtended by the chord. If by a line cutting the circle and a tangent, or by two tangents, it will be measured by half the difference between the convex and concave arcs intercepted between the lines.

notation; thus 6 deg. 15 min. 53 sec. would be represented by 6.1553 deg. and 34 deg. 5 min. 9 sec. by 34.0509.

The length of a degree then on that scale will be less than one on the common scale in the proportion of 90 : 100, or 9 : 10; a minute in the proportion of $90 \times 60 : 100 \times 100$, or 27 : 50, and a second in the proportion of $90 \times 60 \times 60 : 100 \times 100 \times 100$, or 81 : 250.

If then n = the number of degrees on the new scale, the corresponding number on the former scale will be = $\frac{n \times 9}{10} = \frac{n \cdot (10 - 1)}{10} = n - \frac{n}{10}$. Hence $\therefore$ the following rule, for reducing the new to the old graduation.

Express the measure in decimals, and from it subtract $\frac{1}{10}$ th of this number; mark off the decimals in the remainder, which multiply by 60, and mark off the decimals again; these again multiply by 60 and mark off as before, and so on; the whole numbers so obtained will be the degrees, minutes, seconds, &c. on the common scale.

Thus to determine how many degrees, minutes, &c. in the old graduation, correspond to 46 deg. 43 min. 15 sec. in the new.

$$\begin{array}{r}
 \text{From } 46.4315 \\
 \text{Subtract } 4.64315 \\
 \hline
 41.78835 \\
 60 \\
 \hline
 47.30100 \\
 60 \\
 \hline
 18.06000
 \end{array}$$

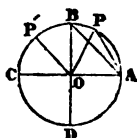
or 41 deg. 47 min. 18.06 sec.

(8.) Degrees, Minutes, Seconds &c. are expressed by $^{\circ}$, $'$, $''$; thus $10^{\circ} 15' 45''$ represent an arc of ten degrees, fifteen minutes and forty five seconds.

(9.) Arcs and angles are expressed without distinction in trigonometrical calculations by degrees, minutes, and seconds.

(10.) A right angle is measured by an arc of 90° ; two right angles by 180° ; half a right angle by 45° ; and each angle of an equilateral triangle by 60° .

(11.) DEF. 2. With the centre O and radius OA describe a circle, and draw the diameters AOC , BOD at right angles to each other. These will divide the circumference into four equal parts called *Quadrants*.



(12.) DEF. 3. The *Complement* of any arc or angle is the difference between that arc or angle and a quadrant or 90° . Thus taking any point P in the quadrant AB ; the complement of the arc AP is the arc BP ; and the complement of the angle AOP is the angle BOP . If $AP = 62^{\circ} 16'$, BP its complement $= 27^{\circ} 44'$. Also the complement of $35^{\circ} 15' 45''$ is an arc of $54^{\circ} 44' 15''$. And in general A being any arc whatever, its complement is $90^{\circ} - A$; or $A - 90^{\circ}$, if A be greater than 90° .

The two acute angles of a right-angled triangle, being together equal to a right angle, are complements to each other.

(13.) DEF. 4. The *Supplement* of any arc or angle is its defect from a semicircle or 180° . Thus, taking any point P , the supplement of the arc AP is PBC , and of the angle AOP is the angle POC ; the supplement of ABP is CP , and of the angle AOP is the angle COP . The supplement of $32^{\circ} 42' 18''$ is an arc of $147^{\circ} 17' 42''$.

if A be any arc, its supplement is $(180 - A)$. Also the supplement of $(90 - A)$ is $(90 + A)$.

In every triangle, the sum of the three angles being equal to two right angles, each angle is the supplement to the sum of the other two.

(14.) DEF. 5. The *Chord* of an arc is a straight line joining the extremities of the arc. Thus AP is the chord of the arc AP . AB is the chord of the quadrant AB . The diameter is the chord of a semicircle; and the chord of the whole circumference is $= 0$.

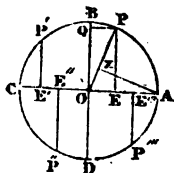
(15.) If the chord does not pass through the centre, as AP , it is the chord of two unequal arcs, AP and the remaining part of the circumference $ADCBP$. In speaking however of the chord of an arc of a circle, the less arc is generally meant.

(16.) COR. The chord of $60^\circ =$ radius of the circle. For the triangle AOP is equiangular and $\therefore$ equilateral.

The chord of a quadrant $= R\sqrt{2}$.

(17.) DEF. 6. The *Sine* or *Right Sine* of an arc is a straight line drawn from one end of the arc perpendicular to the diameter passing through the other end of the arc.

Thus, if from P , PE be drawn perpendicular to AC , it will be the sine of the arc AP less than a quadrant. If PQ be perpendicular to BO , it will be the sine of the arc BP . Also $P'E$, $P''E''$, $P'''E'''$ are the sines of arcs AP' , AP'' , AP''' measured from A and terminating in the second, third and fourth quadrants respectively.



If from A , Ax be drawn perpendicular to OP , it will be the sine of AP , and be equal to PE .

(18.) To trace the changes in magnitude of the Right Sine.

The sine begins with the arc; when the arc is nothing, P coinciding with A , E also coincides with it. But as P moves from A towards B , PE continually increases, *i. e.* as the arc increases, the sine increases; and when P coincides with B , E coincides with O , and the sine of 90° becomes radius. When the arc is greater than 90° , and terminates in the second quadrant, as the arc increases, the sine decreases, till it becomes $= 0$, at the end of the second quadrant, P' and E' both coinciding with C . If the arc terminates in the third quadrant, the sine continually increases, till at the end of it, it becomes radius; and if in the fourth quadrant, it continually decreases again till it becomes $= 0$.

Hence it appears that the sine never exceeds radius.

(19.) To trace the changes in the Algebraic Sign of the right sine.

If the right sines of arcs in the first quadrant be reckoned positive, they will be positive or negative for arcs which terminate in the other quadrants, according as they are drawn in the same or contrary directions with the former, *i. e.* according as they fall on the same or contrary sides of the diameter AC . Hence during the two first quadrants the algebraic sign is positive, and for arcs which terminate in the two next it is negative.

(20.) COR. The sine of an arc is equal to the sine of its supplement.

For PE is equal to the sine of CBP which is the supplement of AP . Thus $48^\circ. 15'$ being the supplement of $131^\circ. 45'$, the sine of $131^\circ. 45' =$ the sine of $41^\circ. 15'$. Also $\sin. (180 - A) = \sin. A$, and $\sin. (90 + A) = \sin. (90 - A)$.

The exterior angle of a triangle being the supplement of its interior adjacent angle, their sines are equal.

(21.) DEF. 7. The *Versed Sine* is that part of the diameter passing through the beginning of the arc, which is intercepted between the beginning of the arc and the right sine.

Thus AE is the versed sine of the arc AP . AE' , AE'' , AE''' are the versed sines of the arcs AP' , AP'' , AP''' respectively.

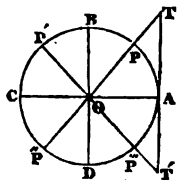
(22.) To trace the changes in magnitude of the versed sine.

The versed sine begins with the arc, and when the arc is nothing, the versed sine $= 0$, E coinciding with A . It increases with the arc, till the arc becoming a semi-circle, the versed sine becomes the diameter; after which it again decreases, whilst the arcs terminate in the third and fourth quadrants, till at the end of the fourth quadrant it becomes $= 0$. But as it is always measured in the same direction from A , it is positive through all the four quadrants.

(23.) COR. Hence (Eucl. vi. 8.) the chord of an arc is a mean proportional between the diameter and the versed sine of that arc.

(24.) DEF. 8. The *Tangent* of an arc is a straight line which touches the circle at one end of the arc, and is terminated by the radius produced through the other end of the arc.

Thus, if the line TAT' touch the circle in A , and AP be an arc less than a quadrant, and OP joined; since TAO , AOP are less than two right angles, OP and AT will meet above AO ; and AT



will be the tangent of AP . But if AP be taken an arc greater than a quadrant, and less than a semicircle, TAO , AOP are greater than two right angles, and $\therefore TA$ and PO will meet on the contrary side of AO , or AT will be the tangent of ABP . In the same manner it may be shewn that AT is the tangent of an arc $ABCP'$ terminating in the third quadrant; and AT of one terminating in the fourth.

(25.) To trace the changes in magnitude of the tangent.

The tangent begins with the arc; when the arc is nothing, P coinciding with A , T coincides with it also. Whilst the arc is less than a quadrant, as AP increases, the angle AOT , and $\therefore AT$ increases, or the tangent increases; and as the arc approximates to a quadrant, OP approximates to the direction OB which is parallel to AT , and $\therefore$ the tangent approximates to a line greater than any that can be assigned. But if the arc be greater than a quadrant, and less than a semicircle, as the arc increases, BP increases, $\therefore$ the angle BOP or DOT increases, and AOT decreases, $\therefore PT$ intersects AT in points continually nearer and nearer to A , or the tangent decreases, and at the end of the second quadrant, the tangent $= 0$. In the same manner it may be shewn that the tangent of an arc terminating in the third quadrant continually increases till it approximates to a line greater than any that can be assigned; and for arcs terminating in the fourth quadrant, it decreases till it becomes $= 0$.

(26.) To trace the changes in the algebraic sign of the tangent.

If the tangent of an arc in the first quadrant be reckoned positive, it will be positive or negative for arcs

terminating in the other quadrants, according as it is drawn in the same or contrary directions with the tangents of arcs ending in the first quadrant, *i. e.* according as it falls on the same or opposite sides of the diameter *AC*. Hence the tangents of arcs ending in the first and third quadrants have a positive sign, and those of arcs ending in the second and fourth, a negative one.

(27.) COR. 1. The tangent of an arc is equal to the tangent of its supplement;

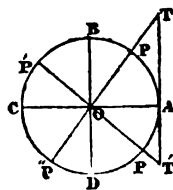
For the angle *POC* is equal to the vertically opposite angle *AOP''*; and (24) *AT* is the tangent of *ADP''* which measures *AOP''*. As one of the arcs is less, and the other greater than a quadrant, they will be affected with contrary signs.

Hence if *A* be any angle, $\text{tang. } (180 - A) = -\text{tang. } A$ and $\text{tang. } (90 + A) = -\text{tang. } (90 - A)$.

(28.) COR. 2. $\text{Tang. } 45^\circ = \text{radius}$. For in the triangle *OAT*, the angle *OAT* is a right angle, and *AOT* = 45° , $\therefore ATO = 45^\circ = AOT$ and *AT* = *AO*.

(29.) DEF. 9. The *Secant* of an arc is the line drawn from the centre through the end of the arc, and produced till it meets the tangent;

Thus, the construction remaining as in Art. 24. *OT* is the secant of the arc *AP*; and the secant of the arc *AP'* terminating in the second quadrant is *OT'*. The secants also of arcs *AP''*, *AP'''* terminating in the third and fourth quadrants will be *OT*, *OT'* respectively.



(30.) To trace the changes in magnitude of the secant.

When *P* coincides with *A*, the secant coincides with

the radius OA , *i. e.* in the beginning of the circle the secant = radius. In the first quadrant as AP increases, AT increases (25.) $\therefore OT$ increases (Eucl. i. 47.) *i. e.* as the arc increases, the secant increases; and when the arc approximates to a quadrant, the secant approximates to a line greater than any that can be assigned. But if the arc be greater than a quadrant and less than a semicircle, as the arc increases, T' approaches nearer to A , and the secant decreases, till at the end of the second quadrant, it becomes equal to radius. In the same manner, the secant of an arc ending in the third quadrant increases from radius till it approximates to a line greater than any that can be assigned; and for arcs terminating in the fourth quadrant, decreases again till it becomes = radius.

Hence the secant is never less than the radius.

(31.) To trace the changes in the algebraic sign of the secant.

If the secants of arcs in the first quadrant are *reckoned* positive, those of arcs ending in the other quadrants will be positive or negative according as they are drawn from the centre through the end of the arc, or from the end of the arc through the centre. Hence the secants of arcs ending in the first and fourth quadrants have a positive sign; and of those ending in the second and third, a negative sign.

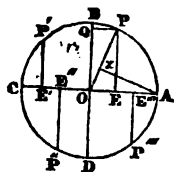
(32.) COR. 1. The secant of an arc is equal to the secant of its supplement.

For OT is the secant of ADP' which is the supplement of AP . As one of the arcs is greater and the other less than a quadrant, the secants will have contrary signs. If A be any angle, $\sec. (180 - A) = - \sec. A$, and $\sec. (90 + A) = - \sec. (90 - A)$.

(33.) COR. 2. $\sec. 45^\circ = R \sqrt{2}$.

(34.) DEF. 10. The *Cosine* of an arc is that part of the diameter passing through the beginning of the arc, which is intercepted between the centre and the right sine.

Thus OE is the cosine of the arc AP in the first quadrant; OQ the cosine of the arc BP . Also OE' , OE'' , OE''' are the cosines of arcs AP' , AP'' , AP''' respectively terminating in the second, third and fourth quadrants.



(35.) To trace the changes in magnitude of the cosine.

When P coincides with A , E also coincides with it, and in the beginning of the arc, cosine = radius. In the first quadrant, as AP increases, EO decreases, *i. e.* as the arc increases, the cosine decreases; till at the end of the first quadrant, the cosine = 0. But if the arc be greater than a quadrant, and less than a semicircle, as AP' increases, OE' increases, and $\therefore$ as the arc increases the cosine increases, till at the end of the second quadrant it = radius. In the same manner the cosines of arcs terminating in the third quadrant decrease from radius, till at the end of the third quadrant the cosine = 0, and in the fourth quadrant increase till at the end of it, cosine = radius.

Hence the cosine is never greater than radius.

(36.) To trace the changes in the algebraic sign of the cosine.

If the cosines of arcs ending in the first quadrant be reckoned positive, those of arcs terminating in the other quadrants will be positive or negative according as they are drawn in the same or opposite directions with those

of arcs ending in the first quadrant, *i. e.* according as they are measured on the same or opposite sides of the centre with the cosines of arcs in the first quadrant. Hence the cosine has a positive sign in the first and fourth quadrants, and a negative one in the second and third.

(37.) COR. 1. The cosine of an arc is equal to the cosine of its supplement.

For OE is the cosine of the arc CBP which is the supplement of AP . But as one of the arcs is less and the other greater than a quadrant, they will be affected with contrary signs. Hence if A be any arc, $\cos. (180 - A) = -\cos. A$; and $\cos. (90 + A) = -\cos. (90 - A)$.

(38.) COR. 2. The cosine of an arc is equal to the sine of its complement.

For OE the cosine of AP is equal to PQ the sine of BP . Hence $\cos. A = \sin. (90 - A)$ and $\cos. (90 - A) = \sin. A$. If $A = 30^\circ$, $\cos. 30^\circ = \sin. 60^\circ$, and $\cos. 60^\circ = \sin. 30^\circ$.

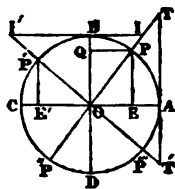
(39.) COR. 3. $\cos. 45^\circ = \sin. 45^\circ = \frac{R}{\sqrt{2}}$. (Eucl. i. 47.)

(40.) COR. 4. The versed sine of an arc less than a quadrant is equal to the difference of the radius and cosine; and of an arc greater than a quadrant is equal to their sum.

(41.) DEF. 11. The *Cotangent* of an arc is a line touching the circle at the end of the first quadrant and meeting the radius produced through the end of the arc.

Thus, if IBI' be drawn touching the circle in B the end of the quadrant AB , BI will be the cotangent of the arc AP which is less than a quadrant. But if the arc

arc ABP be greater than a quadrant and less than a semicircle, BI' is its cotangent. Also the cotangents of the arcs AP'' , AP''' ending in the third and fourth quadrants are respectively BI and BI' .



(42.) To trace the changes in magnitude of the cotangent.

-When P coincides with A , OPI falls in the direction OA , and being parallel to BI , the cotangent is greater than any assignable line. In the first quadrant, as the arc AP increases, OI intersects the line BI in points continually nearer to B . Hence in arcs less than a quadrant, as the arc increases, the cotangent decreases, till at the end of the first quadrant it $= 0$. In arcs greater than a quadrant and less than a semicircle, as the arc ABP increases, BI' the cotangent increases; till as the arc approximates to a semicircle, the cotangent approximates to a line greater than any that can be assigned. In the same manner the cotangents of arcs ending in the third quadrant continually decrease, till at the end of it, the cotangent $= 0$; and if the arcs terminate in the fourth quadrant, the cotangent increases again, till it approximates to a line greater than any that can be assigned.

(43.) To trace the changes of the cotangent in the Algebraic Sign.

If the cotangents of arcs ending in the first quadrant be *reckoned* positive, those of arcs ending in the other quadrants will be positive or negative according as they are drawn in the same or contrary directions with those of arcs ending in the first quadrant, *i. e.* according as they fall on the same side of the diameter BD with the cotangents of arcs ending in the first quadrant or the contrary.

Hence the cotangents of arcs ending in the first and third quadrants have a positive, those of arcs ending in the second and fourth, a negative sign.

(44.) COR. 1. The cotangent of an arc is equal to the cotangent of its supplement.

For BI is the cotangent of CBP which is the supplement of AP . But these cotangents have contrary signs. If $\therefore A$ be any arc, $\cot. (180 - A) = -\cot. A$, and $\cot. (90 + A) = -\cot. (90 - A)$.

(45.) COR. 2. The cotangent of an arc is equal to the tangent of its complement.

For BI the cotangent of AP is the tangent of BP which is the complement of AP . If $\therefore A$ be any arc, $\cot. A = \tan. (90 - A)$ and $\cot. (90 - A) = \tan. A$.

(46.) COR. 3. $\cot. 45 = \text{radius}$.

(47.) DEF. 12. The *Cosecant* of an arc is a straight line drawn from the centre through the end of the arc and produced till it meets the cotangent.

Thus the arc AP being less than a quadrant, OI is its cosecant. Also OI' is the cosecant of the arc ABP' greater than a quadrant and less than a semicircle. And OI , OI' are the cosecants respectively of the arcs AP' , AP'' terminating in the third and fourth quadrants.

(48.) To trace the changes in magnitude of the cosecant.

When P coincides with A , OPI falls in the direction of OA which is parallel to BI , and therefore at the beginning of the arc the cosecant is greater than any assignable line. In the first quadrant, as the arc AP increases, the line OI cuts BI in points continually nearer

to B , *i.e.* as the arc increases, the cosecant decreases, till at the end of the first quadrant it becomes equal to radius. As the arc increases from a quadrant to a semicircle, IB increases (42) and the cosecant increases, till as the arc approximates to a semicircle, the cosecant approximates to a line greater than any that can be assigned. If the arc terminates in the third quadrant, as the arc increases, the cosecant decreases, till it becomes equal to radius; and during the fourth quadrant it increases again, till it approximates to a line greater than any that can be assigned.

Hence the cosecant is never less than radius.

(49.) To trace the changes in the algebraic sign of the cosecant.

The cosecants of arcs terminating in the first quadrant being *reckoned* positive, and drawn from the centre through the ends of the arcs; the cosecants of arcs ending in the other quadrants will be positive or negative according as they are drawn from the centre through the ends of the arcs, or from the ends of the arcs through the centre. And hence the cosecants of arcs ending in the two first quadrants have a positive sign, and those of arcs ending in the two last a negative sign.

(50.) COR. 1. The cosecant of an arc is equal to the cosecant of its supplement, and they have the same sign. If $\therefore A$ be any arc, cosec. $(180 - A) = \text{cosec. } A$, and cosec. $(90 + A) = \text{cosec. } (90 - A)$.

(51.) COR. 2. The cosecant of an arc is equal to the secant of its complement; for OI the cosecant of the arc AP is the secant of BP the complement of AP . Hence cosec. $A = \text{sec. } (90 - A)$ and cosec. $(90 - A) = \text{sec. } A$.

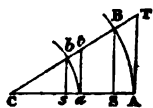
(52.) Cosec. $45^\circ = R \sqrt{2}$. (33.)

(53.) In trigonometrical calculations, when the quantity required is deduced in terms of the sine, the case is ambiguous, since the sine of any arc and its supplement (19) have the same algebraic sign. The ambiguity may however be removed by some other consideration. But if an expression be deduced in terms of a cosine or a tangent, there will be no ambiguity, since a positive cosine (36) or tangent (26) denotes an arc less than 90° , and a negative cosine or tangent indicates an arc greater than 90° and less than 180° . In calculations arcs and angles are generally less than 180° .

PROP. I.

(54.) *The sines, cosines, tangents, &c. of the same angles at the centres of different circles are proportional to the radii of those circles.*

Let AB, ab be two arcs described with the centre C and radii CA, Ca , subtending the same angle at C . Draw BS the sine and AT the tangent of AB ; bs the sine and at the tangent of ab . Then the triangles CBS, Cbs being similar,



$$BS : bs :: CB : Cb,$$

i. e. the sines are proportional to the radii.

$$\text{Also } CS : Cs :: CB : Cb,$$

or the cosines are proportional to the radii.

$$\text{And since } CS : Cs :: CB : Cb :: CA : Ca;$$

$$\therefore (\text{Eucl. v. 19.}) AS : as :: CA : Ca,$$

or the versed sines are proportional to the radii.

$$\text{Again from the similar triangles } ATC, atC,$$

$$AT : at :: AC : aC.$$

or the tangents are proportional to the radii.

and $CT : Ct :: AC : aC$,
or the secants are proportional to the radii.

(55.) COR. 1. Hence $\frac{BS}{BC} = \frac{bs}{bc}$ and $\therefore$ if S and s represent the sines of the same angle to radii R and r , $\frac{S}{R} = \frac{s}{r}$, and $S = \frac{R}{r} \times s$. If $r = 1$, $S = Rs$, and $\frac{S}{R} = s$. And the same is true of any other corresponding lines. Hence if any expressions be calculated for rad. = 1; the corresponding expressions in a circle whose radius is R will be determined by taking each line an R^{th} part of the former.

(56.) COR. 2. If a given radius be divided into any number of equal parts, and the sines &c. of every angle be given in such parts, the sines &c. of any given angle may be found, corresponding to another given radius.

PROP. II.

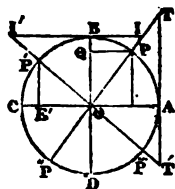
(57.) *To find the relation between the sines, cosines, tangents &c. of the same angle.*

Let AP be any arc, PE and OE its sine and cosine, draw the tangent AT , secant OT and cotangent Bl . Then PE being parallel to AT , the triangles OPE , OAT are equiangular;

$$\therefore OE : EP :: OA : AT,$$

$$\text{or cos. : sine} :: \text{rad. : tang.}$$

$$\therefore \text{tang.} = \text{rad.} \times \frac{\text{sine}}{\text{cos.}} = \frac{\text{sine}^*}{\text{cos.}}, \text{ if } R=1.$$



* In the second quadrant the sine is positive and the cosine negative.
Hence

Also from the same triangles, $OE : OP :: OA : OT$,
 or $\cos : R :: R : \sec. \therefore \sec. = \frac{R^2}{\cos.} = \frac{1^*}{\cos.}$ if $R = 1$.

Also from the similar triangles OPE, BOI ,
 $PE : PO :: OB : OI$, or $\sin. : R :: R : \operatorname{cosec.}$;

$$\therefore \operatorname{cosec.} = \frac{R^2}{\sin.} = \frac{1^\dagger}{\sin.} \text{ (if } R = 1\text{)}.$$

And from the same triangles, $PE : EO :: BO : BI$,
 or $\sin. : \cos. :: R : \cotang.$;

$$\therefore \cotang. = R \times \frac{\cos.}{\sin.} = \frac{\cos.^\ddagger}{\sin.}, \text{ if } R = 1.$$

Also from the similar triangles OAT, OBI ,

$TA : AO :: OB : BI$, or $\tan. : R :: R : \cotang.$;

$$\therefore \cotang. = \frac{R^2}{\tan.} = \frac{1}{\tan.}, \text{ if } R = 1.$$

(58.) COR. 1. If either the sine or cosine of an arc A be known, all the rest may be found.

Hence $\tan. (180 - A) = \frac{\sin. A}{-\cos. A} = -\frac{\sin. A}{\cos. A} = -\tan. A$, which confirms what was shewn in Art. 26.

When the sine and cosine have the same algebraic sign, the tangent will have a positive sign, and when different, a negative one. Also since $\cos. 90^\circ = 0$ and $\sin. 90^\circ = R$, $\tan. 90^\circ = \frac{1}{0} = \infty$, which confirms what was shewn in Art. 25.

* Hence the secant will have the same algebraic sign as the cosine.

† Hence the cosecant will have the same algebraic sign as the right sine.

‡ Hence the cotangent will have a positive sign, when the cosine and sine have the same algebraic signs, and a negative one, when different; or as also appears from the next equation, it will have the same sign as the tangent.

For (Eucl. i. 47.) $PO^2 = PE^2 + EO^2$, i. e. $R^2 = \sin^2. A + \cos^2. A$, $\therefore \sin. A = \sqrt{(R^2 - \cos^2. A)}$, and $\cos. A = \sqrt{(R^2 - \sin^2. A)}$, which values may be substituted in the preceding expressions.

(59.) COR. 2. Since $OT^2 = OA^2 + AT^2$, $\sec^2. A = R^2 + \tan^2. A$, and $\therefore \sec. A = \sqrt{(R^2 + \tan^2. A)}$, whence $\cos. A = \frac{R}{\sqrt{(R^2 + \tan^2. A)}}$, and $\sin. A = \frac{R \times \tan. A}{\sqrt{(R^2 + \tan^2. A)}}$.

(60.) COR. 3. Also $\tan. A = \sqrt{(\sec^2. A - R^2)}$, and $\operatorname{cosec}^2. A = R^2 + \cotang^2. A$.

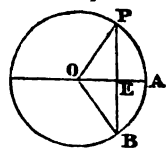
(61.) COR. 4. If the radius be represented by unity, the expressions become, $\sin. A = \sqrt{(1 - \cos^2. A)}$, $\cos. A = \sqrt{(1 - \sin^2. A)}$, $\sec. A = \sqrt{(1 + \tan^2. A)}$, $\tan. A = \sqrt{(\sec^2. A - 1)}$, $\cos. A = \frac{1}{\sqrt{(1 + \tan^2. A)}}$

and $\sin. A = \frac{\tan. A}{\sqrt{(1 + \tan^2. A)}}$, $\tan. A \times \cotang. A = 1$.

PROP. III.

(62.) *The sine of any arc is equal to half the chord of double the arc.*

Let the arc PB be double of PA . Join OA, PB intersecting each other in E . Since PB is double of PA , $PA = AB$, and $\therefore$ the angles POE, BOE are equal, but OPE is equal to OBE , and



OE is common to the triangles OPE, OBE , $\therefore$ (Eucl. i. 26.) PE is equal to EB , and $\therefore$ is half of PB ; and the angles OEP, OEB are equal, i. e. each is a right angle;

hence PE is the sine of the arc AP , and is half the chord of double the arc.

(63.) COR. 1. Hence the $\sin. 30^\circ = \frac{1}{2}$ radius. For (16) the chord of $60^\circ =$ radius.

$$\text{Also } \cos. 30^\circ = \sqrt{(R^2 - \sin^2. 30)} = \sqrt{\left(R^2 - \frac{R^2}{4}\right)} = \frac{R\sqrt{3}}{2};$$

$$\text{tang. } 30^\circ = \frac{R}{\sqrt{3}}, \text{ and } \sec. 30^\circ = \frac{2R}{\sqrt{3}}.$$

Hence $\sec. 30^\circ$ is double the $\text{tang. } 30^\circ$.

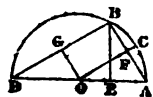
$$(64.) \text{ COR. 2. } \sin. 60^\circ = \frac{R\sqrt{3}}{2} (38), \text{ and } \cos. 60^\circ = \frac{R}{2}.$$

$$\text{Versed sine } 60^\circ = \frac{R}{2}, \text{ tang. } 60^\circ = R\sqrt{3}, \text{ and } \sec. 60^\circ = 2R.$$

PROP. IV.

(65.) *The diameter is to the versed sine of any arc as the square of radius is to the square of the sine of half that arc.*

Let ABD be a semicircle, AD its diameter, ACB any arc, whose chord is AB and versed sine AE . Join DB , and from the centre O draw OFC perpendicular to the chord AB , and meeting the circumference in C ; then AB is bisected in F , and the arc AB in C ; and AF is the sine of $\frac{1}{2} AB$. Also the triangles DAB , OAF are similar,



$$\text{whence } DA : AB :: OA : AF.$$

But (Eucl. vi. 8. and v. Def. 10.)

$$\begin{aligned} DA : AE &:: DA^2 : AB^2, \\ \therefore DA : AE &:: OA^2 : AF^2. \end{aligned}$$

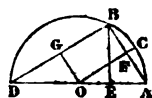
(66.) COR. 1. Hence the rectangle contained by the radius and the versed sine of any arc is equal to twice the square of the sine of half that arc.

(67.) COR. 2. And the square of the sine of any arc varies as the versed sine of double that arc.

PROP. V.

(68.) *Radius is to the cosine of any arc as twice the sine of that arc is to the sine of double that arc.*

The same construction remaining, OF is the cosine of AC ; and the triangles OAF , BAE , having the angle at A common, and right angles at F and E , are equiangular;



whence $OA : OF :: (BA =) 2AF : BE$.

(69.) COR. 1. Hence the rectangle contained by the sine and cosine of any arc $\propto$ the sine of double that arc; for $BE \propto 2AF \times OF \propto AF \times OF$, i. e. $\sin. 2A \propto \sin. A \times \cos. A$.

Also $R \times \sin. A = 2 \sin. \frac{1}{2} A \times \cos. \frac{1}{2} A$.

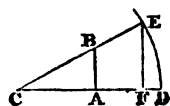
(70.) COR. 2. If from O , OG be drawn perpendicular to DB , it bisects it, $\therefore DB = 2BG = 2OF = 2 \cos. \frac{1}{2} AB$; or the chord of an arc is equal to twice the cosine of half the supplemental arc.

PROP. VI.

(71.) *In any right-angled triangle, radius is to the sine of either of the acute angles, as the hypotenuse is to the side opposite to that angle.*

Let ABC be a right-angled triangle, having the angle at

A a right angle. With the centre *C*, and radius *CD* = the tabular radius, describe an arc *DE*, meeting *CB* produced in *E*; it will be a measure of the angle *C*.



Let fall the perpendicular *EF*, which will be equal to the tabular sine of the angle *C*. The triangles *CEF*, *CBA* being similar,

$$CE : EF :: CB : BA,$$

$$\text{or rad.} : \sin. C :: CB : BA.$$

In the same manner, if *BA* be produced till it be equal to the tabular radius, and a circular arc be described with *B* as a centre, it may be shewn that $\text{rad.} : \sin. B :: BC : CA$.

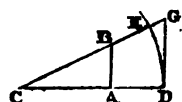
(72.) COR. 1. $\text{Rad.} : \cos. C :: CB : CA$;
and $\text{rad.} : \cos. B :: CB : BA$.

(73.) COR. 2. If $\text{rad.} = 1$, $BA = CB \times \sin. C$, or $CB \times \cos. B$,
and $CA = BC \times \sin. B$, or $BC \times \cos. C$.

PROP. VII.

(74.) *In any right-angled triangle, radius is to the tangent of either of the acute angles as the side adjacent to that angle is to the opposite side.*

Let *CAB* be a right-angled triangle, having the angle at *A* a right angle. With the centre *C*, and radius *CD* = the tabular radius, describe a circular arc *DE*



meeting *CB* produced in *E*; then *DE* is the measure of the angle *C*; and drawing *DG* perpendicular to *DC* meeting *CE* in *G*, *DG* will be the tangent of *DE* or of the angle *ACB*. Now the triangles *DGC* *ABC* being similar,

$$DC : DG :: AC : AB,$$

$$\text{or rad. : tang. } C :: AC : AB,$$

In the same manner it may be shewn that $\text{rad. : tang. } B :: AB : AC$.

$$(75) \text{ COR. Rad. : sec. } C :: AC : BC,$$

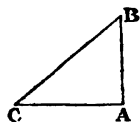
$$\text{and rad. : sec. } B :: AB : BC,$$

PROP. VIII.

(76.) *Having given two parts of a right-angled triangle; to find the other parts.*

This resolves itself into four cases.

1. Having given the hypotenuse CB , and the angle C ; to find the rest.



Since (71) tabular radius : $\sin. C :: BC : BA$, and the angle C being given, its sine may be found from the tables, the three first terms in the foregoing proportion are known, $\therefore BA = \frac{BC \times \sin. C}{R}$ may be determined.

Also $AC = \sqrt{(BC^2 - BA^2)} = \sqrt{\{(BC + BA) \cdot (BC - BA)\}}$ may be computed, or three terms in the proportion (72) $\text{rad. : cos. } C :: BC : CA$ being known, $CA = \frac{BC \times \cos. C}{R}$ may be also determined.

And the angle $B = 90^\circ - C$.

Ex. Suppose the angle $C = 30^\circ$ and $BC = 10$ chains.

Then $\sin. 30^\circ = \frac{1}{2} R$, and $\cos. 30^\circ = \frac{\sqrt{3}}{2} R$, $\therefore BA =$

$$\frac{10 \times \frac{1}{2} R}{R} = 5, \text{ and } CA = \sqrt{(15 \times 5)} = 5\sqrt{3}, \text{ or } CA =$$

$$\frac{10 \times \sqrt{3} \cdot R}{2 \cdot R} = 5\sqrt{3}, \text{ and } B = 90^\circ - 30^\circ = 60^\circ.$$

By the same method the triangle may be solved, if the hypotenuse BC and the angle B be given; the point B being now considered as the centre.

2. Having given one side AC and the angle C ; to find the rest.

Since (74) tabular radius : tang. $C :: AC : AB$, and the angle C being given, its tangent may be found from the tables; the three first terms in the preceding proportion are known, and $\therefore BA = \frac{AC \times \text{tang. } C}{R}$ may be determined.

Also (75) tabular radius : sec. $C :: AC : CB$, in which proportion three terms being known, the fourth, $CB = \frac{AC \times \text{sec. } C}{R}$ may be determined.

Or if in the tables the secants are not computed, tab. cos. $C : R :: AC : CB$, $\therefore CB = \frac{AC \times R}{\cos. C}$, which may be determined.

Also the angle $B = 90^\circ - C$.

Ex. If $AC = 6$ chains, and the angle $C = 60^\circ$;

$$\text{tang. } 60^\circ = \sqrt{3} \cdot R, \therefore AB = \frac{6 \times \sqrt{3} \cdot R}{R} = 6\sqrt{3}.$$

$$\text{Also sec. } 60^\circ = 2R, \therefore CB = \frac{6 \times 2R}{R} = 12,$$

$$\text{or cos. } 60^\circ = \frac{1}{2}R, \therefore CB = \frac{6 \times R}{\frac{1}{2}R} = 12;$$

and the angle $B = 90^\circ - 60^\circ = 30^\circ$.

In the same manner the triangle might be solved, if AB and the angle B were given.

If AC and the angle B were given; since the angle $C = 90^\circ - B$, C is also known, and $\therefore$ this is reducible to the second case.

3. Having given the hypotenuse BC , and one side AC ; to find the rest.

Here (72) $BC : CA :: \text{rad.} : \cos. C$, in which proportion the three first terms being known, $\cos. C = \frac{R \times CA}{BC}$ may be computed, and $\therefore$ the angle C determined from the tables.

Also the angle $B = 90^\circ - C$ and is $\therefore$ known.

And $AB = \sqrt{(BC^2 - CA^2)} = \sqrt{\{(BC + CA).(BC - CA)\}}$ which may be found;

Or since $\text{rad.} : \sin. C :: BC : BA$, $\therefore BA = \frac{BC \times \sin. C}{R}$ may be computed.

In the same manner the triangle may be solved, if BC and AB be given.

4. Having given the two sides BA and AC containing the right angle; to find the rest.

Here (74) $AC : AB :: \text{rad.} : \text{tang. } C$, in which proportion the three first terms being known, the $\text{tang. } C = \frac{AB \times R}{AC}$ may be computed, and the angle C determined from the tables.

Also the angle $B = 90^\circ - C$.

And tabular radius : $\sec. C :: AC : CB$, whence $CB = \frac{AC \times \sec. C}{R}$ may be determined;

or $CB = \frac{AC \times R}{\cos. C}$ may be computed.

(77.) By Case 2. the height of any object may be determined, the base of which is accessible. From the base A , measure along the horizontal plane any length AC . At C let the angle be observed which the object subtends. Then its height $BA = \frac{AC \times \text{tang. } C}{R}$.

PROP. IX.

(78.) *The sides of any triangle are to one another as the sines of the angles opposite to them.*

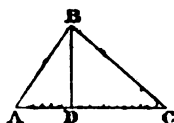
Let ABC be any triangle, BD a perpendicular let fall on AC , and

1. Let it fall within the triangle, then

$$(71) AB : BD :: \text{rad.} : \sin. A,$$

$$\text{and } BD : BC :: \sin. C : \text{rad.}$$

$$\therefore \text{ex æquo per. } AB : BC :: \sin. C : \sin. A$$



2. But if BD falls without the triangle,

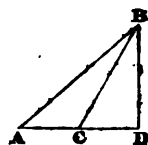
$$AB : BD :: \text{rad.} : \sin. A$$

$$\text{and } BD : BC :: \sin. BCD : \text{rad.}$$

$$\therefore AB : BC :: \sin. BCD : \sin. A$$

$$:: \sin. BCA : \sin. A, \text{ since the}$$

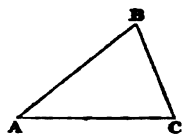
angle BCA is the supplement of BCD (20).



PROP. X.

(79.) *Having given, in any triangle, a side and an angle opposite to it, and also one other angle or one other side, the remaining sides and angles may be found.*

1. Having given the angles at A and C , and the side BC ; to find the rest.



Since the angles at A and C are given, their sines may be found from the tables, and (78) $\sin. A : \sin. C :: BC : BA$, in which proportion the three first terms being known, $BA = \frac{BC \times \sin. C}{\sin. A}$ may be determined.

Also the angle $B = 180^\circ - (A + C)$ and may $\therefore$ be found.

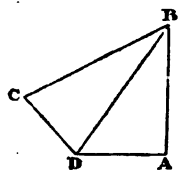
Whence also $\sin. A : \sin. B :: BC : CA$,

and $\therefore CA = \frac{BC \times \sin. B}{\sin. A}$ may be computed.

In the same manner, if the angles A and B and the side BC ; or the angles B and C and the side AC ; or the angles A and C and the side AC were given; the other sides and angles might be found.

Ex. 1. Hence may be determined the height of an object, the base of which is inaccessible.

Let AB be the height; along the horizontal base measure any distance CD , and at C let the angle BCD be observed, and at D the angles BDC , BDA ; then in the triangle BDC , the angle $CBD = 180^\circ - (BCD + BDC)$ and $\therefore$ is known;



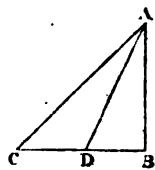
and $\sin. CBD : \sin. BCD :: CD : DB$,

$\therefore DB = CD \times \frac{\sin. BCD}{\sin. CBD}$, which may $\therefore$ be computed.

Hence in the right-angled triangle BAD , $\text{rad.} : \sin. BDA :: BD : BA$, and $\therefore BA = \frac{BD \times \sin. BDA}{\text{rad.}}$

which may $\therefore$ be determined.

Ex. 2. To determine the height of a cloud A or any object in the air; let two observers C and D in the same vertical plane take the angles of elevation ACB , ADB ($=a^\circ$ and b°) and measure CD ($=c$ yards) the distance between them; the height and the distances from them may be found; for the angle ADB being $=b^\circ$, $ACB = a^\circ$, $ADC = 180^\circ - b$, and $CAD = b^\circ - a^\circ = d$.



Hence in the triangle CAD , $\sin. CAD : \sin. C :: CD : DA$,

or $\sin. d : \sin. a :: c : DA = \frac{c \cdot \sin. a}{\sin. d}$, which is $\therefore$ known.

And $\sin. CAD : \sin. CDA :: CD : CA$,

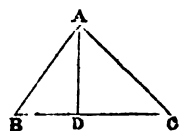
i. e. $\sin. d : \sin. (180 - b) :: c : CA = \frac{c \cdot \sin. b}{\sin. d}$, which

is also known. Hence in the right-angled triangle ADB , $\text{rad.} : \sin. ADB :: AD : AB$,

$$\therefore AB = \frac{AD \times \sin. ADB}{R} = \frac{c \cdot \sin. a \times \sin. b}{R \cdot \sin. d},$$

which may be determined.

Ex. 3. To determine the breadth of a river and the distance of an object B by its side from another object A on the opposite bank, let a line BC be measured ($=a$ yards) along its bank, and by means of a Theodolite let the angles CBA , BCA be measured ($=b^\circ$ and c°); then the angle $BAC = 180^\circ - (B + C) = d^\circ$.



And $\sin. BAC : \sin. C :: BC : AB$,

or $\sin. d : \sin. c :: a : AB = \frac{a \sin. c}{\sin. d}$, which may be

determined. Whence in the right-angled triangle ADB ,

$$\begin{aligned} \text{rad.} : \sin. B &:: AB : AD, \\ \therefore AD &= \frac{AB \cdot \sin. B}{R} = \frac{a \cdot \sin. c \times \sin. b}{R \cdot \sin. d}, \end{aligned}$$

which may be computed.

Ex. 4. From *B* the top of a tower, the angle of depression of a vessel at anchor ($HBS = a^\circ$) was observed, and at *C* the bottom of the tower, the angle of depression ($ECS = b^\circ$) was again observed. The height of the tower ($BC = c$) being known; to determine the horizontal distance *AS*, and the height *CA* of the bottom of the tower above the level of the sea.

The angle $BSC = BSA - CSA$
 $= HBS - ECS = a - b = d$ and
 $SBC = 90 - HBS = 90 - a$,

$\therefore$ in the triangle *SBC*,

$$\begin{aligned} \sin. BSC : \sin. SBC &:: BC : CS \\ \text{or } \sin. d : \cos. a &:: c : CS, \end{aligned}$$

$$\therefore CS = \frac{c \cdot \cos. a}{\sin. d}, \text{ which may be found.}$$

And in the triangle *CSA*, $\text{rad.} : \sin. CSA :: CS : CA$,

$$\therefore CA = \frac{CS \cdot \sin. CSA}{R} = \frac{c \cdot \cos. a \times \sin. b}{R \times \sin. d},$$

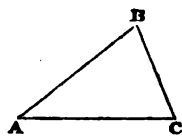
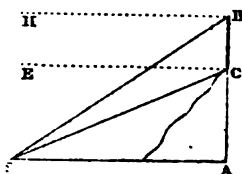
and $\text{rad.} : \cos. CSA :: CS : SA$,

$$\therefore SA = \frac{CS \times \cos. CSA}{R} = \frac{c \cdot \cos. a \times \cos. b}{R \times \sin. d}.$$

2. Having given the angle at *A* and the sides *AC*, *CB*; to find the rest.

The angle at *A* being given, its sine may be found from the tables; and (78)
 $CB : CA :: \sin. A : \sin. B$;

hence $\sin. B = \frac{CA}{CB} \times \sin. A$, and may $\therefore$ be determined.



But as the sine of an angle is equal to the sine of its supplement, the angle B may be greater or less than a right angle, unless BC be greater than AC and consequently the angle A greater than the angle B^* .

The angle B being found, the angle $C = 180 - (A + B)$, and may $\therefore$ be determined.

Also $\sin. A : \sin. C :: CB : BA$,

whence $BA = CB \times \frac{\sin. C}{\sin. A}$ may be computed.

In the same manner may be solved any case, wherein are given two sides and an angle opposite to one of them.

(80.) If all the angles of any triangle be given, the ratio of the sides to each other may be found.

For the angles A, B and C being given, their sines may be found from the tables,

and (78) $\sin. C : \sin. A :: AB : BC$,

$\sin. B : \sin. C :: AC : AB$

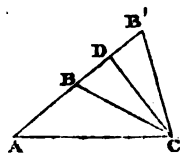
$\sin. A : \sin. B :: BC : AC$,

in which proportions the two first terms being given, the ratio of the third to the fourth in each is given.

(81.) In any rectilinear triangle, it is sufficient if three out of the six parts which belong to it are known, provided that one of these parts be a side; for if only

* This ambiguity occurs in oblique-angled triangles whenever the side opposite to the given angle is less than the side adjacent to it.

For if CB be less than AC , from C draw CD perpendicular to AB , produced if necessary. Make $DB' = DB$, and join CB' . (Eucl. i. 4.) $CB' = CB$ and the angle $CB'D$ is equal to CBD , which is the supplement of ABC . Hence there are two triangles $CBA, CB'A$, which have one angle and the two sides answering the conditions.



three angles be given, any triangle equiangular to the given one will satisfy the conditions.

LEMMA III.

(82.) *If the semi-sum of two quantities be added to their semi-difference, the aggregate will be the greater of the two ; if the semi-difference be subtracted from the semi-sum, the remainder will be the less of the two.*

Let AB and BC representing the two quantities be placed in the same straight line, which bisect in D , and cut off $AE=BC$ and $\therefore DE = DB$. Then $AB - BC = AB - AE = 2DB$, whence DB = the semi-difference of the lines ; and since AD = the semi-sum, $AD + DB = AB$ the greater, and $AD - DB = DC - DB = BC$ the less.

(83.) COR. If the semi-sum, be subtracted from the greater quantity, the remainder will be the semi-difference.

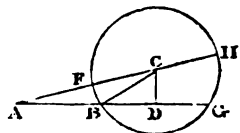
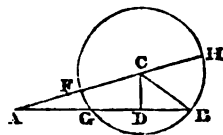
For $AB - AD = DB$, the semi-difference.

PROP. XI.

(84.) *In any triangle, if a perpendicular from the vertex be drawn upon the base ; the base will be to the sum of the sides as the difference of the sides to the difference or sum of the segments of the base made by the perpendicular, according as the perpendicular falls within or without the triangle.*

Let ABC be the proposed triangle, C the vertex, and AB the base ; draw the perpendicular CD cutting

the base or base produced in D . With the centre C , and radius CB the less of the two sides, describe a circle cutting the side AC in F , and the base or base produced in G ; produce AC to H . Then AH is equal to the sum of the sides AC , CB ; and AF to their difference. And because $BD = DG$, (Eucl.



III. 3.) AG is the difference of the segments AD , DB in one case, and their sum in the other. And (Eucl. iii. 36.) the rectangle AB , AG is equal to the rectangle AH , AF ,

$$\therefore AB : AH :: AF : AG.$$

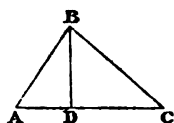
(85.) COR. Hence if the three sides of any triangle are given, the three angles may be found. For the three sides being given, the three first terms in the preceding proportion are given, and consequently the fourth, *i. e.* the difference of the segments of the base made by the perpendicular when it falls within the triangle, may be found; and the base or sum of the segments is also given, $\therefore$ the segments themselves AD , DB may be found (82). Hence in each of the right-angled triangles ACD , BCD , the two sides being known, the angles at A and B may be determined (76), and $\therefore$ the remaining angle at C (13).

But if the perpendicular fall without the triangle, the fourth term in the proportion or sum of the segments, will be found; and the base AB , which is their difference, is given, $\therefore$ the segments may be determined, and the angles as before.

PROP. XII.

(86.) *In any plane triangle, the cosine of an angle is to radius as the difference between the sum of the squares of the sides which contain that angle and the square of the third side, is to twice the rectangle contained by the two first sides.*

Let ABC be any triangle; from B let fall the perpendicular BD on AC ; then (Eucl. ii. 13.)



$$BC^2 = AB^2 + AC^2 - 2AC \times AD,$$

$$\text{or } 2AC \times AD = AB^2 + AC^2 - BC^2.$$

$$\begin{aligned} \text{Now } \cos. A : \text{rad.} :: DA : AB :: 2AC \times AD : 2AC \times AB \\ :: AB^2 + AC^2 - BC^2 : 2AC \times AB. \end{aligned}$$

Nearly in the same manner it may be shewn, if the perpendicular falls without the triangle.

(87.) COR. 1. Let a, b, c be the three sides opposite to the angles A, B, C ; then $\cos. A = \frac{b^2 + c^2 - a^2}{2bc}$,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}, \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Hence the sides of a triangle being given, the angles may be found*.

* The preceding expressions not being easy for calculation, values may be deduced for the sines, $\sin^2 A = 1 - \cos^2 A = 1 - \left(\frac{b^2 + c^2 - a^2}{2bc} \right)^2$

$$= \frac{(a+b+c) \cdot (b+c-a) \cdot (a+b-c) \cdot (a+c-b)}{4b^2c^2},$$

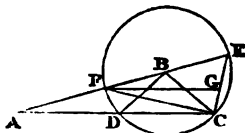
$$\text{and } \therefore \sin. A = \frac{1}{2bc} \times \sqrt{\{ (a+b+c) \cdot (a+b-c) \cdot (a+c-b) \cdot (b+c-a) \}}$$

$$= \frac{2}{bc} \times \sqrt{\{ P \cdot (P-a) \cdot (P-b) \cdot (P-c) \}}, \text{ if } 2P = a+b+c.$$

PROP. XIII.

(88.) *In any triangle, the sum of any two sides is to their difference as the tangent of the semi-sum of the angles at the base is to the tangent of their semi-difference.*

Let ABC be any triangle, whose base is AC . With the centre B , and radius BC the less of the two sides, describe a circle meeting the side AB in F and AB produced in E and AC in D . Join BD , CE , CF ; and draw FG parallel to AC .



Then since CBE , CFE are on the same base CE , (Eucl. iii. 20.) $CFE = \frac{1}{2}CBE = \frac{1}{2}(CAB + BCA)$; and since $BD = BC$, the angle $BCD = BDC$, and $\therefore DBA$ is equal to the difference of the angles BCA , BAC . Hence (Eucl. iii. 20.) FCD and $\therefore$ its equal $CFG = \frac{1}{2}(BCA - BAC)$. If now with the centre F , and radius FC , a circle be described,

$$CE : CG :: \text{tang. } CFE : \text{tang. } CFG.$$

But FG being parallel to AC one of the sides of the triangle AEC , (Eucl. vi. 2.)

$$\begin{aligned} CE : CG &:: AE : AF :: AB + BC : AB - BC, \\ \therefore AB + BC : AB - BC &:: \text{tang. } CFE : \text{tang. } CFG \\ &:: \text{tang. } \frac{1}{2}(BCA + BAC) : \text{tang. } \frac{1}{2}(BCA - BAC). \end{aligned}$$

(89.) COR. 1. Hence if the sides AB , BC of any triangle, and the angle ABC included between those sides, be given, the other sides and angles may be found.

For, from the included angle ABC , its supplement EBC may be determined; half of which is the semi-sum of the angles at the base; and their semi-difference may be found by the Proposition. Having $\therefore$ the semi-sum and semi-difference of the angles at the base, the angles

themselves BAC , BCA may be found (82). The three angles and two sides, $\therefore$ being obtained, the third side may be found, (78).

Or the side may be calculated by means of a subsidiary angle, thus

$$\begin{aligned}
 (87), \quad b^2 &= a^2 + c^2 - 2ac \cdot \cos. B \\
 &= (a-c)^2 + 2ac \cdot (1 - \cos. B) \\
 &= (a-c)^2 + 4ac \cdot \sin^2. \frac{1}{2} B \quad (70) \\
 &= (a-c)^2 \times \left\{ 1 + \frac{4ac \cdot \sin^2. \frac{1}{2} B}{(a-c)^2} \right\}.
 \end{aligned}$$

Now since tangents admit of every degree of magnitude, find an angle x whose tangent is $\frac{2\sqrt{ac} \cdot \sin. \frac{1}{2} B}{a-c}$, and

$$\therefore b = (a-c) \cdot \sqrt{(1 + \tan^2. x)} = (a-c) \cdot \sec. x = \frac{a-c}{\cos. x},$$

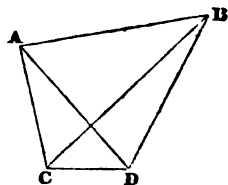
or to radius $R = \frac{(a-c) \cdot R}{\cos. x}$ which (since x is known

from the equation $\tan. x = \frac{2\sqrt{ac} \cdot \sin. \frac{1}{2} B}{a-c}$), may

easily be computed.

Ex. To determine the distance between two visible inaccessible objects.

Let A and B be the two objects. Measure a given line $CD (=a)$ along a horizontal base, from the extremities of which A and B are visible. At C observe the angles ACB , BCD , and at D the angles ADC , ADB . Then in the triangle ACD , the angle $CAD = 180^\circ - (ACD + ADC) = b$,



$$\text{and } \sin. DAC : \sin. ADC :: CD : CA$$

$$\text{or } \sin. b : \sin. c :: a : CA,$$

$$\therefore CA = \frac{a \cdot \sin. c}{\sin. b}, \text{ which may be computed.}$$

And in the triangle BCD , the angle $CBD = 180^\circ - (\angle CDB + \angle BCD) = d$, whence

$$\sin. CBD : \sin. CDB :: CD : CB,$$

$$\text{or } \sin. d : \sin. e :: a : CB,$$

$$\therefore CB = \frac{a \cdot \sin. e}{\sin. d}, \text{ which may be computed.}$$

And in the triangle ACB , $AC + CB : AC \sim CB$

$$:: \tan. \frac{1}{2}(BAC + ABC) : \tan. \frac{1}{2}(BAC \sim ABC)$$

$$:: \tan. \frac{1}{2}(180 - ACB) : \tan. \frac{1}{2}(BAC \sim ABC),$$

$$\therefore \tan. \frac{1}{2}(BAC \sim ABC) = \frac{AC \sim CB}{AC + CB} \times \cot. \frac{1}{2} ACB,$$

which may be calculated. Hence from the tables the difference of the angles may be found, and the sum being known, the angles themselves may be determined

$$(82). \text{ And } \sin. ABC : \sin. ACB :: AC : AB,$$

$$\therefore AD = \frac{\sin. ACB}{\sin. ABC} \times AC \text{ may be calculated.}$$

Or AB may be ascertained as before by means of the subsidiary angle x ; and from the three sides being known the two remaining angles may be determined.

(90). COR. 2. Hence also the sum of the sines of two angles is to the difference of the sines, as the tangent of half the sum of those angles is to the tangent of half their difference.

$$\text{For } AC : BC :: \sin. B : \sin. A,$$

$$\therefore AC + BC : AC \sim BC :: \sin. B + \sin. A : \sin. B \sim \sin. A$$

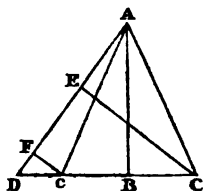
$$\text{and } \therefore \sin. A + \sin. B : \sin. A \sim \sin. B :: \tan. \frac{1}{2}(A + B)$$

$$[: \tan. \frac{1}{2}(A \sim B).]$$

PROP. XIV.

(91.) *The sum of the tangents of any two angles is to their difference as the sine of their sum is to the sine of their difference.*

Let DAB , BAC be the angles, and through any point B in AB , draw DC perpendicular to AB . Make the angle $BAC = BAC$; then will $Ac = AC$. From C and c draw CE , cF perpendicular to AD .



Then $CE : cF :: \sin. CAD : \sin. cAD$

$:: \sin. (CAB + BAD) : \sin. (BAD \sim BAC).$

And $BD : BC :: \text{tang. } BAD : \text{tang. } BAC,$

$\therefore CD : cD :: \text{tang. } BAD + \text{tang. } BAC :$

$[\text{tang. } BAD \sim \text{tang. } BAC.$

And cF , CE being perpendicular to AD , are parallel,

whence $CE : cF :: CD : cD$

or $\sin. (DAB + BAC) : \sin. (DAB \sim BAC) ::$

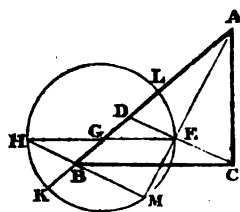
$\text{tang. } DAB + \text{tang. } BAC : \text{tang. } DAB \sim \text{tang. } BAC.$

PROP. XV.

(92.) *In any plane triangle, the rectangle contained by any two sides is to the rectangle contained by the excesses of half the perimeter above each of them respectively as the square of radius is to the square of the sine of half the angle contained by those sides.*

Let ABC be any triangle, BC its base, AB the greater of the sides and AC the less; $P = \frac{1}{2}$ the perimeter of the triangle, then $AB \times AC : (P - AB) \cdot (P - AC) :: R^2 : \sin^2. \frac{1}{2} A.$

In AB take $AD = AC$, join DC ; and draw AE perpendicular to DC , and EG parallel to BC , cutting AB in G . With the centre G , and radius GE , describe a circle cutting AB in L , and AB and EG produced in K and H . Join HB ; and produce AE , HB till they meet in M .



Since $AD = AC$, and the angles at E are right angles, the squares of AE , ED are equal to the squares of AE , EC ; and $\therefore ED = EC$, or $DC = 2DE$. Hence by similar triangles DGE , DBC , $BC = 2GE = EH$, and BC is also parallel to EH , $\therefore HBM$ is parallel to CED , and (Eucl. i. 29.) the angle $BME = DEA$, or it is a right angle; and HE being a diameter, M is a point in the circle.

Now (71) $AB : BM :: \text{rad.} : (\sin. DAE =) \sin. \frac{1}{2} BAC$,

and $AD : DE :: \text{rad.} : \sin. \frac{1}{2} BAC$,

$\therefore \text{comp. } AB \times AD : BM \times DE :: R^2 : \sin^2. \frac{1}{2} BAC$.

But from the similar triangles DGE , DBC , $DB = 2DG$, to each of these equals $AD + AC = 2AD$,

and $BA + AC = 2AG$,

to each of which equals add $BC = 2GE = 2GK$,

$\therefore AB + AC + BC = 2AK$, $\therefore AK = P$,

and $BK = AK - AB = P - AB$,

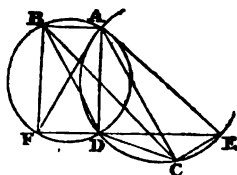
and $BL = DK = AK - AD = P - AC$.

And the rectangle $BM \times DE = BM \times EC = BM \times BH$ = (Eucl. iii. 35.) $BK \times BL = (P - AB) \cdot (P - AC)$, hence $AB \times AC : (P - AB) \cdot (P - AC) :: R^2 : \sin^2. \frac{1}{2} BAC$.

(93.) COR. 1. Hence if the three sides of any triangle be given, the angles may be found. For if the sides opposite to the angles A , B , C be a , b , c , respectively,

then $\sin^2 \frac{1}{2} A = r^2 \times \frac{(p-c) \cdot (p-b)}{cb}$, may be determined, or the sine of half the vertical angle, and consequently the vertical angle itself may be found. Hence the other angle (78).

Ex. Three points A, B, C being given in a plane, to determine the position of a fourth D , where the angles ADB, ADC , subtended by A and B, A and C are a° and b° respectively.



On AB describe a segment of a circle containing an angle a° , and on AC a segment containing an angle b° ; the point of intersection D is the point required. Let

$$AB=c, AC=d, BC=e, \sin \frac{1}{2} BAC = \frac{R \sqrt{(p-c) \cdot (p-d)}}{\sqrt{cd}},$$

which may be computed, p being $= \frac{1}{2} \cdot (c+d+e)$. Draw the diameter AF , and join FB . In the triangle BAF , $AF =$

$$\frac{BA \times R}{\sin. BFA} = \frac{c \times R}{\sin. a} \text{ may be found.}$$

Draw the diameter AE , and join CE . In the triangle ACE ,

$$\text{the angle } CAE = ADC - ADE = b - 90^\circ = g,$$

$$\text{and } AE = \frac{R \times AC}{\cos. CAE} = \frac{R \cdot AC}{\cos. g}.$$

Join DE, DF , they are in the same straight line (ii. 5.)

Hence in the triangle FAE , AE and AF having been computed, and the angle $FAE = BAC + CAE - FAB$

$$= h, \text{ and } \tan g. \frac{1}{2} (EFA \sim AEF) = \frac{AE \sim AF}{AE + AF} \times \cotang.$$

$\frac{1}{2} FAE$; whence the angle AFE may be determined.

And in the right-angled triangle DAF , $AD =$

$\frac{AF \times \sin. AFD}{\text{rad.}}$ may be found; which line and the angle

BAD determine the position of D .

(94.) COR. 2. Also $AB : AM :: \text{rad.} : \cos. \frac{1}{2} BAC$,

and $AD : AE :: \text{rad.} : \cos. \frac{1}{2} BAC$,

$\therefore AB \times AD : AM \times AE :: R^2 : \cos^2. \frac{1}{2} BAC$.

and since $AD=AC$, and $AM \times AE = AK \times AL = P \times [(P - BC)]$

$AB \times AC : P \cdot (P - BC) :: R^2 : \cos^2. \frac{1}{2} BAC$,

$$\therefore \cos. \frac{1}{2} A = r \cdot \sqrt{\left\{ \frac{p \times (p - a)}{bc} \right\}}.$$

(95.) COR. 3. Also $AE : ED :: R : \text{tang.} \frac{1}{2} BAC$,

and $AM : MB :: R : \text{tang.} \frac{1}{2} BAC$,

$\therefore AE \times AM : ED \times MB :: R^2 : \text{tang}^2. \frac{1}{2} BAC$,

or $P \cdot (P - BC) : (P - AB) \cdot (P - AC) :: R^2 : \text{tang}^2. \frac{1}{2} BAC$,

$$\therefore \text{tang.} \frac{1}{2} A = r \cdot \sqrt{\left\{ \frac{(p - b) \cdot (p - c)}{p \cdot (p - a)} \right\}}.$$

PROP. XVI.

(96.) *Having given the sines and cosines of two arcs; to find the sine of an arc which is equal to their sum.*

Let AB, AC be the two arcs; take AD, AE their doubles; draw the diameter AF . Join AD, DF, AE, EF, ED .

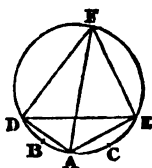
Then the chord $AD = 2 \cdot \sin. AB$ (62)

$AE = 2 \cdot \sin. AC$,

$DE = 2 \cdot \sin. BC$,

and $DF = 2 \cos. \frac{1}{2} AD = 2 \cos. AB$ (70).

$EF = 2 \cos. AC$.



Now $AF \times ED = AD \times FE + AE \times FD$ (Eucl. vi. D.)

or $2 \text{ rad.} \times 2 \sin. BC = 2 \sin. AB \times 2 \cos. AC$

$+ 2 \sin. AC \times 2 \cos. AB,$

$\therefore \text{rad.} \times \sin. BC = \sin. AB \times \cos. AC + \sin. AC \times \cos. AB.$

(97.) COR. 1. If $\text{rad.} = 1$, $AB = a$, $AC = b$,

$\sin. (a + b) = \sin. a \times \cos. b + \sin. b \times \cos. a.$

(98.) COR. 2. If $a = b$, $\sin. 2a = 2 \sin. a \times \cos. a.$

Hence also $\sin. 3a = 3 \sin. a - 4 \sin^3. a$

$\sin. 4a = (4 \sin. a - 8 \sin^3. a) \cdot \cos. a$

$\sin. 5a = 5 \sin. a - 20 \sin^3. a + 16 \sin^5. a.$

PROP. XVII.

(99.) *Given the sines and cosines of two arcs; to find the sine of an arc which is equal to their difference.*

Let AB, AC be the two arcs; take AD, AE the doubles of them; draw the diameter AF ;

join AE, AD, FE, FD, DE

Then the chord $AD = 2 \cdot \sin. AB$ (62)

$AE = 2 \cdot \sin. AC$

$DE = 2 \cdot \sin. BC,$

and $FD = 2 \cdot \cos. AB$ (70)

$EF = 2 \cdot \cos. AC;$

and $AF \times ED = AD \times EF - AE \times FD$ (Eucl. vi. D.)

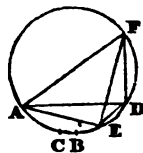
or $2 \text{ rad.} \times 2 \sin. BC = 2 \sin. AB \times 2 \cos. AC -$

$[2 \sin. AC \times 2 \cos. AB,$

$\therefore \text{rad.} \times \sin. BC = \sin. AB \times \cos. AC - \sin. AC \times \cos. AB.$

(100.) COR. If $\text{rad.} = 1$, and $AB = a$, $AC = b$,

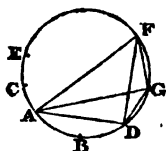
$\sin. (a - b) = \sin. a \times \cos. b - \sin. b \times \cos. a.$



PROP. XVIII.

(101.) *Given the sines and cosines of two arcs; to find the cosine of an arc which is equal to their sum.*

Let AB, AC be the two arcs. Take AD, AE the doubles of them. Draw the diameter AF , and make $FG = AE$. Join FG, GD, DA, AG, FD . Then GD is the supplement of DE , and $\therefore$ its chord $DG = 2 \cos. \frac{1}{2} DE = 2 \cos. BC$ (70).



$$DF = 2 \cos. AB.$$

$$\text{and } AG = 2 \cos. \frac{1}{2} FG = 2 \cos. \frac{1}{2} AE = 2 \cos. AC.$$

$$AD = 2 \sin. AB,$$

$$\text{and } FG = 2 \sin. \frac{1}{2} FG = 2 \sin. AC.$$

$$\text{And } AF \times DG = FD \times AG - AD \times FG \text{ (Eucl. vi. D.)}$$

$$\text{or } 2 \text{ rad.} \times 2 \cos. BC = 2 \cos. AB \times 2 \cos. AC - 2 \sin.$$

$$[AB \times 2 \sin. AC.$$

$$\therefore \text{rad.} \times \cos. BC = \cos. AB \times \cos. AC - \sin. AB \times \sin. AC.$$

$$(102.) \text{ COR. 1. If rad.} = 1, AB = a, AC = b,$$

$$\cos. (a + b) = \cos. a \times \cos. b - \sin. a \times \sin. b.$$

$$(103.) \text{ COR. 2. If } a = b, \cos. 2a = \cos^2 a - \sin^2 a$$

$$= 2 \cos^2 a - 1,$$

$$\text{or } \cos. 2a = 1 - 2 \sin^2 a$$

$$\text{Also } \cos. 3a = 4 \cos^3 a - 3 \cos. a$$

$$\cos. 4a = 8 \cos^4 a - 8 \cos^2 a + 1.$$

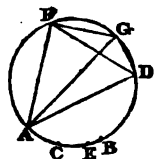
$$\cos. 5a = 16 \cos^5 a - 20 \cos^3 a + 5 \cos. a$$

$$\cos. 6a = 32 \cos^6 a - 48 \cos^4 a + 18 \cos^2 a - 1.$$

PROP. XIX.

(104.) *Given the sines and cosines of two arcs; to find the cosine of an arc which is equal to their difference.*

Let AB, AC be the two arcs, take AD, AE the doubles of them; draw the diameter AG , and make GF (on the opposite side) $= AE$. Join AD, AF, DF, DG, FG . The arc DGF is the supplement of DE , which $\therefore$ is equal to $AD - AE = 2AB - 2AC = 2BC$, and $\therefore DF = 2 \cos. \frac{1}{2} DE = 2 \cos. BC$.



$$DG = 2 \cos. AB \quad (70) \qquad AD = 2 \sin. AB \quad (62)$$

$$AF = 2 \cos. AC \qquad FG = 2 \sin. AC.$$

Now $AG \times DF = DG \times AF + AD \times GF$ (Eucl. vi. D.)

$$\text{or } 2 \text{ rad.} \times 2 \cos. BC = 2 \cos. AB \times 2 \cos. AC + 2 \sin. AB$$

$$[\times 2 \sin. AC.$$

$$\therefore \text{rad.} \times \cos. BC = \cos. AB \times \cos. AC + \sin. AB \times \sin. AC,$$

(105.) COR. If $\text{rad.} = 1$, $AB = a$, $AC = b$,

$$\cos. (a - b) = \cos. a \times \cos. b + \sin. a \times \sin. b.$$

(106.) It may perhaps be objected that the preceding propositions are proved only when the arcs (a) and (b) or even $(a + b)$ are less than quadrants. Assuming them to be proved within such a limit that (a) does not exceed a value α , and (b) a value β , it may be proved by means of what has been shewn before, that the values of the sines and cosines of the sums are equally true, when (a) does not exceed $90^\circ + \alpha$ and (b) does not exceed β .

For let $a = 90^\circ + m$, $\therefore m = a - 90^\circ$, which is less than 90° .

$$\text{Now } \sin. (90^\circ + m + b) = \sin. \{90^\circ - (m + b)\}$$

$$= \cos. (m + b)$$

$$= \cos. m \times \cos. b - \sin. m \times \sin. b$$

$$= \sin. (90^\circ - m) \times \cos. b - \cos. (90^\circ - m) \times \sin. b$$

$$= \sin. (180^\circ - a) \times \cos. b - \cos. (180^\circ - a) \times \sin. b$$

$$= \sin. a \times \cos. b + \cos. a \times \sin. b.$$

$$\begin{aligned}
 \text{Also } \cos. (90 + m + b) &= -\cos. \{90 - (m + b)\} \\
 &= -\sin. (m + b) \\
 &= -\sin. m \times \cos. b - \cos. m \times \sin. b \\
 &= -\cos. (90 + m) \times \cos. b - \sin. (90 - m) \times \sin. b \\
 &= -\cos. (180 - a) \times \cos. b - \sin. (180 - a) \times \sin. b \\
 &= \cos. a \times \cos. b - \sin. a \times \sin. b.
 \end{aligned}$$

Hence $\therefore$ these expressions which were demonstrated for (a) less than a and (b) less than β , are also true when (a) does not exceed $90 + a$ and (b) does not exceed β . In the very same manner from the preceding, it might be proved that they are true, when (a) does not exceed $90 + a$ and (b) $90 + \beta$, and so on, i.e. they are true whatever be the values of (a) and (b). The same is equally applicable to the values of $\sin. (a - b)$ and $\cos. (a - b)$.

$$(107.) \text{ Since } \sin. (30^\circ + a) = \sin. 30^\circ \times \cos. a + \cos. 30^\circ \times \sin. a,$$

$$\text{and } \sin. (30^\circ - a) = \sin. 30^\circ \times \cos. a - \cos. 30^\circ \times \sin. a,$$

$$\therefore \sin. (30^\circ + a) + \sin. (30^\circ - a) = 2 \sin. 30^\circ \times \cos. a \\ = \cos. a \quad (63).$$

and $\sin. (30^\circ + a) = \cos. a - \sin. (30^\circ - a)$. If $\therefore$ the sines and cosines of all arcs less than 30° be known, the sines of all arcs above 30° and less than 60° might be found by subtraction.

$$(108.) \text{ Since } \sin. (60^\circ + a) = \sin. 60^\circ \times \cos. a + \cos. 60^\circ \times \sin. a,$$

$$\text{and } \sin. (60^\circ - a) = \sin. 60^\circ \times \cos. a - \cos. 60^\circ \times \sin. a,$$

$\therefore \sin. (60^\circ + a) - \sin. (60^\circ - a) = 2 \cos. 60^\circ \times \sin. a = \sin. a$,
and $\sin. (60^\circ + a) = \sin. a + \sin. (60^\circ - a)$. If $\therefore$ the sines of arcs less than 60° be known, the sines of arcs greater than 60° may be found, by addition.

(109.) Let $a = nA$ and $b = A$.

Then $\sin. (n + 1). A = \sin. nA \times \cos. A + \cos. nA$
 $[\times \sin. A,$

and $\sin. (n - 1). A = \sin. nA \times \cos. A - \cos. nA \times \sin. A,$

$\therefore \sin. (n + 1). A + \sin. (n - 1). A = 2 \sin. nA \times \cos. A^*,$

and $\sin. (n + 1). A = 2 \sin. nA \times \cos. A - \sin. (n - 1). A.$

Also $\sin. (n + 1). A = \sin. (n - 1). A + 2 \sin. A \times \cos. nA.$

and $\cos. (n + 1). A = 2 \cos. nA \times \cos. A - \cos. (n - 1). A$

or $= \cos. (n - 1). A - 2 \sin. nA \times \cos. A.$

(110.) If in the equations (97) and (100)

$a + b = A$ and $a - b = B$, then $a = \frac{1}{2}(A + B)$ and $b = \frac{1}{2}(A - B)$

and $\sin. (a + b) = \sin. a \times \cos. b + \cos. a \times \sin. b$

$\sin. (a - b) = \sin. a \times \cos. b - \cos. a \times \sin. b,$

$\therefore \sin. (a + b) + \sin. (a - b) = 2 \sin. a \times \cos. b$

and $\sin. (a + b) - \sin. (a - b) = 2 \cos. a \times \sin. b,$

$\therefore \sin. A + \sin. B = 2 \sin. \frac{1}{2}(A + B) \times \cos. \frac{1}{2}(A - B)$

and $\sin. A - \sin. B = 2 \cos. \frac{1}{2}(A + B) \times \sin. \frac{1}{2}(A - B)$

Hence also $\sin. A + \sin. B = \frac{\cos. \frac{1}{2}(A - B)}{\cos. \frac{1}{2}(A + B)} \times \sin. (A + B),$

and $\sin. A - \sin. B = \frac{\sin. \frac{1}{2}(A - B)}{\sin. \frac{1}{2}(A + B)} \times \sin. (A + B).$

Hence also $\sin^2. A - \sin^2. B = \sin. (A + B) \times \sin. (A - B).$

* Hence if a series of arcs be taken in arithmetic progression, radius is to twice the cosine of the common difference as the sine of any one arc taken as a mean is to the sum of the sines of any two equidistant extremes.

Also radius is to twice the cosine of the common difference as the cosine of any one arc taken as a mean is to the sum of the cosines of any two equidistant extremes.

And in a similar manner,

$$\cos^2 B - \cos^2 A = \sin. (A+B) \times \sin. (A-B).$$

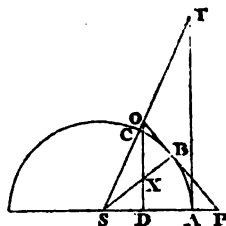
(111.) To find the $\sin. (A+B+C)$.

$$\begin{aligned} \sin. (A+B+C) &= \sin. (A+B) \times \cos. C + \cos. (A+B) \times \\ &\quad [\sin. C \\ &= \sin. A \times \cos. B \times \cos. C + \sin. B \times \cos. A \times \cos. C \\ &\quad + \sin. C \times \cos. A \times \cos. B - \sin. A \times \sin. B \times \sin. C. \end{aligned}$$

PROP. XX.

(112.) *Having given the tangents of two arcs; to determine the tangents of two arcs which may be equal to their sum and difference.*

Let AB, BC be the given arcs, AB being the greater, then AC is the sum of the arcs (in Fig. 1.) and their difference (in Fig. 2.); AT the tangent of their sum (1) or difference (2), BP, BO the tangents of the respective arcs AB, BC .



Draw OD perpendicular to SA cutting SB or SB produced in x , $\therefore$ the triangle OBx is similar to SxD and $\therefore$ to SBP ; hence $OB : Bx :: SB : BP$, $\therefore OB \times BP = SB \times Bx$. Also by the similar triangles TAS, ODS ,

$$AT : OD :: SA : SD$$

$$OD : OP :: SD : Sx$$

by the sim. triangles SxD, ODP ,

$$\therefore AT : OP :: SA : Sx$$

$$:: SA : SB - Bx \text{ (fig. 1.)}$$

$$:: SA^2 : SB^2 - SB \times Bx$$

$$:: SA^2 : SA^2 - OB \times BP$$

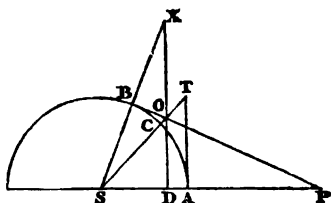
Again (in Fig. 2.) $AT : OP$

$$\therefore SA : Sx$$

$$\therefore SA : SB + Bx$$

$$\therefore SA^2 : SB^2 + SB \times Bx$$

$$\therefore SA^2 : SA^2 + OB \times BP.$$



$$\text{Hence } AT = SB^2 \times \frac{OP}{SA^2 \mp OB \times BP}$$

$$= SA^2 \times \frac{OB \pm BP}{SA^2 \mp OP \times BP}.$$

(113.) COR. 1. If rad. = 1, $AB = a$, $BC = b$,

$$\text{tang. } (a \pm b) = \frac{\text{tang. } a \pm \text{tang. } b}{1 \mp \text{tang. } a \times \text{tang. } b}^*.$$

(114.) COR. 2. If $a = b$, $\text{tang. } 2a = \frac{2 \cdot \text{tang. } a}{1 - \text{tang.}^2 a}$

$$\text{Tang. } 3a = \frac{3 \cdot \text{tang. } a - \text{tang.}^3 a}{1 - 3 \cdot \text{tang.}^2 a},$$

$$\text{tang. } 4a = \frac{4 \cdot \text{tang. } a - 4 \text{ tang.}^3 a}{1 - 6 \text{ tang.}^2 a + \text{tang.}^4 a}.$$

$$\text{tang. } 5a = \frac{5 \text{ tang. } a - 10 \text{ tang.}^3 a + \text{tang.}^5 a}{1 - 10 \text{ tang.}^2 a + 5 \text{ tang.}^4 a},$$

$$\text{tang. } 6a = \frac{6 \text{ tang. } a - 20 \text{ tang.}^3 a + 6 \text{ tang.}^5 a}{1 - 15 \text{ tang.}^2 a + 15 \text{ tang.}^4 a - \text{tang.}^6 a}, \text{ \&c.}$$

(115.) If (in Art. 113.) $a = 45^\circ$,

$$\text{tang. } (45^\circ + b) = \frac{1 + \text{tang. } b}{1 - \text{tang. } b},$$

$$* \text{ Tang. } (a+b) = \frac{\sin. (a+b)}{\cos. (a+b)} = \frac{\sin. a \times \cos. b + \cos. a \times \sin. b}{\cos. a \times \cos. b - \sin. a \times \sin. b}$$

$$= \frac{\frac{\sin. a}{\cos. a} + \frac{\sin. b}{\cos. b}}{1 - \frac{\sin. a \times \sin. b}{\cos. a \times \cos. b}} = \frac{\text{tang. } a + \text{tang. } b}{1 - \text{tang. } a \times \text{tang. } b}.$$

$$\text{and tang. } (45^\circ - b) = \frac{1 - \text{tang. } b}{1 + \text{tang. } b};$$

$$\begin{aligned} \therefore \text{tang. } (45^\circ + b) - \text{tang. } (45^\circ - b) &= \frac{1 + \text{tang. } b}{1 - \text{tang. } b} - \frac{1 - \text{tang. } b}{1 + \text{tang. } b} \\ &= 2 \cdot \frac{2 \text{ tang. } b}{1 - \text{tang.}^2 b} = 2 \text{ tang. } 2b, \end{aligned}$$

$$\text{and tang. } (45^\circ + b) = \text{tang. } (45^\circ - b) + 2 \text{ tang. } 2b.$$

If $\therefore$ the tangents of arcs less than 45° be known, the tangents of arcs greater than 45° may be determined by addition.

$$(116.) \text{ Tang. } a + \text{cotang. } a = 2 \text{ cosec. } 2a.$$

$$\text{For tang. } a + \text{cotang. } a = \frac{\sin. a}{\cos. a} + \frac{\cos. a}{\sin. a} =$$

$$\frac{\sin^2. a + \cos^2. a}{\sin. a \times \cos. a} = \frac{2}{\sin. 2a} = 2 \text{ cosec. } 2a.$$

In the same way it may be shewn that

$$\text{cotang. } a - \text{tang. } a = 2 \text{ cotang. } 2a.$$

$$\text{and sec. } a + \text{tang. } a = \text{tang. } (45^\circ + \frac{1}{2}a)$$

$$\text{that tang. } a = \frac{\sin. 2a}{1 + \cos. 2a}, \text{ cotang. } a = \frac{\sin. 2a}{1 - \cos. 2a},$$

$$\cos. 2a = \frac{1 - \text{tang.}^2. a}{1 + \text{tang.}^2. a}, \text{ and sec. } 2a = \frac{\sec^2. a}{1 - \text{tang.}^2. a}.$$

PROP. XXI.

(117.) *The base and altitude of a triangle being given; to find its area.*

(Eucl. I. 41.) The area of the triangle $ABC = \frac{1}{2} AC \times BD$. (See next Fig.).

(118.) COR. 1. If a = the altitude BD , and b = the

base AC , the area $S = \frac{ab}{2}$ and $a = \frac{2S}{b}$ and $b = \frac{2S}{a}$; any two of the terms $\therefore$ being known, the third may be found.

PROP. XXII.

(119.) *Two sides of a triangle, and the included angle being given; to find its area.*

Let AB , AC and the angle BAC be given. Let fall the perpendicular BD ,

$$AB : BD :: \text{rad.} : \sin. A,$$

$$\therefore BD = \frac{AB \times \sin. A}{R} = AB \times \sin. A,$$

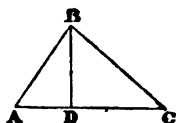
(if $R = 1$),

$$\therefore \text{the area} = \frac{1}{2} AC \times BD = \frac{1}{2} AB \times AC \times \sin. A.$$

(120.) COR. 1. Hence the areas of triangles which have one angle in each equal, are as the products of the sides containing those angles. Which is also true of parallelograms.

$$(121.) \text{ COR. 2. } AC = \frac{2S}{AB \times \sin. A}, \quad AB = \frac{2S}{AC \times \sin. A},$$

$$\text{and } \sin. A = \frac{2S}{AB \times AC}.$$



PROP. XXIII.

(122.) *Given two angles and a side of a triangle; to find its area.*

Two angles being given, the third is also known.

Let AC be the given side.

$$\text{Then } \sin. B : \sin. C :: AC : AB = \frac{AC \times \sin. C}{\sin. B},$$

$$\text{and } BD = AB \times \sin. A = \frac{AC \times \sin. A \times \sin. C}{\sin. B};$$

$$\text{whence the area} = \frac{1}{2} AC^2 \times \frac{\sin. A \times \sin. C}{\sin. B}.$$

(123.) Cor. If the angles ABC , ACB be equal, the area will be $= \frac{1}{2} AC^2 \times \sin. A$. If the angles A and C are equal, the area will be $= \frac{1}{2} AC^2 \times \frac{\sin^2. A}{\sin. B}$.

But since the angle $B = 180^\circ - 2A$,

$$\sin. B = \sin. 2A = 2 \sin. A \times \cos. A,$$

$$\text{whence the area} = \frac{1}{4} AC^2 \times \frac{\sin^2. A}{\sin. A \times \cos. A}$$

$$= \frac{1}{4} AC^2 \times \frac{\sin. A}{\cos. A} = \frac{1}{4} AC^2 \times \text{tang. } A.$$

PROP. XXIV.

(124) *Given the three angles, and the altitude of a triangle; to find its area.*

Since (122) $BD \times \sin. B = AC \times \sin. A \times \sin. C$, and $AC = \frac{2S}{BD}$, $\therefore 2S \times \sin. A \times \sin. C = BD^2 \times \sin. B$.

$$\text{or } S = \frac{1}{2} BD^2 \times \frac{\sin. B}{\sin. A \times \sin. C}.$$

PROP. XXV.

(125.) *Given the three sides of a triangle; to find its area.*

The area of the triangle $ABC = \text{area } ADC + \text{area } BDC$
 $= AE \times DE + ME \times DE$, (BM being parallel to DC)
 $= AM \times DE$.

The sine and cosine of $1'$ being ascertained, the sines of $2'$, $3'$, $4'$, &c. may be determined (109) by making $n = 1, 2, 3$, &c. and the cosines from (58). In this manner the sines and cosines of arcs as far as 30° may be computed. When the arc exceeds 30° , the sines may be computed by Art. (107), and the cosines as before, till the arc is 45° . And since the sine of an arc is equal to the cosine of its complement, the sines and cosines of arcs as far as 90° are determined. Also since the sines and cosines of arcs are equal to the sines and cosines of their supplements; the sines and cosines of all arcs up to 180° are known.

Since $\text{tang. } A = \frac{\sin. A}{\cos. A}$, the tangents of all arcs may be computed. When however they exceed 45° , they are more readily computed from (115) by addition. And the tangent of an arc being equal to the tangent of its supplement, the tangents of all arcs may be determined.

Hence also the cotangents (45).

Also the $\text{sec. } A = \frac{1}{\cos. A}$, and $\therefore$ the cosines being known, the secants may be determined. And the secants being known, the cosecants are also determined.

Also the versed sine ($= 1 \mp \text{cosine}$) may be determined.



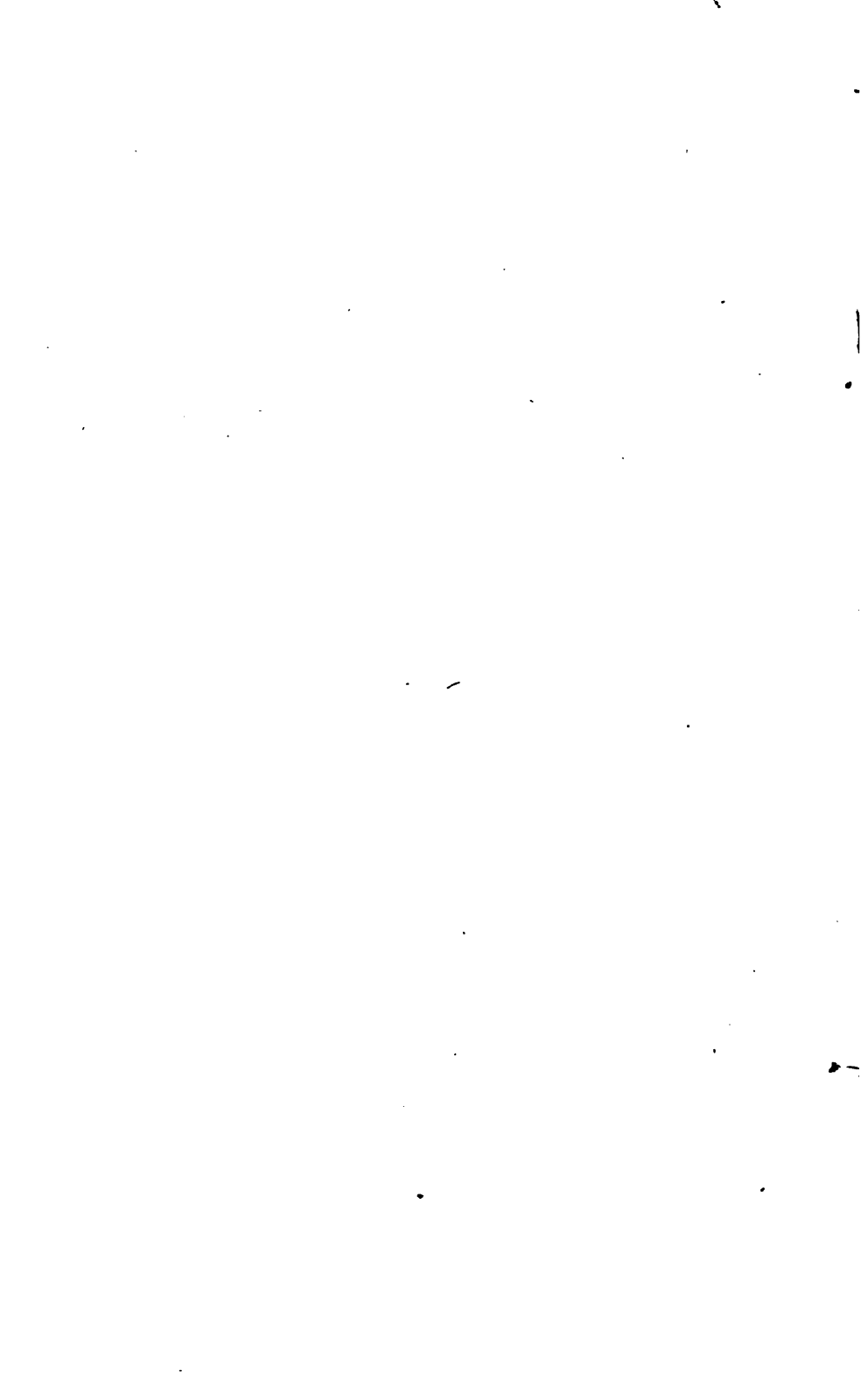
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ALGEBRAICAL PROBLEMS, producing **SIMPLE** and **QUADRATIC EQUATIONS** with their **SOLUTIONS**, designed as an Introduction to the higher branches of Analytics. By **M. BLAND, B. D.** Fellow and Tutor of St. John's College. 3d Edit. price 10s. 6d.

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THE following pages contain a collection of Algebraical Problems, designed solely to point out the various methods employed by Analysts in the Solution of Equations. They are arranged in the usual manner: 1. Simple Equations; 2. Pure Quadratics, and others which may be solved without completing the square; and 3. Affecting Quadratics. Utility being the sole object of this Publication, wherever a proper Example occurred, it has been taken without hesitation, or altered to suit the purpose. At the head of each Section are given the common Rules; and the whole concludes with a Collection of Problems without Solutions, for the Exercise of the Learner.

To the Sixth Edition has been added an Appendix, containing a Collection of Problems in Arithmetical, Geometrical, and Harmonical Progressions; and another on the nature of Equations, and the solution of those of higher dimensions.



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ALGEBRAICAL PROBLEMS.

SECTION I.

DEFINITIONS.

(1.) AN Equation is a proposition, which declares the equality of two quantities, expressed algebraically.

This is done by connecting these quantities by the sign ($=$); thus, $x - 4 = 6 - x$ is an equation expressing the equality of the quantities $x - 4$ and $6 - x$. Also $x - 5 = 0$ is an equation which asserts that $x - 5$ is equal to nothing, and therefore that the positive part of the expression is equal to the negative part.

(2.) A Simple Equation is one which, when cleared of fractions and surds, contains only the first power of the unknown quantity.

(3.) A Quadratic Equation, or an equation of two dimensions, is one into which the square of the unknown quantity enters, with or without the simple power.

(4.) A Cubic Equation, or an equation of three dimensions, is one into which the cube of the unknown quantity enters, with or without the simple and quadratic powers.

(5.) In general, the index of the highest power of the unknown quantity denotes the number of dimensions of the equation.

(6.) A Pure Quadratic is one into which only the square of the unknown quantity enters.

(7.) An Adfected Quadratic is one which involves the square of the unknown quantity, and also the simple power and known quantities.

Thus, $ax^2 + b = 0$ is a pure quadratic,

and $ax^2 + bx + c = 0$ is an adfected quadratic.

(8.) The Resolution of Equations is the determining, from some quantities given, the values of others which are unknown, so that these latter may answer certain conditions proposed.

(9.) And these values are called Roots of the Equation.

(10.) Known quantities are usually expressed by the first letters of the alphabet, a, b, c , &c.; and unknown quantities by the last, v, x, y , &c. And this must be always understood, unless the contrary be expressed.

AXIOMS.

(11.) If equal quantities be added to equal quantities, the sums will be equal.

(12.) If equal quantities be taken from equal quantities, the remainders will be equal.

(13.) If equal quantities be multiplied by the same or equal quantities, the products will be equal.

(14.) If equal quantities be divided by the same or equal quantities, the quotients will be equal.

(15.) If the same quantity be added to and subtracted from another, the value of the latter will not be altered.

(16.) If a quantity be both multiplied and divided by another, its value will not be altered.

(17.) Any quantity may be transposed from one side of an equation to the other, by changing its sign :

Because, in this transposition, the same quantity is merely subtracted from each side of the equation ; and (12) if equals be taken from equals, the remainders are equal.

Thus, if $x + 9 = 15$, and 9 be subtracted from each side, $x = 15 - 9$, or 6. Also, if $x + b = a$, and b be subtracted from each side, $x = a - b$. And if $x - c = d$, and c be added to each side, $x = d + c$.

Also, if $5x - 7 = 2x + 2$, and $2x$ be taken from each side, $5x - 2x - 7 = 2$, or $3x - 7 = 2$; and if -7 be subtracted, or (which is the same thing) if $+7$ be added to each side, $3x = 2 + 7 = 9$.

Also, if $x - a + b = c - 3x$, then, by subtracting $-a + b - 3x$ from each side, we have $x + 3x = a - b + c$.

COR. 1. Hence, if the signs of all the terms on each side of an equation be changed, the two sides still remain equal; because in this change every term is transposed.

COR. 2. Hence, when the known and unknown quantities are connected in an equation by the signs $+$ or $-$, they may be separated by transposing the known quantities to one side, and the unknown to the other.

COR. 3. Hence also, if any quantity be found on both sides of an equation, it may be taken away from each; thus, if $x + y = 5 + y$, then $x = 5$. If $a - b = c + d - b$, then $a = c + d$.

(18.) If every term on each side of an equation be multiplied by the same quantity, the results will be equal:

Because in multiplying every term on each side by any quantity, the value of the whole side is multiplied by that quantity; and (13) if equals be multiplied by the same quantity, the products will be equal.

Thus, if $x = 5 + a$, then $6x = 30 + 6a$, by multiplying every term by 6.

COR. 1. Hence an equation, of which any part is fractional, may be reduced to an equation expressed in integers, by multiplying every term by the denominator of the fraction. If there be more fractions than one in the given equation, it may be so reduced by multiplying every term either by the product of the denominators, or by a common multiple of

them; and if the least common multiple be used, the equation will be in its lowest terms.

Thus, if $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 13$; if every term be multiplied by 12, which is the least common multiple of 2, 3, 4; $6x + 4x + 3x = 156$.

COR. 2. Hence also, if every term on both sides have a common multiplier or divisor, that common multiplier or divisor may be taken away;

Thus, if $ax^2 + abx = cdx$; each term being divided by the common multiplier x , $ax + ab = cd$.

Also, if $\frac{5x}{4} + \frac{a+6}{4} = \frac{4x-7}{4}$, then also $5x + a + 6 = 4x - 7$;

Also, if $\frac{ax+ab}{c} = \frac{ad}{c} + \frac{4ax}{c}$, then, multiplying by $\frac{c}{a}$, $x + b = d + 4x$.

Also, if $(a^2 + x^2)^{\frac{1}{2}} = 3x^{\frac{1}{2}}$, $(a^2 + x^2)^{\frac{1}{2}}$, then dividing by $(a^2 + x^2)^{\frac{1}{2}}$, $a^2 + x^2 = 3x^2$.

COR. 3. Also, if each member of the equation have a common divisor, the equation may be reduced by dividing both sides by that common divisor;

Thus, if $ax^2 - a^2x = abx - a^2b$, each side is divisible by $ax - a^2$, whence $x = b$.

COR. 4. Hence also any term of an equation may be made a square, by multiplying all the terms of the equation by the quantities necessary; as, if $ax^2 + bcx = cd^2$, the first term may be made a square by multiplying each term by a , and $a^2x^2 + abcx = acd^2$.

(19.) If each side of an equation be raised to the same power, the results are equal;

Thus, if $x = 6$, $x^2 = 36$; if $x + a = y - b$, then $x^2 + 2ax + a^2 = y^2 - 2by + b^2$;

And if the same roots be extracted on each side, the results are equal:

Thus, if $x^2 = 49$, $x = 7$; if $x^3 = a^3 b^3$, then $x = ab$; if $x^2 + 2x + 1 = y^2 - y + \frac{1}{4}$, then $x + 1 = y - \frac{1}{4}$, and if $x^2 - 4ax + 4a^2 = y^2 + 6by + 9b^2$, then $x - 2a = y + 3b$.

For (13 and 14) when equal quantities on each side of an equation are multiplied or divided by equal quantities, the results will be equal.

COR. Hence, if that side of the equation which contains the unknown quantity be a perfect square, cube, or other power, by extracting the square root, cube root, &c. of both sides, the equation will be reduced to one of lower dimensions:

Thus, if $x^2 + 8x + 16 = 36$, $x + 4 = 6$,
 if $x^3 + 3x^2 + 3x + 1 = 27$, $x + 1 = 3$,
 if $x^4 + 2x^3 + x^2 = 100$, $x^2 + x = 10$.

(20.) Any equation may be cleared of a single radical quantity by transposing all the other terms to the contrary side, and raising each side to the power denominated by the surd. If there are more than one surd, the operation must be repeated.

Thus, if $x = \sqrt{ax + b^2}$, by squaring each side $x^2 = ax + b^2$, which is free from surds.

Also, if $\sqrt{x^2 + 7} + x = 7$,

then (17) by transposition, $\sqrt{x^2 + 7} = 7 - x$;
 and (19) by squaring each side, $x^2 + 7 = 49 - 14x + x^2$, which is free from surds.

Also, if $x + \sqrt[3]{a^2 x} = b$,

then (17) by transposition, $\sqrt[3]{a^2 x} = b - x$;
 and (19) by cubing each side, $a^2 x = b^3 - 3b^2 x + 3bx^2 - x^3$, which is free from surds.

Also, if $\sqrt{x^2 + \sqrt{x^2 + 21}} - 1 = x$,

then (17) by transposition, $\sqrt{x^2 + \sqrt{x^2 + 21}} = x + 1$,

and (19) by squaring each side, $x^2 + \sqrt{x^2 + 21} = x^2 + 2x + 1$;

$$\therefore (17. \text{ Cor. } 3.) \sqrt{x^2 + 21} = 2x + 1,$$

and (19) by squaring each side, $x^2 + 21 = 4x^2 + 4x + 1$, which is free from surds.

$$\text{And, if } \sqrt[3]{a^3x} + \sqrt{a^3x^2} = c,$$

$$(19) \text{ by cubing each side, } a^3x + \sqrt{a^3x^2} = c^3,$$

$$\text{and (17) by transposition, } \sqrt{a^3x^2} = c^3 - a^3x;$$

$\therefore$ (19) by squaring each side, $a^2x^2 = c^6 - 2a^3c^3x + a^4x^2$, which is free from surds.

(21.) Any proportion may be converted into an equation; for the product of the extremes is equal to the product of the means.

$$\text{Let } a : b :: c : d, \text{ by the nature of proportion } \frac{a}{b} = \frac{c}{d};$$

$$\therefore (18. \text{ Cor. } 1.) ad = bc.$$

(22.) *EXAMPLES in which the preceding Rules are applied, in the Solution of Equations.*

1. Given $4x + 36 = 5x + 34$, to find the value of x .

$$(17) \text{ By transposition, } 36 - 34 = 5x - 4x,$$

$$\text{and } \therefore 2 = x.$$

2. Given $x - 7 = \frac{x}{5} + \frac{x}{3}$, to find the value of x .

Here 15, the product of 3 and 5, being their least common multiple, every term must be multiplied by it (18. Cor. 1.), and $15x - 105 = 3x + 5x$;

$$\therefore (17) \text{ by transposition, } 15x - 3x - 5x = 105,$$

$$\text{or } 7x = 105;$$

$$\text{and } \therefore (18. \text{ Cor. } 2.) x = \frac{105}{7} = 15.$$

3. Given $3ax - 4ab = 2ax - 6ac$, to find the value of x in terms of b and c .

(18. Cor. 2.) dividing every term by a , $3x - 4b = 2x - 6c$;

$\therefore$ (17) by transposition, $3x - 2x = 4b - 6c$,

$$\text{or } x = 4b - 6c.$$

4. Given $3x^2 - 10x = 8x + x^2$, to find the value of x .

(18. Cor. 2.) dividing every term by x , $3x - 10 = 8 + x$;

$\therefore$ by transposition, $3x - x = 8 + 10$,

$$\text{or } 2x = 18;$$

$$\therefore (18. \text{ Cor. } 2.) \ x = \frac{18}{2} = 9.$$

5. Given $\frac{x}{2} + \frac{x}{3} = \frac{x}{4} + 7$, to find the value of x .

Here 12 is the least common multiple of 2, 3, and 4;

(18. Cor. 1.) multiplying both sides of the equation therefore by 12, $6x + 4x = 3x + 84$;

$\therefore$ (17) by transposition, $6x + 4x - 3x = 84$,

$$\text{or } 7x = 84;$$

$$\therefore (18. \text{ Cor. } 2.) \ x = \frac{84}{7} = 12.$$

6. Given $\frac{x-5}{4} + 6x = \frac{284-x}{5}$, to find the value of x .

(18. Cor. 1.) multiplying by 20, the least common multiple of 4 and 5,

$$5x - 25 + 120x = 1136 - 4x;$$

$\therefore$ (17) by transposition, $5x + 120x + 4x = 1136 + 25$,

$$\text{or } 129x = 1161;$$

$$\therefore (18. \text{ Cor. } 2.) \ x = \frac{1161}{129} = 9.$$

7. Given $x + \frac{11-x}{3} = \frac{19-x}{2}$, to find the value of x .

(18. Cor. 1.) multiplying by 6, the least common multiple of 2 and 3.

$$6x + 22 - 2x = 57 - 3x;$$

$$\therefore (17) \text{ by transposition, } 6x - 2x + 3x = 57 - 22,$$

$$\text{or } 7x = 35;$$

$$\therefore (18. \text{ Cor. 2.}) \ x = \frac{35}{7} = 5.$$

8. Given $3x + \frac{2x+6}{5} = 5 + \frac{11x-37}{2}$, to find the value of x .

(18. Cor. 1.) multiplying by 10, the least common multiple of 2 and 5,

$$30x + 4x + 12 = 50 + 55x - 185;$$

$$\therefore (17) \text{ by transposition, } 12 - 50 + 185 = 55x - 30x - 4x,$$

$$\text{or } 147 = 21x;$$

$$\therefore (18. \text{ Cor. 2.}) \ \frac{147}{21} = 7 = x.$$

9. Given $\frac{6x-4}{3} - 2 = \frac{18-4x}{3} + x$, to find the value of x .

(18. Cor. 1.) multiplying every term by 3,

$$6x - 4 - 6 = 18 - 4x + 3x;$$

and $\therefore (17) \text{ by transposition, } 6x + 4x - 3x = 18 + 6 + 4,$

$$\text{or } 7x = 28;$$

$$\therefore (18. \text{ Cor. 2.}) \ x = \frac{28}{7} = 4.$$

10. Given $21 + \frac{3x-11}{16} = \frac{5x-5}{8} + \frac{97-7x}{2}$, to find the value of x .

Since 16 contains 8 and 2, a certain number of times exactly, it will be the least common multiple of 16, 8, and 2; and therefore (18. Cor. 1.) multiplying both sides of the equation by 16,

$$336 + 3x - 11 = 10x - 10 + 776 - 56x;$$

∴ (17) by transposition, $3x - 10x + 56x = 11 - 10 + 776 - 336$,
or $49x = 441$;

$$\therefore (18. \text{ Cor. 2.}) x = \frac{441}{49} = 9.$$

11. Given $x + \frac{3x - 5}{2} = 12 - \frac{2x - 4}{3}$, to find the value of x .

(18. Cor. 1.) multiplying both sides of the equation by 6, the product of 2 and 3,

$$6x + 9x - 15 = 72 - 4x + 8;$$

∴ (17) by transposition, $6x + 9x + 4x = 72 + 8 + 15$,
or $19x = 95$;

$$\therefore (18. \text{ Cor. 2.}) x = \frac{95}{19} = 5.$$

12. Given $3x - \frac{x - 4}{4} - 4 = \frac{5x + 14}{3} - \frac{1}{12}$, to find the value of x .

Since 12 is a multiple of 3 and 4, it is the least common multiple of 3, 4, and 12; therefore (18. Cor. 1.) multiplying both sides of the equation by 12,

$$36x - 3x + 12 - 48 = 20x + 56 - 1;$$

∴ (17) by transposition, $36x - 3x - 20x = 56 + 48 - 1 - 12$,
or $13x = 91$;

$$\therefore (18. \text{ Cor. 2.}) x = \frac{91}{13} = 7.$$

13. Given $\frac{x-1}{7} + \frac{23-x}{5} = 7 - \frac{4+x}{4}$, to find the value of x .

(18. Cor. 1.) multiplying both sides of the equation by $4 \times 5 \times 7 = 140$,

$$20x - 20 + 644 - 28x = 980 - 140 - 35x;$$

∴ (17) by transⁿ, $20x - 28x + 35x = 980 - 140 + 20 - 644$,
or $27x = 216$;

$$\therefore (18. \text{ Cor. 2.}) \ x = \frac{216}{27} = 8.$$

14. Given $\frac{7x+5}{3} - \frac{16+4x}{5} + 6 = \frac{3x+9}{2}$, to find the value of x .

(18. Cor. 1.) multiplying both sides by $2 \times 3 \times 5 = 30$,
 $70x + 50 - 96 - 24x + 180 = 45x + 135$;

$\therefore$ (17) by transposition, $70x - 24x - 45x = 135 + 96 - 50 - 180$,
 or $x = 1$.

15. Given $\frac{3x+4}{5} - \frac{7x-3}{2} = \frac{x-16}{4}$, to find the value of x .

(18. Cor. 1.) multiplying by 20, the least common multiple of 2, 4, and 5,

$$12x + 16 - 70x + 30 = 5x - 80;$$

$\therefore$ (17) by transposition, $16 + 30 + 80 = 5x + 70x - 12x$,
 or $126 = 63x$;

$$\therefore (18. \text{ Cor. 2.}) \ \frac{126}{63} = 2 = x.$$

16. Given $\frac{17-3x}{5} - \frac{4x+2}{3} = 5 - 6x + \frac{7x+14}{3}$, to find the value of x .

(18. Cor. 1.) multiplying both sides of the equation by $3 \times 5 = 15$,

$$51 - 9x - 20x - 10 = 75 - 90x + 35x + 70;$$

$\therefore$ (17) by transⁿ, $90x - 35x - 20x - 9x = 75 + 70 + 10 - 51$,
 or $26x = 104$;

$$\therefore (18. \text{ Cor. 2.}) \ x = \frac{104}{26} = 4.$$

17. Given $x - \frac{3x-3}{5} + 4 = \frac{20-x}{2} - \frac{6x-8}{7} + \frac{4x-4}{5}$,
 to find the value of x .

(18. Cor. 1.) multiplying both sides of the equation by $2 \times 5 \times 7 = 70$,

$$70x - 42x + 42 + 280 = 700 - 35x - 60x + 80 + 56x - 56;$$

$\therefore$ (17) by transposition,

$$70x + 35x + 60x - 42x - 56x = 700 + 80 - 56 - 280 - 42,$$

$$\text{or } 67x = 402;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{402}{67} = 6.$$

18. Given $\frac{4x-21}{9} + 3\frac{3}{4} + \frac{57-3x}{4} = 241 - \frac{5x-96}{12} - 11x$,
to find the value of x .

(18. Cor. 1.) multiplying by 36, the least common multiple of 4, 9, and 12,

$$16x - 84 + 135 + 513 - 27x = 8676 - 15x + 288 - 396x;$$

$\therefore$ (17) by transposition,

$$16x + 15x + 396x - 27x = 8676 + 288 + 84 - 135 - 513,$$

$$\text{or } 400x = 8400;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{8400}{400} = 21.$$

19. Given $\frac{6x+18}{13} - 4\frac{1}{2} - \frac{11-3x}{36} = 5x - 48 - \frac{13-x}{12} - \frac{21-2x}{18}$, to find the value of x .

(18. Cor. 1.) multiplying by 36×13 , the least common multiple of the denominators,

$$216x + 648 - 2262 - 143 + 39x$$

$$= 2340x - 22464 - 507 + 39x - 546 + 52x;$$

$\therefore$ (17) by transposition, $648 + 22464 + 507 + 546 - 2262 - 143$
 $= 2340x + 39x + 52x - 216x - 39x$,

$$\text{or } 21760 = 2176x;$$

$$\therefore \text{ (18. Cor. 2.) } \frac{21760}{2176} = 10 = x.$$

20. Given $ax - \frac{a^2 - 3bx}{a} - ab^2 = bx + \frac{6bx - 5a^2}{2a} - \frac{bx + 4a}{4}$,
to find the value of x .

(18. Cor. 1.) multiplying by $4a$, the least common multiple of the denominators,

$$4a^2x - 4a^2 + 12bx - 4a^2b^2 = 4abx + 12bx - 10a^2 - abx - 4a^2;$$

$$\therefore (17. \text{ Cor. 3.}) 4a^2x - 4a^2b^2 = 3abx - 10a^2;$$

$$\text{by transposition, } (4a^2 - 3ab) \cdot x = 4a^2b^2 - 10a^2;$$

$$(18. \text{ Cor. 2.}) (4a - 3b) \cdot x = 4ab^2 - 10a;$$

$$\therefore x = \frac{4ab^2 - 10a}{4a - 3b}.$$

21. Given $\frac{7x + 16}{21} - \frac{x + 8}{4x - 11} = \frac{x}{3}$, to find the value of x .

Multiplying both sides of the equation by 21,

$$7x + 16 - \frac{21x + 168}{4x - 11} = 7x;$$

$$\therefore (17. \text{ Cor. 3.}) 16 = \frac{21x + 168}{4x - 11};$$

$$\therefore (18. \text{ Cor. 1.}) 64x - 176 = 21x + 168;$$

$$\therefore (17) \text{ by transposition, } 64x - 21x = 168 + 176,$$

$$\text{or } 43x = 344;$$

$$\therefore (18. \text{ Cor. 2.}) x = \frac{344}{43} = 8.$$

22. Given $\frac{6x + 7}{9} + \frac{7x - 13}{6x + 3} = \frac{2x + 4}{3}$, to find the value of x .

Multiplying both sides of the equation by 9,

$$6x + 7 + \frac{21x - 39}{2x + 1} = 6x + 12;$$

$$\therefore (17. \text{ Cor. } 3.) \frac{21x - 39}{2x + 1} = 5;$$

$$\therefore (18. \text{ Cor. } 1.) 21x - 39 = 10x + 5;$$

$$\therefore (17) \text{ by transposition, } 21x - 10x = 39 + 5, \\ \text{or } 11x = 44;$$

$$\therefore (18. \text{ Cor. } 2.) x = \frac{44}{11} = 4.$$

23. Given $\frac{4x + 3}{9} + \frac{7x - 29}{5x - 12} = \frac{8x + 19}{18}$, to find the value of x .

Multiplying both sides of the equation by 18,

$$8x + 6 + \frac{126x - 522}{5x - 12} = 8x + 19;$$

$$\therefore (17. \text{ Cor. } 3.) \frac{126x - 522}{5x - 12} = 13,$$

$$\text{and } (18. \text{ Cor. } 1.) 126x - 522 = 65x - 156;$$

$$(17) \text{ by transposition, } 61x = 366;$$

$$\therefore (18. \text{ Cor. } 2.) x = \frac{366}{61} = 6.$$

24. Given $12 - x : \frac{x}{2} :: 4 : 1$, to find the value of x .

(21) Since the product of the extremes is equal to the product of the means,

$$12 - x = 4 \times \frac{x}{2} = 2x;$$

$$\therefore (17) \text{ by transposition, } 12 = 2x + x = 3x,$$

$$\text{and } (18. \text{ Cor. } 2.) \frac{12}{3} = 4 = x.$$

25. Given $\frac{5x + 4}{2} : \frac{18 - x}{4} :: 7 : 4$, to find the value of x .

20. Given $ax - \frac{a^2 - 3bx}{a} - ab^2 = bx + \frac{6bx - 5a^2}{2a} - \frac{bx + 4a}{4}$,
to find the value of x .

(18. Cor. 1.) multiplying by $4a$, the least common multiple of the denominators,

$$4a^2x - 4a^2 + 12bx - 4a^2b^2 = 4abx + 12bx - 10a^2 - abx - 4a^2;$$

$$\therefore (17. \text{ Cor. 3.}) 4a^2x - 4a^2b^2 = 3abx - 10a^2;$$

$$\text{by transposition, } (4a^2 - 3ab) \cdot x = 4a^2b^2 - 10a^2;$$

$$(18. \text{ Cor. 2.}) (4a - 3b) \cdot x = 4ab^2 - 10a;$$

$$\therefore x = \frac{4ab^2 - 10a}{4a - 3b}.$$

21. Given $\frac{7x + 16}{21} - \frac{x + 8}{4x - 11} = \frac{x}{3}$, to find the value of x .

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$$7x + 16 - \frac{21x + 168}{4x - 11} = 7x;$$

$$\therefore (17. \text{ Cor. 3.}) 16 = \frac{21x + 168}{4x - 11};$$

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$$\therefore (17) \text{ by transposition, } 64x - 21x = 168 + 176,$$

$$\text{or } 43x = 344;$$

$$\therefore (18. \text{ Cor. 2.}) x = \frac{344}{43} = 8.$$

22. Given $\frac{6x + 7}{9} + \frac{7x - 13}{6x + 3} = \frac{2x + 4}{3}$, to find the value of x .

Multiplying both sides of the equation by 9,

$$6x + 7 + \frac{21x - 39}{2x + 1} = 6x + 12;$$

$$\therefore (17. \text{ Cor. 3.}) \frac{21x - 39}{2x + 1} = 5;$$

$$\therefore (18. \text{ Cor. 1.}) 21x - 39 = 10x + 5;$$

$$\therefore (17) \text{ by transposition, } 21x - 10x = 39 + 5, \\ \text{or } 11x = 44;$$

$$\therefore (18. \text{ Cor. 2.}) x = \frac{44}{11} = 4.$$

23. Given $\frac{4x + 3}{9} + \frac{7x - 29}{5x - 12} = \frac{8x + 19}{18}$, to find the value of x .

Multiplying both sides of the equation by 18,

$$8x + 6 + \frac{126x - 522}{5x - 12} = 8x + 19;$$

$$\therefore (17. \text{ Cor. 3.}) \frac{126x - 522}{5x - 12} = 13,$$

$$\text{and } (18. \text{ Cor. 1.}) 126x - 522 = 65x - 156;$$

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24. Given $12 - x : \frac{x}{2} :: 4 : 1$, to find the value of x .

(21) Since the product of the extremes is equal to the product of the means,

$$12 - x = 4 \times \frac{x}{2} = 2x;$$

$$\therefore (17) \text{ by transposition, } 12 = 2x + x = 3x,$$

$$\text{and } (18. \text{ Cor. 2.}) \frac{12}{3} = 4 = x.$$

25. Given $\frac{5x + 4}{2} : \frac{18 - x}{4} :: 7 : 4$, to find the value of x .

$$(21) \frac{5x + 4}{2} \times 4 = \frac{18 - x}{4} \times 7,$$

$$\text{or } 10x + 8 = \frac{126 - 7x}{4};$$

$$\therefore (18. \text{ Cor. 2.}) 40x + 32 = 126 - 7x;$$

$$\therefore (17) \text{ by transposition, } 40x + 7x = 126 - 32,$$

$$\text{or } 47x = 94;$$

$$\therefore x = \frac{94}{47} = 2.$$

26. Given $\sqrt{4x + 16} = 12$, to find the value of x .

(19) squaring both sides of the equation, $4x + 16 = 144$;

$$\therefore (17) \text{ by transposition, } 4x = 144 - 16 = 128;$$

$$\therefore (18. \text{ Cor. 2.}) x = \frac{128}{4} = 32.$$

27. Given $\sqrt[3]{2x + 3} + 4 = 7$, to find the value of x .

$$(17) \text{ by transposition, } \sqrt[3]{2x + 3} = 7 - 4 = 3;$$

$\therefore$ (19) cubing both sides of the equation, $2x + 3 = 27$;

$$\therefore (17) \text{ by transposition, } 2x = 27 - 3 = 24,$$

$$\text{and } (18. \text{ Cor. 2.}) x = \frac{24}{2} = 12.$$

28. Given $\sqrt{12 + x} = 2 + \sqrt{x}$, to find the value of x .

(19) squaring both sides of the equation,

$$12 + x = 4 + 4\sqrt{x} + x;$$

$$\therefore (17. \text{ Cor. 3.}) 8 = 4\sqrt{x};$$

$$\text{and } (18. \text{ Cor. 2.}) 2 = \sqrt{x};$$

$$\therefore (19) 4 = x.$$

29. Given $\sqrt{x + 40} = 10 - \sqrt{x}$, to find the value of x .

(19) squaring both sides of the equation,

$$x + 40 = 100 - 20\sqrt{x} + x;$$

$$\therefore (17. \text{ Cor. 3.}) 20\sqrt{x} = 100 - 40 = 60,$$

$$\text{and } (19. \text{ Cor. 2.}) \sqrt{x} = 3;$$

$$\therefore (19) x = 9.$$

30. Given $\sqrt{x - 16} = 8 - \sqrt{x}$, to find the value of x .

(19) squaring both sides of the equation,

$$x - 16 = 64 - 16\sqrt{x} + x;$$

$$\therefore (17. \text{ Cor. 3.}) 16\sqrt{x} = 64 + 16 = 80;$$

$$\therefore (18. \text{ Cor. 2.}) \sqrt{x} = 5,$$

$$\text{and } (19) x = 25.$$

31. Given $\sqrt{x - 24} = \sqrt{x} - 2$, to find the value of x .

(19) squaring both sides of the equation,

$$x - 24 = x - 4\sqrt{x} + 4;$$

$$\therefore (17. \text{ Cor. 3.}) 4\sqrt{x} = 24 + 4 = 28;$$

$$\therefore (18. \text{ Cor. 2.}) \sqrt{x} = 7,$$

$$\text{and } (19) x = 49.$$

32. Given $\sqrt{x - a} = \sqrt{x} - \frac{1}{2}\sqrt{a}$, to find the value of x .

(19) squaring both sides of the equation,

$$x - a = x - \sqrt{ax} + \frac{1}{4}a;$$

$$\therefore (17. \text{ Cor. 3.}) \sqrt{ax} = a + \frac{1}{4}a = \frac{5a}{4};$$

$$(19) ax = \frac{25a^2}{16};$$

$$\therefore (18. \text{ Cor. 2.}) x = \frac{25a}{16}.$$

33. Given $\sqrt{5} \times \sqrt{x + 2} = \sqrt{5x} + 2$.

(19) squaring both sides of the equation,

$$5x + 10 = 5x + 4\sqrt{5x} + 4;$$

$$\therefore (17. \text{ Cor. 3.}) \quad 10 - 4 = 6 = 4\sqrt{5x};$$

$$\therefore (18. \text{ Cor. 2.}) \quad \frac{6}{4} = \frac{3}{2} = \sqrt{5x},$$

$$\text{and } (19) \quad \frac{9}{4} = 5x;$$

$$\therefore (18. \text{ Cor. 2.}) \quad \frac{9}{20} = x.$$

34. Given $\sqrt{4a + x} = 2\sqrt{b + x} - \sqrt{x}$, to find the value of x .

(19) squaring both sides of the equation,

$$4a + x = 4 \cdot (b + x) - 4\sqrt{(bx + x^2) + x};$$

$$(17. \text{ Cor. 3.}) \quad 4a + 4\sqrt{(bx + x^2)} = 4 \cdot (b + x);$$

$$(18. \text{ Cor. 2.}) \quad a + \sqrt{(bx + x^2)} = b + x;$$

$$(17) \text{ by transposition, } \sqrt{(bx + x^2)} = b - a + x;$$

$$(19) \quad bx + x^2 = (b - a)^2 + 2 \cdot (b - a) \cdot x + x^2;$$

$$(17. \text{ Cor. 3.}) \quad (2a - b) \cdot x = (b - a)^2;$$

$$\therefore (18. \text{ Cor. 2.}) \quad x = \frac{(b - a)^2}{2a - b}.$$

35. Given $x + a + \sqrt{2ax + x^2} = b$, to find the value of x .

$$(17) \text{ by transposition, } \sqrt{2ax + x^2} = b - a - x;$$

and (19) squaring both sides,

$$2ax + x^2 = (b - a)^2 - 2 \cdot (b - a) \cdot x + x^2;$$

$$\therefore (17. \text{ Cor. 3.}) \quad 2ax + 2 \cdot (b - a) \cdot x = (b - a)^2,$$

$$\text{or } 2bx = (b - a)^2;$$

$$\therefore (18. \text{ Cor. 2.}) \quad x = \frac{(b - a)^2}{2b}.$$

36. Given $\frac{x - ax}{\sqrt{x}} = \frac{\sqrt{x}}{x}$ to find the value of x .

Multiplying both sides of the equation by $\sqrt{x}$,

$$x - ax = \frac{x}{x} = 1,$$

$$(1 - a) \cdot x = 1,$$

$$\text{and (18. Cor. 2.) } x = \frac{1}{1 - a}.$$

37. Given $\frac{\sqrt{x} + 28}{\sqrt{x} + 4} = \frac{\sqrt{x} + 38}{\sqrt{x} + 6}$, to find the value of x .

(18. Cor. 1.) multiplying both sides of the equation by $(\sqrt{x} + 4) \times (\sqrt{x} + 6)$,

$$x + 34\sqrt{x} + 168 = x + 42\sqrt{x} + 152;$$

$\therefore$ (17. Cor. 3.) taking $(x + 34\sqrt{x} + 152)$ from each side of the equation,

$$16 = 8\sqrt{x};$$

$$\therefore \text{ (18. Cor. 2.) } 2 = \sqrt{x};$$

$$\therefore \text{ (19) } 4 = x.$$

38. Given $\frac{\sqrt{x} + 2a}{\sqrt{x} + b} = \frac{\sqrt{x} + 4a}{\sqrt{x} + 3b}$, to find the value of x .

(18. Cor. 1.) multiplying both sides of the equation by $(\sqrt{x} + b) \cdot (\sqrt{x} + 3b)$,

$$x + (2a + 3b) \cdot \sqrt{x} + 6ab = x + (4a + b) \cdot \sqrt{x} + 4ab;$$

$$\therefore \text{ (17. Cor. 3.) } (2a - b) \cdot \sqrt{x} = 2ab,$$

$$\text{or (18. Cor. 2.) } \sqrt{x} = \frac{ab}{a - b};$$

$$\therefore \text{ (19) } x = \left(\frac{ab}{a - b} \right)^2.$$

39. Given $\frac{\sqrt{ax} - b}{\sqrt{ax} + b} = \frac{3\sqrt{ax} - 2b}{3\sqrt{ax} + 5b}$, to find the value of x .

(18. Cor. 1.) multiplying both sides of the equation by $(\sqrt{ax} + b) \cdot (3\sqrt{ax} + 5b)$,

$$3ax + 2b\sqrt{ax} - 5b^2 = 3ax + b\sqrt{ax} - 2b^2;$$

$$\therefore (17. \text{ Cor. 3.}) \quad b\sqrt{ax} = 3b^2;$$

$$(18. \text{ Cor. 2.}) \quad \sqrt{ax} = 3b;$$

$$(19) \text{ squaring both sides, } ax = 9b^2,$$

$$\text{and } (18. \text{ Cor. 2.}) \quad x = \frac{9b^2}{a}.$$

40. Given $\frac{3x - 1}{\sqrt{3x} + 1} = 1 + \frac{\sqrt{3x} - 1}{2}$, to find the value of x .

$$\text{Since } 3x - 1 = (\sqrt{3x} + 1) \times (\sqrt{3x} - 1);$$

$$\therefore \frac{3x - 1}{\sqrt{3x} + 1} = \sqrt{3x} - 1;$$

$$\therefore \sqrt{3x} - 1 = 1 + \frac{\sqrt{3x} - 1}{2},$$

and (17. Cor. 3.) taking $\frac{\sqrt{3x} - 1}{2}$ from each side,

$$\frac{\sqrt{3x} - 1}{2} = 1;$$

$$\therefore (18. \text{ Cor. 1.}) \quad \sqrt{3x} - 1 = 2;$$

$$\therefore (17) \text{ by transposition, } \sqrt{3x} = 2 + 1 = 3;$$

$$\therefore (19) \text{ squaring both sides, } 3x = 9,$$

$$(18. \text{ Cor. 2.}) \quad x = \frac{9}{3} = 3.$$

41. Given $\frac{ax - b^2}{\sqrt{ax} + b} = c + \frac{\sqrt{ax} - b}{c}$, to find the value of x .

Since $ax - b^2 = (\sqrt{ax} + b) \cdot (\sqrt{ax} - b)$;

$$\therefore \frac{ax - b^2}{\sqrt{ax} + b} = \sqrt{ax} - b;$$

$$\therefore \sqrt{ax} - b = c + \frac{\sqrt{ax} - b}{c};$$

and taking $\frac{\sqrt{ax} - b}{c}$ from each side,

$$\frac{c-1}{c} \cdot (\sqrt{ax} - b) = c,$$

$$\text{and (18. Cor. 1.) } \sqrt{ax} - b = \frac{c^2}{c-1};$$

$$\text{by transposition, } \sqrt{ax} = b + \frac{c^2}{c-1};$$

$$(19) \text{ squaring both sides, } ax = \left(b + \frac{c^2}{c-1}\right)^2;$$

$$\therefore (18. \text{ Cor. 2.}) x = \frac{1}{a} \cdot \left(b + \frac{c^2}{c-1}\right)^2.$$

42. Given $x = \sqrt{a^2 + x\sqrt{b^2 + x^2}} - a$, to find the value of x .

$$(17) \text{ by transposition, } x + a = \sqrt{a^2 + x\sqrt{b^2 + x^2}}$$

$$\therefore (19) \text{ squaring both sides, } x^2 + 2ax + a^2 = a^2 + x\sqrt{b^2 + x^2};$$

$$(17. \text{ Cor. 3.}) x^2 + 2ax = x\sqrt{b^2 + x^2};$$

$$(18. \text{ Cor. 2.}) x + 2a = \sqrt{b^2 + x^2};$$

$$\text{and (19) squaring both sides, } x^2 + 4ax + 4a^2 = b^2 + x^2$$

$$\therefore (17. \text{ Cor. 3.}) 4ax + 4a^2 = b^2;$$

$$(17) \text{ by transposition, } 4ax = b^2 - 4a^2;$$

$$\therefore (18. \text{ Cor. 2.}) x = \frac{b^2 - 4a^2}{4a}.$$

43. Given $\sqrt{2+x} + \sqrt{x} = \frac{4}{\sqrt{2+x}}$ to find the value of x .

$$(18. \text{ Cor. 1.}) \quad 2 + x + \sqrt{2x + x^2} = 4;$$

$\therefore$ (17) by transposition, $\sqrt{2x + x^2} = 4 - 2 - x = 2 - x$,

and (19) squaring both sides, $2x + x^2 = 4 - 4x + x^2$;

$$\therefore (17. \text{ Cor. 3.}) \quad 6x = 4;$$

$$\therefore (18. \text{ Cor. 2.}) \quad x = \frac{4}{6} = \frac{2}{3}.$$

44. Given $\sqrt{5+x} + \sqrt{x} = \frac{15}{\sqrt{5+x}}$, to find the value of x .

$$(18. \text{ Cor. 1.}) \quad 5 + x + \sqrt{5x + x^2} = 15;$$

$\therefore$ (17) by transposition, $\sqrt{5x + x^2} = 15 - 5 - x = 10 - x$,

and (19) squaring both sides, $5x + x^2 = 100 - 20x + x^2$;

$$\therefore (17. \text{ Cor. 3.}) \quad 25x = 100;$$

$$\therefore (18. \text{ Cor. 2.}) \quad x = \frac{100}{25} = 4.$$

45. Given $\sqrt{x + \sqrt{x}} - \sqrt{x - \sqrt{x}} = \frac{3}{2} \sqrt{\left(\frac{x}{x + \sqrt{x}}\right)}$, to find the value of x .

$$(18. \text{ Cor. 1.}) \quad x + \sqrt{x} - \sqrt{x^2 - x} = \frac{3\sqrt{x}}{2};$$

$$\therefore \text{ by transposition, } x - \frac{\sqrt{x}}{2} = \sqrt{x^2 - x},$$

$$\text{and (18. Cor. 2.) } \sqrt{x} - \frac{1}{2} = \sqrt{x - 1};$$

(19) squaring both sides, $x - \sqrt{x} + \frac{1}{4} = x - 1$;

$$\therefore (19. \text{ Cor. 3.}) \quad \sqrt{x} = \frac{5}{4};$$

and (19) squaring both sides, $x = \frac{25}{16}$.

46. Given $\frac{1}{x} + \frac{1}{a} = \sqrt{\frac{1}{a^2} + \sqrt{\frac{4}{a^2 x^2} + \frac{9}{x^4}}}$, to find the value of x .

(19) squaring both sides,

$$\frac{1}{x^2} + \frac{2}{ax} + \frac{1}{a^2} = \frac{1}{a^2} + \sqrt{\left(\frac{4}{a^2 x^2} + \frac{9}{x^4}\right)};$$

$$\therefore (17. \text{ Cor. } 3.) \frac{1}{x^2} + \frac{2}{ax} = \sqrt{\left(\frac{4}{a^2 x^2} + \frac{9}{x^4}\right)},$$

$$\text{and } (18. \text{ Cor. } 2.) \frac{1}{x} + \frac{2}{a} = \sqrt{\left(\frac{4}{a^2} + \frac{9}{x^2}\right)};$$

$$\therefore (19.) \text{ squaring both sides } \frac{1}{x^2} + \frac{4}{ax} + \frac{4}{a^2} = \frac{4}{a^2} + \frac{9}{x^2};$$

$$(17. \text{ Cor. } 3.) \frac{4}{ax} = \frac{8}{x^2};$$

$$(18. \text{ Cor. } 2.) \frac{1}{a} = \frac{2}{x};$$

$$\therefore (18. \text{ Cor. } 1.) x = 2a.$$

SECTION II.

On the Solution of Simple Equations which involve more than one unknown Quantity.

(23.) If the equation involve several unknown quantities, and definite values of these are required, there must necessarily be as many independent equations as there are unknown quantities. In which case, the values will be found by exterminating all the unknown quantities except one; and this may be done by either of the three following methods :

1. By equalizing the coefficients of the same unknown quantity in the several equations.
 2. By substitution.
 3. By equating different values of the same unknown quantity.
1. Of exterminating an unknown quantity by the first method in equations where two unknown quantities are concerned. .

If the coefficient of either unknown quantity in one equation be contained a certain number of times exactly in the coefficient of the same unknown quantity in the other, multiply the former equation by that number, then add it to, or subtract it from, the other equation, according as the signs are different or the same, and an equation arises, in which only one unknown quantity is found.

Thus, if $4x + y = 34$ } Here the coefficient of x in the
 and $4y + x = 16$ } second equation is contained 4 times exactly in the first ;
 multiplying therefore the second equation by 4, and sub-
 tracting the first from it,

$$\begin{array}{r} 4x + 16y = 64, \\ \text{and } 4x + y = 34; \\ \hline \therefore 15y = 30, \text{ and } y = 2. \end{array}$$

Having thus obtained a value of one of the unknown quantities, the other may be determined by substituting in either equation the value of the quantity found, and thus reducing the equation to one which contains only the other unknown quantity. Thus, from the second of the preceding equations, $x = 16 - 4y = 16 - 8 = 8$.

The values of x and y might be found in a similar manner, by multiplying the first equation by 4, and subtracting the second from it.

But if neither of the coefficients be a measure of the coefficient of the same unknown quantity in the other equation, multiply the first equation by the coefficient of one of the unknown quantities in the second equation, and the second equation by the coefficient of the same unknown quantity in the first. If the signs of the unknown quantity be alike in both, subtract one equation from the other; if unlike, add them together, and an equation arises in which only one unknown quantity is found.

Thus, if $2x + 3y = 23$ } In this case neither of the coeffi-
 and $5x - 2y = 10$ } cients is a measure of the coefficient of the same unknown
 quantity in the other equation; and therefore, multiplying the
 first equation by 2, and the second by 3,

$$\begin{array}{r} 4x + 6y = 46, \\ \text{and } 15x - 6y = 30; \\ \hline \therefore \text{ by addition, } 19x = 76, \text{ and } x = 4; \end{array}$$

whence, as before, $3y = 23 - 2x = 23 - 8 = 15$,

and $y = 5$.

The values of x and y might also be obtained, by multiplying the first equation by 5, and the second by 2, and then subtracting the second from the first.

2. By substitution.

Find the value of one of the unknown quantities, in terms of the other and known quantities, in the more simple of the two equations; and substitute this value instead of the quantity itself in the other equation; thus an equation is obtained in which there is only one unknown quantity.

Thus in the first of the preceding examples; from the second equation, $x = 16 - 4y$; substituting therefore this value of x in the first equation,

$$4 \cdot (16 - 4y) + y = 34,$$

$$\text{or } 64 - 16y + y = 34;$$

$$\therefore \text{ by transposition, } (64 - 34 =) 30 = 15y,$$

$$\text{and therefore } 2 = y;$$

whence, as before, $x = 8$.

Here a value of x might have been obtained from the second equation, and substituted for it in the first; whence an equation would have arisen, involving only y ; the value of which being found, that of x also might be determined, as before, by substitution.

Or a value of y might be determined from either equation, and substituted in the other; from which would arise an equation involving only x , the value of which might be found; and therefore the value of y also might be obtained by substitution.

Again, in the second example; from the first equation is obtained

$$2x = 23 - 3y; \text{ and therefore } x = \frac{23 - 3y}{2};$$

substituting therefore this value in the second equation,

$$5 \cdot \frac{23 - 3y}{2} - 2y = 10,$$

$$\text{or } 115 - 15y - 4y = 20;$$

$$\therefore \text{ by transposition, } 115 - 20 = 15y + 4y,$$

$$\text{or } 95 = 19y;$$

$$\therefore 5 = y,$$

$$\text{and } x = \frac{23 - 3y}{2} = \frac{23 - 15}{2} = \frac{8}{2} = 4.$$

Here also a value of x might be obtained from the second equation, and substituted in the first, which would give an equation involving only y ; or a value of y might be obtained from either equation, which substituted in the other would give an equation involving only x ; the value of which might therefore be found, and consequently that of y might also be determined.

3. By equating different values of the same unknown quantity.

From each equation find the value of the same unknown quantity in terms of the other and known quantities; then, by equating the values so found, an equation arises containing only one unknown quantity.

Thus in the first of the preceding examples; from the first equation, $y = 34 - 4x$,

and from the second equation,

$$4y = 16 - x; \text{ and therefore } y = \frac{16 - x}{4};$$

$$\therefore \frac{16 - x}{4} = 34 - 4x;$$

$$\text{consequently, } 16 - x = 136 - 16x;$$

$$\therefore \text{ by transposition, } 16x - x = 136 - 16,$$

$$\text{or } 15x = 120;$$

$$\therefore x = 8,$$

$$\text{and } y = 34 - 4x = 34 - 32 = 2, \text{ as before.}$$

In this case also, two values of x are deducible from the two equations, which would give an equation involving y only; and the value of y being determined, that of x might also be found.

Again, in the second of the preceding examples ;

$$\text{from the first equation, } x = \frac{23 - 3y}{2},$$

$$\text{and from the second, } x = \frac{10 + 2y}{5};$$

$$\therefore \frac{10 + 2y}{5} = \frac{23 - 3y}{2},$$

$$\text{and } 20 + 4y = 115 - 15y;$$

$$\text{by transposition, } 4y + 15y = 115 - 20,$$

$$\text{or } 19y = 95;$$

$$\therefore y = 5, \text{ and } x = 4, \text{ as before.}$$

Here again two values of y might have been found, which would have given an equation involving only x ; and from the solution of this new equation, a value of x , and therefore of y , might be found.

EXAMPLES.

1. Given $5x + 4y = 58$ } to find the values of x and y .
and $3x + 7y = 67$ }

Multiplying the second equation by 5, and the first by 3,

$$15x + 35y = 335,$$

$$\text{and } 15x + 12y = 174;$$

$$\therefore \text{ by subtraction, } 23y = 161,$$

$$\text{and } y = 7;$$

$$\text{whence, } 5x = 58 - 4y = 58 - 28 = 30,$$

$$\text{and therefore } x = 6.$$

If the second equation had been multiplied by 4, and subtracted from the first when multiplied by 7, an equation would have arisen, involving only x , the value of which might be determined, and thence, by substitution, the value of y .

Second Method.

From the second equation, $3x = 67 - 7y$;

$$\therefore x = \frac{67 - 7y}{3}.$$

Substituting this value of x in the first equation,

$$5 \cdot \frac{67 - 7y}{3} + 4y = 58,$$

$$\text{and } 335 - 35y + 12y = 174;$$

$$\therefore \text{ by transposition, } 335 - 174 = 35y - 12y,$$

$$\text{or } 161 = 23y;$$

$$\therefore 7 = y;$$

whence, as before, the value of x may be found. In the same manner, a value of x might be found from the first equation, which substituted in the second, would give an equation involving only y . Or a value of y might be obtained from either equation, and substituted for it in the other; whence an equation would arise involving only x , the value of which might be found, and therefore that of y also determined.

Third Method.

From the first equation, $5x = 58 - 4y$;

$$\therefore x = \frac{58 - 4y}{5}.$$

$$\text{From the second, } x = \frac{67 - 7y}{3}.$$

$$\therefore \frac{58 - 4y}{5} = \frac{67 - 7y}{3},$$

$$\text{and } 174 - 12y = 335 - 35y;$$

by transposition, $35y - 12y = 335 - 174$,

$$\text{or } 23y = 161;$$

$$\therefore y = 7;$$

whence, as before, $x = 6$.

In this case, two values of y might be deduced from the two equations; and from equating these, there would arise an equation involving x only; whose value being found, that of y also might be determined by substitution.

$$2. \quad \left. \begin{array}{l} \text{Given } ax + by = m \\ \phantom{\text{Given }} cx + dy = n \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

Multiplying the first equation by c , and the second by a ,

$$acx + bcy = mc,$$

$$acx + ady = na;$$

$$\therefore \text{ by subtraction, } (ad - bc) \cdot y = na - mc,$$

$$\text{and } y = \frac{na - mc}{ad - bc};$$

$$\text{whence } x = \frac{m}{a} - \frac{by}{a} = \frac{m}{a} - \frac{nab - mbc}{a^2d - abc},$$

$$= \frac{mad - mbc}{a^2d - abc} - \frac{nab - mbc}{a^2d - abc},$$

$$= \frac{mad - nab}{a^2d - abc} = \frac{md - nb}{ad - bc}.$$

Or the value of x might be determined from the second equation, $x = \frac{n}{c} - \frac{dy}{c}$.

If the first equation had been multiplied by d , and subtracted from the second multiplied by b , an equation would have arisen involving only x , the value of which might be determined; and this being substituted in either of the equations, the value of y might also be found.

Second Method.

From the first equation, $ax = m - by$;

$$\text{and } \therefore x = \frac{m - by}{a}.$$

Substituting this value of x in the second equation,

$$c \cdot \frac{m - by}{a} + dy = n;$$

$$\therefore mc - bcy + ady = an,$$

$$\text{and } (ad - bc) \cdot y = an - mc;$$

$$\therefore y = \frac{an - mc}{ad - bc};$$

whence, the value of x may be determined, as before.

In the same manner, a value of x might be found from the second equation, which substituted in the first would give an equation involving only y , the value of which being found, that of x might also be determined. Or, a value of y might be obtained from either equation, which substituted in the other would give an equation involving only x , the value of which, and consequently that of y , might be found.

Third Method.

From the first equation, $x = \frac{m - by}{a}$,

and from the second, $x = \frac{n - dy}{c}$;

$$\therefore \frac{m - by}{a} = \frac{n - dy}{c};$$

$$\text{and } mc - bcy = na - ady;$$

$$\therefore \text{ by transposition, } ady - bcy = na - mc;$$

$$\therefore y = \frac{na - mc}{ad - bc};$$

$$\text{whence, as before, } x = \frac{md - nb}{ad - bc}.$$

In this case, two values of y might be deduced from the two equations; and from equating these, there would arise another equation involving only x , the value of which being determined, that of y also might be found by substitution.

$$\left. \begin{array}{l} 3. \text{ Given } 11x + 3y = 100 \\ \text{and } 4x - 7y = 4 \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

Multiplying the first equation by 7, and the second by 3,

$$77x + 21y = 700,$$

$$\text{and } 12x - 21y = 12;$$

$$\therefore \text{ by addition, } \quad 89x = 712,$$

$$\text{and } x = 8;$$

$$\text{whence } 3y = 100 - 11x = 100 - 88 = 12;$$

$$\therefore y = 4.$$

$$\left. \begin{array}{l} 4. \text{ Given } \frac{x}{2} + \frac{y}{3} = 7 \\ \text{and } \frac{x}{3} + \frac{y}{2} = 8 \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

(18. Cor. 1.) clearing the equations of fractions, by multiplying each by 6,

$$3x + 2y = 42,$$

$$\text{and } 2x + 3y = 48;$$

and as the coefficients in this case are not aliquot parts, multiplying the first by 3, and the second by 2;

$$\therefore 9x + 6y = 126,$$

$$\text{and } 4x + 6y = 96;$$

$$\therefore \text{ by subtraction, } \quad 5x = 30,$$

$$\text{and } x = 6;$$

$$\text{whence } 2y = 42 - 3x = 42 - 18 = 24,$$

$$\text{and } y = 12.$$

$$5. \quad \left. \begin{array}{l} \text{Given } \frac{x}{7} + 7y = 99 \\ \text{and } \frac{y}{7} + 7x = 51 \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

(18. Cor. 1.) multiplying each equation by 7,

$$\therefore x + 49y = 693,$$

$$\text{and } 49x + y = 357;$$

$$\therefore \text{ by addition, } 50x + 50y = 1050,$$

$$\text{and } \therefore x + y = 21;$$

$$\text{but since } x + 49y = 693,$$

subtracting the upper equation from the lower,

$$48y = 672;$$

$$\therefore y = 14,$$

$$\text{whence } x = 21 - y = 21 - 14 = 7.$$

$$6. \quad \left. \begin{array}{l} \text{Given } \frac{x+2}{3} + 8y = 31 \\ \text{and } \frac{y+5}{4} + 10x = 192 \end{array} \right\} \begin{array}{l} \text{to find the values of } x \text{ and} \\ y. \end{array}$$

Clearing the first equation of fractions,

$$x + 2 + 24y = 93;$$

$$\therefore \text{ by transposition, } x + 24y = 91.$$

Clearing the second equation of fractions,

$$y + 5 + 40x = 768;$$

$$\therefore \text{ by transposition, } 40x + y = 763.$$

Multiplying the first equation by 40, and subtracting the second from it,

$$40x + 960y = 3640;$$

$$40x + y = 763;$$

$$\therefore 959y = 2877,$$

$$\text{and } y = 3;$$

$$\therefore x = 91 - 24y = 91 - 72 = 19.$$

$$7. \quad \left. \begin{array}{l} \text{Given } \frac{2x - y}{2} + 14 = 18 \\ \text{and } \frac{2y + x}{3} + 16 = 19 \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

By transposition, $\frac{2x - y}{2} = 4$, from the first equation,

$$\text{and } \therefore 2x - y = 8.$$

Also, $\frac{2y + x}{3} = 3$, from the second equation,

$$\text{and } \therefore 2y + x = 9;$$

which multiplied by 2, gives $2x + 4y = 18$;

$$\text{but } 2x - y = 8;$$

$$\therefore \text{ by subtraction, } 5y = 10,$$

$$\text{and } y = 2,$$

$$\text{whence } x = 9 - 2y = 9 - 4 = 5.$$

$$8. \quad \left. \begin{array}{l} \text{Given } \frac{2x + 3y}{6} + \frac{x}{3} = 8 \\ \text{and } \frac{7y - 3x}{2} - y = 11 \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

Clearing the first equation of fractions, $2x + 3y + 2x = 48$,

$$\text{or } 4x + 3y = 48;$$

and clearing the second of fractions, $7y - 3x - 2y = 22$,

$$\text{or } 5y - 3x = 22.$$

Multiplying this by 4, and the preceding one by 3,

$$9y + 12x = 144,$$

$$\text{and } 20y - 12x = 88;$$

$$\therefore \text{ by addition, } 29y = 232,$$

$$\text{and } y = 8,$$

$$\text{whence } 4x = 48 - 3y = 48 - 24 = 24,$$

$$\text{and } x = 6.$$

$$9. \quad \left. \begin{array}{l} \text{Given } 3x + \frac{7y}{2} = 22 \\ \text{and } 11y - \frac{2x}{5} = 20 \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

Clearing the first equation of fractions, $6x + 7y = 44$;

but from the second, $55y - 2x = 100$.

Multiplying this last by 3, $165y - 6x = 300$,

$$\text{but } 7y + 6x = 44$$

$$\therefore \text{ by addition, } 172y = 344,$$

$$\text{and } y = 2.$$

$$\text{Now } 6x = 44 - 7y = 44 - 14 = 30;$$

$$\therefore x = 5.$$

$$10. \quad \left. \begin{array}{l} \text{Given } x + 1 : y :: 5 : 3 \\ \text{and } \frac{2x}{3} - \frac{5-y}{2} = \frac{41}{12} - \frac{2x-1}{4} \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

From the second equation, (18. Cor. 1.) multiplied by 12,

$$8x - 30 + 6y = 41 - 6x + 3;$$

$$\therefore \text{ by transposition, } 14x + 6y = 74,$$

$$\text{and } 7x + 3y = 37.$$

$$\text{But from the first equation, } 5y = 3x + 3,$$

$$\text{or } 5y - 3x = 3.$$

$$\text{Multiplying this equation by 7, } 35y - 21x = 21,$$

$$\text{and the former by 3, } 9y + 21x = 111;$$

$$\therefore \text{ by addition, } 44y = 132;$$

$$\text{and } y = 3;$$

$$\therefore x + 1 = \frac{5y}{3} = 5, \text{ and } x = 4.$$

$$11. \quad \left. \begin{array}{l} \text{Given } \frac{x-2}{5} - \frac{10-x}{3} = \frac{y-10}{4} \\ \text{and } \frac{2y+4}{3} - \frac{2x+y}{8} = \frac{x+13}{4} \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

(18. Cor. 1.) multiplying the first equation by 60,

$$12x - 24 - 200 + 20x = 15y - 150;$$

and by transposition, $32x - 15y = 74$.

Also (18. Cor. 1.) multiplying the second equation by 24,

$$16y + 32 - 6x - 3y = 6x + 78;$$

$\therefore$ by transposition, $13y - 12x = 46$.

Now the coefficients of x have aliquot parts; multiplying therefore this by 8, and the preceding by 3,

$$104y - 96x = 368,$$

$$\text{and } 96x - 45y = 222;$$

$\therefore$ by addition, $59y = 590$; and $y = 10$;

$$\text{and } 32x = 15y + 74 = 150 + 74 = 224;$$

$$\therefore x = 7.$$

$$12. \quad \left. \begin{array}{l} \text{Given } 2y - \frac{x+3}{4} = 7 + \frac{3x-2y}{5} \\ \text{and } 4x - \frac{8-y}{3} = 24\frac{1}{2} - \frac{2x+1}{2} \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

(18. Cor. 1.) from the first equation,

$$40y - 5x - 15 = 140 + 12x - 8y;$$

$\therefore$ by transposition, $48y - 17x = 155$,

and from the second equation,

$$24x - 16 + 2y = 147 - 6x - 3;$$

$\therefore$ by transposition, $30x + 2y = 160$.

Multiplying this by 24, $48y + 720x = 3840$;

$$\text{but } 48y - 17x = 155;$$

$\therefore$ by subtraction, $737x = 3685$,

$$\text{and } x = 5,$$

$$\text{and } 2y = 160 - 30x = 160 - 150 = 10;$$

$$\therefore y = 5.$$

$$13. \quad \left. \begin{array}{l} \text{Given } \frac{2y}{18} - \frac{8x-2}{36} = 1 - \frac{4+y}{3} + \frac{x-y}{6} \\ \text{and } x : 3y :: 4 : 7 \end{array} \right\} \text{ to find the}$$

values of x and y .

Reducing the first equation to lower terms,

$$\frac{y}{9} - \frac{4x-1}{18} = 1 - \frac{4+y}{3} + \frac{x-y}{6};$$

and therefore (18. Cor. 1.) multiplying by 18,

$$2y - 4x + 1 = 18 - 24 - 6y + 3x - 3y;$$

$$\therefore \text{ by transposition, } 7 = 7x - 11y.$$

But from the second equation, $7x = 12y$.

Substituting therefore this value in the preceding equation,

$$7 = 12y - 11y = y,$$

$$\text{and therefore } x = \frac{12y}{7} = 12.$$

$$14. \quad \left. \begin{array}{l} \text{Given } x - \frac{3y-2+x}{11} = 1 + \frac{15x + \frac{4y}{3}}{33} \\ \text{and } \frac{3x+2y}{6} - \frac{y-5}{4} = \frac{11x+152}{12} - \frac{3y+1}{2} \end{array} \right\} \text{ to}$$

find the values of x and y .

(18. Cor. 1.) multiplying the first equation by 33,

$$33x - 9y + 6 - 3x = 33 + 15x + \frac{4y}{3};$$

by transposition, $15x - 9y = 27 + \frac{4y}{3}$;

$$\therefore 45x - 27y = 81 + 4y,$$

$$\text{and } 45x - 31y = 81.$$

(18. Cor. 1.) multiplying the second equation by 12,

$$6x + 4y - 3y + 15 = 11x + 152 - 18y - 6;$$

$$\therefore \text{ by transposition, } 19y - 5x = 131.$$

Multiplying this by 9, $171y - 45x = 1179$;

$$\text{but } 45x - 31y = 81;$$

$$\therefore \text{ by addition, } 140y = 1260;$$

$$\therefore y = 9,$$

$$\text{and } 5x = 19y - 131 = 171 - 131 = 40;$$

$$\therefore x = 8.$$

$$15. \quad \text{Given } \left. \begin{array}{l} \frac{80 + 3x}{15} = 18\frac{1}{3} - \frac{4x + 3y - 8}{7} \\ \text{and } 10y + \frac{6x - 35}{5} = 55 + 10x \end{array} \right\} \begin{array}{l} \text{to find the values} \\ \text{of } x \text{ and } y. \end{array}$$

(18. Cor. 1.) multiplying the first equation by 105, the least common multiple of 3, 7, and 15,

$$560 + 21x = 1925 - 60x - 45y + 120;$$

$$\therefore \text{ by transposition, } 81x + 45y = 1485;$$

$$\text{and dividing by 9, } 9x + 5y = 165.$$

From the second equation, $50y + 6x - 35 = 275 + 50x$;

$$\therefore \text{ by transposition, } 50y - 44x = 310;$$

$$\text{and dividing by 2, } 25y - 22x = 155;$$

$$\left. \begin{array}{l} \text{but multiplying the equation} \\ \text{found above, by 5,} \end{array} \right\} 25y + 45x = 825;$$

$$\therefore \text{ by subtraction, } 67x = 670,$$

$$\text{and } x = 10.$$

$$\text{Now } 5y = 165 - 9x = 165 - 90 = 75;$$

$$\therefore y = 15.$$

$$\left. \begin{array}{l} 16. \text{ Given } y + \frac{5x + 2y}{6} - \frac{3y - 12 + 8x}{5} = 4 - \frac{15 + 2x - 4y}{3} \\ \text{and } \frac{7x + 13 - 5y}{4} + x = 2y - \frac{3x + 2y - 16}{3} \end{array} \right\}$$

to find the values of x and y .

(18. Cor. 1.) multiplying the first equation by 30, the least common multiple of 3, 5, and 6,

$$30y + 25x + 10y - 18y + 72 - 48x = 120 - 150 - 20x + 40y;$$

$$\text{whence, by transposition, } 102 = 18y + 3x;$$

$$\text{and dividing by 3, } 34 = 6y + x.$$

Clearing the second equation of fractions,

$$21x + 39 - 15y + 12x = 24y - 12x - 8y + 64;$$

$$\text{and by transposition, } 45x - 31y = 25.$$

$$\text{Multiplying the former by 45, } 45x + 270y = 1530;$$

$$\therefore \text{ by subtraction, } 301y = 1505.$$

$$\text{and } y = 5;$$

$$\text{whence } x = 34 - 6y = 34 - 30 = 4.$$

$$\left. \begin{array}{l} 17. \text{ Given } 1 + \frac{25 + 5y}{6} - \frac{7x - 6}{3} = 10 - \frac{3x - 10 + 7y}{12} \\ \text{and } \frac{12 - x}{9} : 5x - \frac{14 + y}{3} :: 1 : 8 \end{array} \right\}$$

to find the values of x and y .

(18. Cor. 1.) multiplying the first equation by 12, the least common multiple of 3, 6, and 12,

$$12 + 50 + 10y - 28x + 24 = 120 - 3x + 10 - 7y;$$

$$\therefore \text{ by transposition, } 17y - 25x = 44,$$

and (21) from the second equation,

$$\frac{96 - 8x}{9} = 5x - \frac{14 + y}{3};$$

$$\therefore 96 - 8x = 45x - 42 - 3y;$$

$$\text{and by transposition, } 138 = 53x - 3y.$$

Multiplying this equation by 17, and the one found above by 3,

$$51y - 75x = 132,$$

$$\text{and } -51y + 901x = 2346;$$

$$\therefore \text{ by addition, } 826x = 2478,$$

$$\text{and } x = 3.$$

$$\text{Now } 3y = 53x - 138 = 159 - 138 = 21;$$

$$\therefore y = 7.$$

$$18. \quad \left. \begin{array}{l} \text{Given } \frac{4x}{x^2} + \frac{5y}{y^2} = \frac{9}{y} - 1 \\ \text{and } \frac{5}{x} + \frac{4}{y} = \frac{7}{x} + \frac{3}{2} \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

Reducing the first equation to lower terms,

$$\frac{4}{x} + \frac{5}{y} = \frac{9}{y} - 1;$$

$$\therefore \text{ by transposition, } \frac{4}{x} - \frac{4}{y} = -1;$$

$$\text{from the second equation, by transposition, } -\frac{2}{x} + \frac{4}{y} = \frac{3}{2};$$

$$\therefore \text{ by addition, } \frac{2}{x} = \frac{1}{2};$$

$$\therefore 4 = x,$$

$$\text{and } \frac{4}{y} = \frac{4}{x} + 1 = 2;$$

$$\therefore 2y = 4, \text{ and } y = 2.$$

$$19. \quad \left. \begin{array}{l} \text{Given } \frac{a}{x} + \frac{b}{y} = m \\ \text{and } \frac{c}{x} + \frac{d}{y} = n \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

Multiplying the first equation by c , and the second by a ,

$$\frac{ac}{x} + \frac{bc}{y} = mc,$$

$$\text{and } \frac{ac}{x} + \frac{ad}{y} = na,$$

$$\therefore \text{ by subtraction, } (bc - ad) \cdot \frac{1}{y} = mc - na;$$

$$\therefore y = \frac{bc - ad}{mc - na},$$

$$\begin{aligned} \text{and } \frac{a}{x} &= m - \frac{b}{y} = m - \frac{mbc - nab}{bc - ad} \\ &= \frac{mbc - mad - mbc + nab}{bc - ad} \\ &= \frac{nab - mad}{bc - ad}; \end{aligned}$$

$$\therefore \frac{1}{x} = \frac{nb - md}{bc - ad},$$

$$\text{and } x = \frac{bc - ad}{nb - md}.$$

$$20. \quad \left. \begin{array}{l} \text{Given } x - \frac{2y - x}{23 - x} = 20 - \frac{59 - 2x}{2} \\ \text{and } y + \frac{y - 3}{x - 18} = 30 - \frac{73 - 3y}{3} \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

Multiplying the first equation by 2,

$$2x - \frac{4y - 2x}{23 - x} = 40 - 59 + 2x;$$

$$\therefore \text{ by transposition, } 19 = \frac{4y - 2x}{23 - x},$$

$$\text{and } 437 - 19x = 4y - 2x;$$

$$\text{and by transposition, } 437 = 17x + 4y.$$

$$\text{Also from the 2}^{\text{d}} \text{ equation, } 3y + \frac{3y - 9}{x - 18} = 90 - 73 + 3y,$$

$$\text{and (17. Cor. 3.) } \frac{3y - 9}{x - 18} = 17;$$

$$\therefore 3y - 9 = 17x - 306;$$

$$\text{by transposition, } 297 = 17x - 3y;$$

$$\text{but } 437 = 17x + 4y;$$

$$\therefore \text{ by subtraction, } 140 = 7y,$$

$$\text{and } 20 = y,$$

$$\text{and } 17x = 297 + 3y = 297 + 60 = 357;$$

$$\therefore x = 21.$$

$$\left. \begin{array}{l} 21. \text{ Given } 8x - \frac{16 + 60x}{3y - 1} = \frac{16xy - 107}{5 + 2y} \\ \text{and } 2 + 6y + 9x = \frac{27x^2 - 12y^2 + 38}{3x - 2y + 1} \end{array} \right\} \begin{array}{l} \text{to find the values} \\ \text{of } x \text{ and } y. \end{array}$$

Multiplying the first equation by $5 + 2y$,

$$40x + 16xy - \frac{80 + 300x + 32y + 120xy}{3y - 1} = 16xy - 107;$$

$$\therefore \text{ by trans}^{\text{n}}, 40x + 107 = \frac{80 + 300x + 32y + 120xy}{3y - 1};$$

and multiplying by $3y - 1$,

$$120xy - 40x + 321y - 107 = 80 + 300x + 32y + 120xy;$$

$$\therefore (17. \text{ Cor. 3.}) 289y - 340x = 187.$$

And from the second equation,

$$27x^2 - 12y^2 + 15x + 2y + 2 = 27x^2 - 12y^2 + 38;$$

$$\therefore \text{ by transposition, } 15x + 2y = 36;$$

whence, the coefficients of x having aliquot parts, multiplying the first equation by 3, and the second by 68,

$$867y - 1020x = 561,$$

$$\text{and } 136y + 1020x = 2448;$$

$$\therefore \text{ by addition, } 1003y = 3009,$$

$$\text{and } y = 3,$$

$$\text{and } 15x = 36 - 2y = 36 - 6 = 30;$$

$$\therefore x = 2.$$

$$22. \text{ Given } \left. \begin{aligned} \frac{3x+2y}{5} - \frac{5x-\frac{3y}{4}+1}{3} &= x + \frac{y-2x}{10} - \frac{4x-y}{7} \end{aligned} \right\}$$

and $y + 2x : y - 2x :: 12x + 6y - 3 : 6y - 12x - 1$
to find the values of x and y .

(18. Cor. 1.) multiplying the first equation by 420,

$$252x + 168y - 700x + 105y - 140 = 420x + 42y - 84x - 240x + 60y,$$

$$\text{and by transposition, } 171y - 544x = 140.$$

From the second equation, (*Wood's Alg.* 182.)

$$2y : 4x :: 12y - 4 : 24x - 2,$$

$$\text{and (Alg. 184.) } y : 2x :: 6y - 2 : 12x - 1;$$

$$\therefore (21) \ 12xy - y = 12xy - 4x;$$

$$(17. \text{ Cor. 3.}) \ y = 4x.$$

Which value of y being substituted in the first equation,

$$684x - 544x = 140,$$

$$\text{or } 140x = 140;$$

$$\therefore x = 1,$$

$$\text{and } y = 4x = 4.$$

$$\left. \begin{aligned}
 23. \text{ Given } 3 - \frac{7 + \frac{2x}{y}}{5} &= 5 - \frac{5x + 9}{3y} \\
 \text{and } y - \frac{4 + 15y}{6x - 2} &= \frac{2xy - \frac{107}{8}}{2x + 5}
 \end{aligned} \right\}$$

to find the values of x and y .

(18. Cor. 1.) multiplying the first equation by $15y$,

$$\therefore 45y - 21y - 6x = 75y - 25x - 45;$$

and by transposition, $51y - 19x = 45$.

Multiplying the second equation by $2x + 5$,

$$2xy + 5y - \frac{8x + 20 + 30xy + 75y}{6x - 2} = 2xy - \frac{107}{8};$$

$$\therefore (17. \text{ Cor. 3.}) 5y + \frac{107}{8} = \frac{8x + 20 + 30xy + 75y}{6x - 2};$$

and multiplying by $6x - 2$,

$$30xy - 10y + \frac{321x - 107}{4} = 8x + 20 + 30xy + 75y;$$

$$\therefore (17. \text{ Cor. 3.}) \frac{321x - 107}{4} = 8x + 85y + 20,$$

$$\text{and } 321x - 107 = 32x + 340y + 80;$$

and by transposition, $-187 = 340y - 289x$.

The coefficients of y in this case having aliquot parts; multiplying the first by 20, and the last by 3,

$$1020y - 380x = 900,$$

$$\text{and } 1020y - 867x = -561;$$

$$\therefore \text{ by subtraction, } 487x = 1461,$$

$$\text{and } x = 3;$$

$$\text{consequently, } 51y = 45 + 19x = 45 + 57 = 102;$$

$$\therefore y = 2.$$

(24.) If there be three unknown quantities, their values may be found from three independent equations.

For from two of the equations, a third, which involves only two of the unknown quantities, may be deduced by the preceding rules; and from the remaining equation, and one of the others, another which contains the same two unknown quantities. Having therefore two equations, which involve only two unknown quantities, these may be determined; and, by substituting their values in any of the original equations, that of the third quantity will be obtained. In some particular equations, two unknown quantities may be exterminated at once.

EXAMPLES.

$$\left. \begin{array}{l} 1. \text{ Given } x + y + z = 31 \\ \quad \quad x + y - z = 25 \\ \quad \quad x - y - z = 9 \end{array} \right\} \text{ to find the values of } x, y \text{ and } z.$$

Adding the first and third equations, $2x = 40$;

$$\therefore x = 20.$$

Subtracting the second from the first, $2z = 6$;

$$\therefore z = 3;$$

and subtracting the third from the second, $2y = 16$;

$$\therefore y = 8.$$

$$\left. \begin{array}{l} 2. \text{ Given } x + y + z = 29 \\ \quad \quad x + 2y + 3z = 62 \\ \quad \quad \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 10 \end{array} \right\} \begin{array}{l} \text{to find the values of } x, y \text{ and} \\ \quad \quad \quad \quad \quad \quad \quad \quad z. \end{array}$$

Subtracting the first equation from the second,

$$y + 2z = 33.$$

(18. Cor. 1.) multiplying the third equation by 12, the least common multiple of 2, 3, and 4,

$6x + 4y + 3z = 120;$

multiplying the first equation by 6, $6x + 6y + 6z = 174;$

$$\begin{array}{rcl} \therefore \text{ by subtraction,} & 2y + 3z = & 54; \\ \text{but} & 2y + 4z = & 66; \end{array}$$

$$\therefore \text{ by subtraction,} \quad z = 12,$$

$$\text{and } y = 33 - 2z = 33 - 24 = 9;$$

$$\text{also } x = 29 - y - z = 29 - 9 - 12 = 8.$$

In like manner, had the first equation been multiplied by 2, and subtracted from the second, an equation would have resulted, involving only x and z ; and had it been multiplied by 4, and subtracted from the third when cleared of fractions, another equation would have been obtained, involving also x and z ; whence, by the preceding rules, the values of x and z would be found, and consequently the value of y also, by substitution. Or if the first equation be multiplied by 3, and the second subtracted from it, an equation would arise involving only x and y ; and if the first, when multiplied by 3, be subtracted from the third when cleared of fractions, another would arise involving only x and y ; whence the values of x and y might be determined. And hence the third, that of z , might be found.

Second Method.

From the first equation, $x = 29 - y - z;$

$\therefore$ substituting this value of x in the second equation,

$$29 - y - z + 2y + 3z = 62;$$

$$\therefore \text{ by transposition, } y = 33 - 2z.$$

Also substituting, in the third equation, the value of x found from the first,

$$\frac{29 - y - z}{2} + \frac{y}{3} + \frac{z}{4} = 10;$$

$$\therefore (18. \text{ Cor. 1.}) \quad 174 - 6y - 6z + 4y + 3z = 120,$$

and by transposition, $54 = 3z + 2y$;

in which, substituting the value of y found above,

$$54 = 3z + 66 - 4z;$$

$\therefore$ by transposition, $z = 12$;

$$\text{whence } y = 33 - 2z = 33 - 24 = 9,$$

$$\text{and } x = 29 - y - z = 29 - 9 - 12 = 8.$$

It may be observed, that there will be the same variety of solution, as in the last case, according as x , y , or z , is exterminated.

Third Method.

From the first equation, $x = 29 - y - z$,

and from the second, $x = 62 - 2y - 3z$;

$$\therefore 29 - y - z = 62 - 2y - 3z,$$

and by transposition, $y = 33 - 2z$.

Again, from the third equation, $x = 20 - \frac{2y}{3} - \frac{z}{2}$;

$$\therefore 29 - y - z = 20 - \frac{2y}{3} - \frac{z}{2};$$

and by transposition, $9 - \frac{z}{2} = \frac{y}{3}$;

$$\therefore 27 - \frac{3z}{2} = y;$$

$$\text{whence } 27 - \frac{3z}{2} = 33 - 2z;$$

$$\therefore \text{ by transposition, } \frac{z}{2} = 6,$$

and $z = 12$;

whence $y = 9$, and $x = 8$, as before.

The same observation applies to this solution, as did to the last.

$$\begin{aligned}
 3. \quad & \text{Given } \frac{4x + 3y + z}{10} - \frac{2y + 2z - x + 1}{15} = 5 + \frac{x - z - 5}{5} \\
 & \text{and } \frac{9x + 5y - 2z}{12} - \frac{2x + y - 3z}{4} = \frac{7y + z + 3}{11} + \frac{1}{6} \\
 & \text{and } \frac{5y + 3z}{4} - \frac{2x + 3y - z}{12} + 2z = y - 1 + \frac{3x + 2y + 7}{6}
 \end{aligned}$$

to find the values of x , y , and z .

Multiplying the first equation by 30, the least common multiple of 5, 10, and 15,

$$12x + 9y + 3z - 4y - 4z + 2x - 2 = 150 + 6x - 6z - 30;$$

$$\therefore \text{by transposition, } 8x + 5y + 5z = 122.$$

Again, multiplying the second equation by 132, the least common multiple of 4, 6, 11, 12,

$$99x + 55y - 22z - 66x - 33y + 99z = 84y + 12z + 36 + 22;$$

$$\therefore \text{by transposition, } 33x - 62y + 65z = 58.$$

Again, multiplying the third equation by 12, the least common multiple of 12, 6, 4,

$$15y + 9z - 2x - 3y + z + 24z = 12y - 12 + 6x + 4y + 14;$$

$$\therefore \text{by transposition, } 8x + 4y - 34z = -2;$$

$$\text{but from the first equation, } 8x + 5y + 5z = 122;$$

$$\therefore \text{by subtraction, } \underline{y + 39z = 124}.$$

Also the third equation being divided by 2,

$$4x + 2y - 17z = -1.$$

Multiplying this by 33, and the second by 4,

$$132x + 66y - 561z = -33,$$

$$\text{and } 132x - 248y + 260z = 232;$$

$$\therefore \text{by subtraction, } \underline{314y - 821z = -265};$$

$$\text{but } \underline{314y + 12246z = 35936}$$

(by multiplying the equation found above by 314);

$$\therefore \text{by subtraction, } 13067z = 39201,$$

$$\text{and therefore } z = 3;$$

$$\text{whence } y = 124 - 39z = 124 - 117 = 7,$$

$$\text{and } 4x = 17z - 2y - 1 = 51 - 14 - 1 = 36;$$

$$\therefore x = 9.$$

(25.) If there be four unknown quantities, their values may be found from four independent equations. For from the four given equations, by the preceding rules, three may be deduced which involve only three unknown quantities, the values of which may be found by the last Article; and hence the fourth may be found, by substituting in any of the four given equations, the values of the three quantities determined.

If there be n unknown quantities and n independent equations, the values of those quantities may be found in a similar manner. For from the n given equations, $n - 1$ may be deduced, involving only $n - 1$ unknown quantities; and from these $n - 1$, $n - 2$ may be obtained, involving only $n - 2$ unknown quantities; and so on, till only one equation remains, involving one unknown quantity; which being found, the values of all the rest may be determined by substitution.

(26.) If there be more unknown quantities than independent equations, some of these quantities cannot be found except in terms of the others; and by assuming values of these others, we may obtain an infinite number of corresponding values of the former quantities, which will satisfy the conditions proposed.

But if there be fewer unknown quantities than independent equations, the values of the unknown quantities may be found from the different equations; and if these values be the same, some of the equations are unnecessary; if different, the equations are incongruous.

SECTION III.

On the Solution of Pure Quadratics, and others which may be solved without completing the Square.

(27.) WHEN the terms of an equation involve the square of the unknown quantity only, the value of the square will be found by the preceding articles; and extracting the root on each side of the equation, the unknown quantity itself will be determined.

In the same way any pure equation may be solved; for the power of the unknown quantity standing alone on one side of the equation, the known quantities being transposed to the other, the simple unknown quantity will be determined by extracting the root.

And by the same process, any equation containing the powers of a function of the unknown quantity, or containing the powers of two unknown quantities, may frequently be reduced to lower dimensions.

EXAMPLES.

1. Given $x^2 - 17 = 130 - 2x^2$, to find the values of x .

By transposition, $3x^2 = 147$;

$$\therefore x^2 = 49,$$

$$\text{and } x = \pm 7.*$$

* The square root of a quantity may be either + or —, and consequently all quadratic equations admit of two solutions. Thus, $+7 \times +7$, and -7×-7 , are both equal to 49; and both, when substituted for x in the original equation, answer the conditions required.

2. Given $x^2 + ab = 5x^2$, to find the values of x .

By transposition, $ab = 4x^2$;

$$\therefore \pm \sqrt{ab} = 2x,$$

$$\text{and } \pm \frac{\sqrt{ab}}{2} = x.$$

3. Given $xy = a$
and $\frac{x}{y} = b$ } to find the values of x and y .

From the second equation, $x = by$.

Substituting this value in the first equation,

$$by^2 = a;$$

$$\therefore y^2 = \frac{a}{b},$$

and extracting the square root, $y = \pm \sqrt{\frac{a}{b}}$;

$$\therefore x = by = \pm b \sqrt{\frac{a}{b}} = \pm \sqrt{ab}.$$

4. Given $x + y : x :: 5 : 3$
and $xy = 6$. } to find the values of x and y .

Since $x + y : x :: 5 : 3$;

$$\therefore (\text{Wood's Alg. 180.}) y : x :: 2 : 3;$$

$$\therefore (21) 3y = 2x, \text{ and } y = \frac{2x}{3}.$$

Substituting this value in the second equation,

$$\frac{2x^2}{3} = 6,$$

$$\text{and } x^2 = 9;$$

therefore, extracting the square root, $x = \pm 3$,

$$\text{whence } y = \frac{2x}{3} = \pm 2.$$

Substituting this for x in the second equation,

$$\frac{9y^2}{7} = 63;$$

$$\therefore y^2 = 49,$$

$$\text{and } y = \pm 7;$$

$$\therefore x = \frac{9y}{7} = \pm 9.$$

10. Given $x^2 + xy = 12$
and $y^2 + xy = 24$ } , to find the values of x and y .

Adding the two equations together,

$$x^2 + 2xy + y^2 = 36;$$

$$\therefore \text{extracting the square root, } x + y = \pm 6.$$

$$\text{Now } x^2 + xy = x \cdot (x + y) = \pm 6x;$$

$$\therefore \pm 6x = 12,$$

$$\text{and } x = \pm 2;$$

$$\text{and therefore } y = \pm 6 \mp 2 = \pm 4.$$

11. Given $x + y = s$
and $x^2 - y^2 = d^2$ } , to find the values of x and y .

$$\text{Since } x^2 - y^2 = (x + y) \cdot (x - y) = s \cdot (x - y);$$

$$\therefore s \cdot (x - y) = d^2,$$

$$\text{and } x - y = \frac{d^2}{s};$$

$$\text{but } x + y = s;$$

$$\therefore \text{by addition, } 2x = s + \frac{d^2}{s} = \frac{s^2 + d^2}{s},$$

$$\text{and } x = \frac{s^2 + d^2}{2s},$$

and by subtraction, $2y = s - \frac{d^2}{s} = \frac{s^2 - d^2}{s}$;

$$\therefore y = \frac{s^2 - d^2}{2s}.$$

12. Given $x + y = s$
and $xy = a^2$ } , to find the values of x and y .

Squaring the first equation, $x^2 + 2xy + y^2 = s^2$,

and from the second, $4xy = 4a^2$;

$$\therefore \text{by subtraction, } x^2 - 2xy + y^2 = s^2 - 4a^2,$$

and extracting the square root, $x - y = \pm \sqrt{s^2 - 4a^2}$,

but $x + y = s$;

$$\therefore \text{by addition, } 2x = s \pm \sqrt{s^2 - 4a^2},$$

and $x = \frac{1}{2} (s \pm \sqrt{s^2 - 4a^2})$;

by subtraction, $2y = s \mp \sqrt{s^2 - 4a^2}$;

$$\therefore y = \frac{1}{2} (s \mp \sqrt{s^2 - 4a^2}).$$

13. Given $x + y = s$
and $x^2 + y^2 = a^2$ } , to find the values of x and y .

Squaring the first equation, $x^2 + 2xy + y^2 = s^2$,

and doubling the second, $2x^2 + 2y^2 = 2a^2$;

$$\therefore \text{by subtraction, } x^2 - 2xy + y^2 = 2a^2 - s^2,$$

and extracting the square root, $x - y = \pm \sqrt{2a^2 - s^2}$;

but $x + y = s$;

$$\therefore \text{by addition, } 2x = s \pm \sqrt{2a^2 - s^2},$$

and $x = \frac{1}{2} (s \pm \sqrt{2a^2 - s^2})$;

also by subtraction, $2y = s \mp \sqrt{2a^2 - s^2}$;

$$\therefore y = \frac{1}{2} (s \mp \sqrt{2a^2 - s^2}).$$

14. Given $\sqrt[3]{x} + \sqrt[3]{y} = 5$
and $\sqrt[3]{x} - \sqrt[3]{y} = 1$ } , to find the values of x
and y .

Adding the two equations, $2\sqrt[3]{x} = 6$;

$$\therefore \sqrt[3]{x} = 3,$$

$$\text{and } x = 27,$$

and subtracting the equations, $2\sqrt[3]{y} = 4$;

$$\therefore \sqrt[3]{y} = 2,$$

$$\text{and } y = 8.$$

15. Given $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{a^2 + x^2}}$, to find the values
of x .

$$(18. \text{ Cor. 1.}) \quad x\sqrt{a^2 + x^2} + a^2 + x^2 = 2a^2;$$

$$\text{by transposition, } x\sqrt{a^2 + x^2} = a^2 - x^2,$$

$$\text{and squaring both sides, } a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4;$$

$$\therefore 3a^2x^2 = a^4,$$

$$\text{and } x^2 = \frac{a^2}{3};$$

$$\therefore x = \pm \frac{a}{\sqrt{3}}.$$

16. Given $\sqrt{\frac{a^2}{x^2} + b^2} - \sqrt{\frac{a^2}{x^2} - b^2} = b$, to find the
values of x .

$$\text{By transposition, } \sqrt{\frac{a^2}{x^2} + b^2} = \sqrt{\frac{a^2}{x^2} - b^2} + b;$$

$$\text{squaring both sides, } \frac{a^2}{x^2} + b^2 = \frac{a^2}{x^2} - b^2 + 2b\sqrt{\frac{a^2}{x^2} - b^2} + b^2;$$

$$(17. \text{ Cor. 3.}) \quad b^2 = 2b\sqrt{\frac{a^2}{x^2} - b^2},$$

$$\text{or } \frac{b}{2} = \sqrt{\frac{a^2}{x^2} - b^2};$$

$$\text{squaring both sides, } \frac{b^2}{4} = \frac{a^2}{x^2} - b^2;$$

$$\therefore \frac{5b^2}{4} = \frac{a^2}{x^2};$$

$$\text{and extracting the square root, } \pm \frac{\sqrt{5 \cdot b}}{2} = \frac{a}{x};$$

$$\therefore x = \pm \frac{2a}{\sqrt{5 \cdot b}}.$$

17. Given $\frac{a}{x} + \frac{\sqrt{a^2 - x^2}}{x} = \frac{x}{b}$, to find the values of x .

$$\text{The given equation becomes } \frac{a}{x} + \sqrt{\frac{a^2}{x^2} - 1} = \frac{x}{b};$$

$$\text{by transposition, } \sqrt{\frac{a^2}{x^2} - 1} = \frac{x}{b} - \frac{a}{x};$$

$$\therefore \text{squaring both sides, } \frac{a^2}{x^2} - 1 = \frac{x^2}{b^2} - \frac{2a}{b} + \frac{a^2}{x^2},$$

$$\text{and by transposition, } \frac{2a}{b} - 1 = \frac{x^2}{b^2};$$

$$\therefore 2ab - b^2 = x^2,$$

$$\text{and extracting the square root, } \pm \sqrt{2ab - b^2} = x.$$

18. Given $x^2 + y^2 = \frac{13}{x - y}$ } to find the values of x and
and $xy = \frac{6}{x - y}$ } y .

From the first equation subtracting twice the second,

$$(x^2 - 2xy + y^2 =) \overline{x - y}^2 = \frac{1}{x - y};$$

$$\therefore (x - y)^2 = 1, \text{ and } x - y = 1;$$

$$\therefore x^2 + y^2 = 13,$$

$$\text{and } 2xy = 12;$$

$$\therefore \text{ by addition, } x^2 + 2xy + y^2 = 25,$$

$$\text{and } x + y = \pm 5;$$

$$\text{but } x - y = 1;$$

$$\therefore \text{ by addition, } 2x = 6, \text{ or } -4,$$

$$\text{and } x = 3, \text{ or } -2,$$

$$\text{and by subtraction, } 2y = 4, \text{ or } -6;$$

$$\therefore y = 2, \text{ or } -3.$$

19. Given $x^2 - xy = 48y$
and $xy - y^2 = 3x$ } , to find the values of x and y .

Dividing the first equation by x , $x - y = \frac{48y}{x}$;

and the second by y , $x - y = \frac{3x}{y}$;

$$\therefore \frac{48y}{x} = \frac{3x}{y};$$

$$\therefore 48y^2 = 3x^2,$$

$$\text{and } 16y^2 = x^2;$$

$$\text{consequently, } \pm 4y = x;$$

$$\text{and first, suppose } +4y = x;$$

$$\therefore (x - y) = 3y = \left(\frac{48y}{x} = \frac{48y}{4y} = 12 \right) 12; \therefore y = 4;$$

$$\therefore x = 4y = 16;$$

But if $x = -4y$,

$$(x - y) = -5y = \left(\frac{48y}{x} = \frac{48y}{-4y} = -12 \right) -12;$$

$$\therefore y = \frac{12}{5},$$

$$\text{and } x = -4y = -\frac{48}{5}.$$

20. Given $\frac{xy}{x^{\frac{1}{2}}y^{\frac{1}{2}}} = 48$ and $\frac{xy}{\sqrt{x}} = 24$ } , to find the values of x and y .

Dividing the first equation by the second,

$$\left(\frac{\sqrt{x}}{x^{\frac{1}{2}}y^{\frac{1}{2}}} \right) y^{\frac{1}{2}} = 2;$$

$$\therefore y = 4;$$

whence, from the second equation, $\frac{4x}{\sqrt{x}} = 4\sqrt{x} = 24;$

$$\therefore \sqrt{x} = 6,$$

$$\text{and } x = 36.$$

21. Given $\frac{x^3 + 3x - 7}{x + 2 + \frac{18}{x}} = 1$, to find the values of x .

Clearing the equation of fractions,

$$x^3 + 3x - 7 = x + 2 + \frac{18}{x};$$

$$\therefore \text{ by transposition, } x^3 + 2x = 9 + \frac{18}{x},$$

$$\text{or } x \cdot (x + 2) = 9 \cdot \left(1 + \frac{2}{x}\right) = \frac{9}{x} \cdot (x + 2);$$

$$\therefore x = \frac{9}{x},$$

$$\text{and } x^2 = 9;$$

$$\therefore x = \pm 3.$$

22. Given $\sqrt{\frac{x+a}{x}} + 2\sqrt{\frac{a}{x+a}} = b$, to find the values of x .

Multiplying the equation by $\sqrt{\frac{x+a}{x}}$,

$$\frac{x+a}{x} + 2\sqrt{\frac{a}{x}} = b^2,$$

$$\text{or } 1 + \frac{a}{x} + 2\sqrt{\frac{a}{x}} = b^2,$$

and extracting the square root, $1 + \sqrt{\frac{a}{x}} = \pm b$;

by transposition, $\sqrt{\frac{a}{x}} = \pm b - 1$,

and squaring both sides, $\frac{a}{x} = (b \mp 1)^2$;

$$\therefore x = \frac{a}{(b \mp 1)^2}.$$

23. Given $\frac{\sqrt{a+x}}{a} + \frac{\sqrt{a+x}}{x} = \frac{\sqrt{x}}{c}$, to find the values of x .

The equation by reduction becomes $\frac{a+x}{ax} \cdot \sqrt{a+x} = \frac{\sqrt{x}}{c}$,

$$\text{and } \therefore \sqrt{a+x}^{\frac{3}{2}} = \frac{a}{c} \cdot x^{\frac{1}{2}};$$

extracting the $(\frac{3}{2})^{\text{th}}$ root, $a+x = \left(\frac{a}{c}\right)^{\frac{2}{3}} \cdot x$;

$\therefore$ by transposition, $a = \left(\left(\frac{a}{c}\right)^{\frac{2}{3}} - 1\right) \cdot x$,

$$\text{and } \frac{a}{\left(\frac{a}{c}\right)^{\frac{2}{3}} - 1} = x.$$

24. Given $\frac{(a+x)^{\frac{1}{n}}}{a} + \frac{(a+x)^{\frac{1}{n}}}{x} = \frac{x^{\frac{1}{n}}}{c}$, to find the value of x .

The equation is $(a + x)^{\frac{1}{n}} \cdot \left(\frac{1}{a} + \frac{1}{x}\right) = \frac{x^{\frac{1}{n}}}{c}$,

$$\text{or } (a + x)^{\frac{1}{n}} \cdot \frac{a + x}{ax} = \frac{x^{\frac{1}{n}}}{c},$$

$$\text{or } (a + x)^{\frac{1}{n} + 1} = \frac{a}{c} \cdot x^{\frac{1}{n} + 1};$$

$$\therefore \left(\frac{a + x}{x}\right)^{\frac{n+1}{n}} = \frac{a}{c};$$

and extracting the root, $\frac{a + x}{x} = \sqrt[n]{\frac{a}{c}^{n+1}}$;

$$\text{or } \frac{a}{x} + 1 = \sqrt[n]{\frac{a}{c}^{n+1}};$$

by transposition, $\frac{a}{x} = \sqrt[n]{\frac{a}{c}^{n+1}} - 1$;

$$\therefore x = \frac{a}{\sqrt[n]{\frac{a}{c}^{n+1}} - 1};$$

25. Given $\frac{m}{n} \cdot x^{\frac{m}{n}-1} = \frac{r}{s} \cdot x^{\frac{r}{s}-1}$, to find the value of x .

$$(18. \text{ Cor. 2.}) \quad \frac{m}{n} \cdot x^{\frac{m}{n}} = \frac{r}{s} \cdot x^{\frac{r}{s}};$$

$$\therefore \frac{x^{\frac{m}{n}}}{x^{\frac{r}{s}}} = \frac{nr}{ms},$$

$$\text{or } x^{\frac{ms - nr}{ns}} = \frac{nr}{ms};$$

$\therefore$ extracting the root, $x = \left(\frac{nr}{ms}\right)^{\frac{ns}{ms - nr}}$.

26. Given $x^2 + y^2 = 13$ } , to find the values of x and y .
 and $x^2 + y^2 = 5$ }

Squaring the second equation, $x^2 + 2x^2y^2 + y^2 = 25$;

and doubling the first, $2x^2 + 2y^2 = 26$;

$\therefore$ by subtraction, $x^2 - 2x^2y^2 + y^2 = 1$;

and extracting the root, $x^2 - y^2 = \pm 1$;

but $x^2 + y^2 = 5$;

$\therefore$ by addition, $2x^2 = 6$, or 4 ,

and $x^2 = 3$, or 2 ,

whence $x = 27$, or 8 ,

but by subtraction, $2y^2 = 4$, or 6 ,

and $y^2 = 2$, or 3 ;

$\therefore y = 8$, or 27 .

27. Given $xy^2 + y = 21$ } , to find the values of x and y .
 and $x^2y^4 + y^2 = 333$ }

Squaring the first equation, $x^2y^4 + 2xy^2 + y^2 = 441$;

and doubling the second, $2x^2y^4 + 2y^2 = 666$;

$\therefore$ by subtraction, $x^2y^4 - 2xy^2 + y^2 = 225$;

extracting the root, $xy^2 - y = \pm 15$;

but $xy^2 + y = 21$;

$\therefore$ by subtraction, $2y = 6$, or 36 ,

and $y = 3$, or 18 ;

but by addition, $2xy^2 = 36$, or 6;

$$\therefore xy^2 = 18, \text{ or } 3,$$

$$\text{and therefore } x = \frac{18}{9}, \text{ or } \frac{3}{324},$$

$$= 2, \text{ or } \frac{1}{108}.$$

28. Given $x^3y + xy^3 = 180$
and $x^3 + y^3 = 189$ } , to find the values of x and y .

Adding 3 times the first equation to the second,

$$x^3 + 3x^2y + 3xy^2 + y^3 = 729;$$

whence, extracting the cube root, $x + y = 9$.

$$\text{Now } x^3y + xy^3 = (x + y) \cdot xy = 180;$$

$$\therefore \text{ by substitution, } 9xy = 180,$$

$$\text{and } xy = 20.$$

$$\text{But } x^3 + 2xy + y^3 = 81,$$

$$\text{and } \quad \quad \quad 4xy \quad = 80;$$

$$\therefore \text{ by subtraction, } x^3 - 2xy + y^3 = 1;$$

$$\therefore \text{ extracting the square root, } x - y = \pm 1,$$

$$\text{and } x + y = 9;$$

$$\therefore \text{ by addition, } 2x = 10, \text{ or } 8;$$

$$\therefore x = 5, \text{ or } 4;$$

$$\text{and by subtraction, } 2y = 8, \text{ or } 10;$$

$$\therefore y = 4, \text{ or } 5.$$

29. Given $x + \sqrt{xy} + y = 19$
and $x^2 + xy + y^2 = 133$ } , to find the values of x

and y .

Dividing the second equation by the first,

$$x - \sqrt{xy} + y = 7;$$

$$\text{but } x + \sqrt{xy} + y = 19;$$

$$\therefore \text{ by addition, } 2x + 2y = 26;$$

$$\text{and } x + y = 13;$$

$$\text{and by subtraction, } 2\sqrt{xy} = 12;$$

$$\therefore \sqrt{xy} = 6,$$

$$\text{and } xy = 36.$$

$$\text{Now from the second equation, } x^2 + xy + y^2 = 133;$$

$$\text{and from the last, } 3xy = 108;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 25;$$

$$\text{and extracting the square root, } x - y = \pm 5;$$

$$\text{but } x + y = 13;$$

$$\therefore \text{ by addition, } 2x = 18, \text{ or } 8,$$

$$\text{and } x = 9, \text{ or } 4;$$

$$\text{but by subtraction, } 2y = 8, \text{ or } 18,$$

$$\text{and } y = 4, \text{ or } 9.$$

$$30. \text{ Given } \frac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} = b, \text{ to find the values of } x.$$

Since the value of a fraction is not altered if the numerator and denominator be multiplied by the same quantity, (*Alg.* 89.)

Multiply therefore by $a - \sqrt{a^2 - x^2}$,

$$\text{and } \frac{(a - \sqrt{a^2 - x^2})^2}{x^2} = b.$$

$$\text{Extracting the square root, } \frac{a - \sqrt{a^2 - x^2}}{x} = \pm \sqrt{b},$$

$$\text{and } a - \sqrt{a^2 - x^2} = \pm \sqrt{b} \cdot x;$$

$$\therefore \text{ by transposition, } a \mp \sqrt{b} \cdot x = \sqrt{a^2 - x^2};$$

$$\text{and squaring both sides, } a^2 \mp 2a\sqrt{b} \cdot x + bx^2 = a^2 - x^2;$$

∴ by transposition, $(b + 1) \cdot x^2 = \pm 2a\sqrt{b} \cdot x$;

$$\text{and therefore } x = \pm \frac{2a\sqrt{b}}{1 + b}.$$

31. Given $\frac{\sqrt{x} + \sqrt{x-a}}{\sqrt{x} - \sqrt{x-a}} = \frac{n^2 a}{x-a}$, to find the values of x .

Multiplying the numerator and the denominator of the first fraction by $\sqrt{x} + \sqrt{x-a}$,

$$\frac{(\sqrt{x} + \sqrt{x-a})^2}{a} = \frac{n^2 a}{x-a};$$

$$(\sqrt{x} + \sqrt{x-a})^2 = \frac{n^2 a^2}{x-a};$$

extracting the square root, $\sqrt{x} + \sqrt{x-a} = \pm \frac{na}{\sqrt{x-a}}$,

$$\text{and } \therefore \sqrt{x^2 - ax} + x - a = \pm na;$$

by transposition, $\sqrt{x^2 - ax} = (1 \pm n) \cdot a - x$,

and squaring both sides,

$$x^2 - ax = (1 \pm n)^2 \cdot a^2 - 2 \cdot (1 \pm n) \cdot ax + x^2;$$

∴ by transposition, $(1 \pm n) \cdot ax = (1 \pm n)^2 \cdot a^2$;

$$\therefore x = \frac{(1 \pm n)^2 \cdot a}{1 \pm 2n}.$$

32. Given $\frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = b$, to find the values of x .

Multiplying the numerator and denominator by

$$\sqrt{a+x} + \sqrt{a-x}, \quad \frac{(\sqrt{a+x} + \sqrt{a-x})^2}{2x} = b,$$

$$\text{or } \frac{2a + 2\sqrt{a^2 - x^2}}{2x} = b;$$

$$\therefore \frac{a}{x} + \sqrt{\frac{a^2}{x^2} - 1} = b;$$

by transposition, $\sqrt{\frac{a^2}{x^2} - 1} = b - \frac{a}{x}$;

squaring both sides, $\frac{a^2}{x^2} - 1 = b^2 - \frac{2ab}{x} + \frac{a^2}{x^2}$;

$\therefore$ by transposition, $\frac{2ab}{x} = b^2 + 1$,

$$\text{and } x = \frac{2ab}{b^2 + 1}.$$

33. Given $\frac{a + x + \sqrt{2ax + x^2}}{a + x} = b$, to find the values of x .

The equation by reduction becomes $1 + \frac{\sqrt{2ax + x^2}}{a + x} = b$,

$$\text{or } 1 + \sqrt{1 - \left(\frac{a}{a+x}\right)^2} = b;$$

by transposition, $\sqrt{1 - \left(\frac{a}{a+x}\right)^2} = b - 1$,

squaring both sides, $1 - \left(\frac{a}{a+x}\right)^2 = b^2 - 2b + 1$;

$\therefore$ (17. Cor. 1. and 3.) $\left(\frac{a}{a+x}\right)^2 = 2b - b^2$;

extracting the square root, $\frac{a}{a+x} = \pm \sqrt{2b - b^2}$,

$$\text{and } \therefore \frac{a+x}{a} = \pm \frac{1}{\sqrt{2b - b^2}};$$

$$\therefore a + x = \pm \frac{a}{\sqrt{2b - b^2}},$$

$$\text{and } x = \pm \frac{a}{\sqrt{2b - b^2}} - a = \pm a \left(\frac{1 \pm \sqrt{2b - b^2}}{\sqrt{2b - b^2}} \right).$$

34. Given $\frac{\sqrt{a+x}}{\sqrt{a}} + \frac{\sqrt{a-x}}{\sqrt{x}} = \sqrt{\frac{x}{b}}$, to find the values of x .

The equation by reduction, is

$$\sqrt{\frac{a}{x} + 1} + \sqrt{\frac{a}{x} - 1} = \sqrt{\frac{x}{b}};$$

by transposition, $\sqrt{\frac{a}{x} + 1} = \sqrt{\frac{x}{b}} - \sqrt{\frac{a}{x} - 1},$

and squaring both sides,

$$\frac{a}{x} + 1 = \frac{x}{b} - 2\sqrt{\frac{x}{b}} \cdot \sqrt{\frac{a}{x} - 1} + \frac{a}{x} - 1;$$

by transposition, $2\sqrt{\frac{x}{b}} \cdot \left(\frac{a}{x} - 1\right) = \frac{x}{b} - 2,$

and $\sqrt{\frac{x}{b}} \cdot \left(\frac{a}{x} - 1\right) = \frac{x}{2b} - 1;$

squaring both sides, $\frac{a}{b} - \frac{x}{b} = \frac{x^2}{4b^2} - \frac{x}{b} + 1;$

$\therefore$ by transposition, $\frac{a}{b} - 1 = \frac{x^2}{4b^2};$

and $4ab - 4b^2 = x^2;$

$\therefore$ extracting the square root, $\pm 2\sqrt{ab - b^2} = x.$

35. Given $3x - \frac{3x}{y} = y^2 - y$ and $y^2 + x = 4$ $\left. \vphantom{\begin{matrix} 3x - \frac{3x}{y} = y^2 - y \\ y^2 + x = 4 \end{matrix}} \right\}$ to find the values of x and y .

Reducing the first equation, $\frac{3x \cdot (y - 1)}{y} = y \cdot (y - 1);$

and therefore (18. Cor. 2.) $\frac{3x}{y} = y,$

and $3x = y^2.$

Substituting this value in the second equation, $4x = 4;$

$\therefore x = 1,$

and $y^2 = 3x = 3;$

$\therefore y = \pm \sqrt{3}.$

by transposition, $\sqrt{\frac{a^2}{x^2} - 1} = b - \frac{a}{x};$

squaring both sides, $\frac{a^2}{x^2} - 1 = b^2 - \frac{2ab}{x} + \frac{a^2}{x^2};$

$\therefore$ by transposition, $\frac{2ab}{x} = b^2 + 1,$

and $x = \frac{2ab}{b^2 + 1}.$

33. Given $\frac{a + x + \sqrt{2ax + x^2}}{a + x} = b,$ to find the values of $x.$

The equation by reduction becomes $1 + \frac{\sqrt{2ax + x^2}}{a + x} = b,$

or $1 + \sqrt{1 - \left(\frac{a}{a+x}\right)^2} = b;$

by transposition, $\sqrt{1 - \left(\frac{a}{a+x}\right)^2} = b - 1,$

squaring both sides, $1 - \left(\frac{a}{a+x}\right)^2 = b^2 - 2b + 1;$

$\therefore$ (17. Cor. 1. and 3.) $\left(\frac{a}{a+x}\right)^2 = 2b - b^2;$

extracting the square root, $\frac{a}{a+x} = \pm \sqrt{2b - b^2},$

and $\therefore \frac{a+x}{a} = \pm \frac{1}{\sqrt{2b - b^2}};$

$\therefore a + x = \pm \frac{a}{\sqrt{2b - b^2}},$

and $x = \pm \frac{a}{\sqrt{2b - b^2}} - a = \pm a \left(\frac{1 \pm \sqrt{2b - b^2}}{\sqrt{2b - b^2}} \right).$

34. Given $\frac{\sqrt{a+x}}{\sqrt{a}} + \frac{\sqrt{a-x}}{\sqrt{x}} = \sqrt{\frac{x}{b}},$ to find the values of $x.$

The equation by reduction, is

$$\sqrt{\frac{a}{x} + 1} + \sqrt{\frac{a}{x} - 1} = \sqrt{\frac{x}{b}};$$

by transposition, $\sqrt{\frac{a}{x} + 1} = \sqrt{\frac{x}{b}} - \sqrt{\frac{a}{x} - 1},$

and squaring both sides,

$$\frac{a}{x} + 1 = \frac{x}{b} - 2 \sqrt{\frac{x}{b}} \cdot \sqrt{\frac{a}{x} - 1} + \frac{a}{x} - 1;$$

by transposition, $2 \sqrt{\frac{x}{b}} \cdot \left(\frac{a}{x} - 1\right) = \frac{x}{b} - 2,$

and $\sqrt{\frac{x}{b}} \cdot \left(\frac{a}{x} - 1\right) = \frac{x}{2b} - 1;$

squaring both sides, $\frac{a}{b} - \frac{x}{b} = \frac{x^2}{4b^2} - \frac{x}{b} + 1;$

$\therefore$ by transposition, $\frac{a}{b} - 1 = \frac{x^2}{4b^2};$

and $4ab - 4b^2 = x^2;$

$\therefore$ extracting the square root, $\pm 2\sqrt{ab - b^2} = x.$

35. Given $3x - \frac{3x}{y} = y^2 - y$ and $y^2 + x = 4$ } to find the values of x and y .

Reducing the first equation, $\frac{3x \cdot (y - 1)}{y} = y \cdot (y - 1);$

and therefore (18. Cor. 2.) $\frac{3x}{y} = y,$

and $3x = y^2.$

Substituting this value in the second equation, $4x = 4;$

$\therefore x = 1,$

and $y^2 = 3x = 3;$

$\therefore y = \pm \sqrt{3}.$

36. Given $x^2 + y\sqrt{xy} = 9$ and $y^2 + x\sqrt{xy} = 18$ } to find the values of x and y .

The first equation is $x^{\frac{1}{2}} \times (x^{\frac{1}{2}} + y^{\frac{1}{2}}) = 9$,

and the second, $y^{\frac{1}{2}} \times (x^{\frac{1}{2}} + y^{\frac{1}{2}}) = 18$.

Dividing the second by the first, $\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} = 2$.

$$\therefore y^{\frac{1}{2}} = 2x^{\frac{1}{2}},$$

$$\text{and } y = 4x.$$

Substituting this value in the first equation,

$$x^2 + 4x\sqrt{4x^3} = 9,$$

$$\text{or } 9x^2 = 9;$$

$$\therefore x = \pm 1;$$

consequently, $y = 4x = \pm 4$.

37. Given $\frac{x^2 + 2y^2}{\sqrt{y}} = x + 2xy$ and $x^2 - 2y^2 = 256 - x\sqrt{y}$ } to find the values of x and y .

From the first equation, $x^2 + 2y^2 = x\sqrt{y} + 2xy^{\frac{3}{2}}$;

$\therefore$ by transposition, $x^2 - x\sqrt{y} = 2xy^{\frac{3}{2}} - 2y^2$,

$$\text{or } x \cdot (x - \sqrt{y}) = 2y^{\frac{1}{2}} \cdot (x - \sqrt{y});$$

$$\therefore x = 2y^{\frac{1}{2}},$$

$$\text{and } x\sqrt{y} = 2y^2;$$

$\therefore$ adding these equals to the second equation,

$$x^2 = 256,$$

and therefore $x = \pm 16$;

$$\therefore 2y^{\frac{1}{2}} = \pm 16,$$

$$\text{and } y^{\frac{1}{2}} = \pm 8;$$

$$\therefore y^{\frac{1}{4}} = \pm 2,$$

$$\text{and } y = 4.$$

38. Given $x^2y + xy^2 = 30$ and $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$ } to find the values of x and y .

From the second equation, $x + y = \frac{5xy}{6}$;

but from the first, $xy = \frac{30}{x + y}$.

Substituting this value in the other equation,

$$x + y = \frac{5}{6} \times \frac{30}{x + y} = \frac{25}{x + y};$$

$$\therefore (x + y)^2 = 25;$$

and extracting the root, $x + y = \pm 5$.

Let $x + y = +5$; then $xy = +6$;

whence $x^2 + 2xy + y^2 = 25$,

$$\text{and } 4xy = 24;$$

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 1$,

$$\text{and } x - y = \pm 1;$$

$$\text{but } x + y = 5;$$

$\therefore$ by addition, $2x = 6$, or 4 ; and $x = 3$, or 2 ;

by subtraction, $2y = 4$, or 6 ; and $y = 2$, or 3 .

But if $x + y = -5$, then $xy = -6$;

whence $x^2 + 2xy + y^2 = 25$,

$$\text{and } 4xy = -24;$$

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 49$;

and extracting the root, $x - y = \pm 7$;

$$\text{but } x + y = -5;$$

$\therefore$ by addition, $2x = 2$, or -12 ; and $x = 1$, or -6 ;

by subtraction, $2y = -12$, or 2 ; and $y = -6$, or 1 .

39. Given $x^2y + xy^2 = 6$ }
and $x^2y^2 + x^2y^2 = 12$ }, to find the values of x and y .

Dividing the second equation by the first, $xy = 2$.

But $x^2y + xy^2 = (x + y) \cdot xy = 6$;

$$\therefore 2 \cdot (x + y) = 6,$$

$$\text{and } x + y = 3;$$

$$\text{whence } x^2 + 2xy + y^2 = 9;$$

$$\text{but } \quad 4xy \quad = 8;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 1;$$

$$\text{and extracting the root, } x - y = \pm 1;$$

$$\text{also } x \quad + y = 3;$$

$$\therefore \text{ by addition, } 2x = 4, \text{ or } 2;$$

$$\therefore x = 2, \text{ or } 1,$$

$$\text{and by subtraction, } 2y = 2, \text{ or } 4;$$

$$\therefore y = 1, \text{ or } 2.$$

$$40. \text{ Given } x^3 - y^3 : (x - y)^3 :: 61 : 1 \left. \vphantom{\begin{matrix} x^3 - y^3 \\ (x - y)^3 \end{matrix}} \right\} \text{, to find the values of}$$

$$\text{and } xy = 320$$

x and y .

$$\text{Since } x^3 - y^3 : x^3 - 3x^2y + 3xy^2 - y^3 :: 61 : 1;$$

$$\therefore (\text{Alg. 180.}) 3x^2y - 3xy^2 : (x - y)^3 :: 60 : 1,$$

$$\text{or } 3xy \times (x - y) : (x - y)^3 :: 60 : 1;$$

$$\therefore (\text{Alg. 184.}) 960 : (x - y)^3 :: 60 : 1, \text{ dividing the first and second terms by } x - y;$$

$$\text{and } 16 : (x - y)^3 :: 1 : 1, \text{ dividing the first and third terms by } 60;$$

$$\therefore (x - y)^3 = 16,$$

$$\text{and } x - y = \pm 4.$$

$$\text{But since } x^2 - 2xy + y^2 = 16,$$

$$\text{and } \quad 4xy \quad = 1280;$$

$$\therefore \text{ by addition, } x^2 + 2xy + y^2 = 1296;$$

$$\therefore \text{ extracting the root, } x + y = \pm 36;$$

$$\text{but } x - y = \pm 4;$$

$$\therefore \text{ by addition, } 2x = \pm 40, \text{ or } \pm 32,$$

$$\text{and } x = \pm 20, \text{ or } \pm 16;$$

but by subtraction, $2y = \pm 32$, or ± 40 ,

and $y = \pm 16$, or ± 20 .

41. Given $(x^3 + y^3) \times (x + y) = 2336$
and $(x^3 - y^3) \cdot (x - y) = 576$ } to find the values
of x and y .

From the first equation, $x^3 + x^2y + xy^2 + y^3 = 2336$;

and from the second, $x^3 - x^2y - xy^2 + y^3 = 576$;

$\therefore$ by subtraction, $2x^2y + 2xy^2 = 1760$;

adding this to the first equation,

$$x^3 + 3x^2y + 3xy^2 + y^3 = 4096;$$

$\therefore$ extracting the cube root, $x + y = 16$,

$$\text{and } 2xy \cdot (x + y) = 1760,$$

$$\text{or } 16 \times 2xy = 1760,$$

$$\therefore xy = 55.$$

$$\text{Now } x^3 + 2xy + y^3 = 256,$$

$$\text{and } 4xy = 220;$$

$\therefore$ by subtraction, $x^3 - 2xy + y^3 = 36$,

and therefore $x - y = \pm 6$;

$$\text{but } x + y = 16;$$

$\therefore$ by addition, $2x = 22$, or 10 ,

and $x = 11$, or 5 ;

but by subtraction, $2y = 10$, or 22 ;

$\therefore y = 5$, or 11 .

42. Given $x^3 + y^3 = (x + y) \cdot xy$
and $x^2y + xy^2 = 4xy$ } to find the values of x
and y .

Dividing the second equation by xy , $x + y = 4$;

$$\therefore x^3 + 3x^2y + 3xy^2 + y^3 = 64.$$

But from the first equation, $x^3 - x^2y - xy^2 + y^3 = 0$;

$$\therefore \text{by subtraction, } 4x^2y + 4xy^2 = 64;$$

$$\therefore (x + y) \cdot xy = 16,$$

$$\text{and } xy = 4.$$

$$\text{But } x^2 + 2xy + y^2 = 16,$$

$$\text{and } 4xy = 16;$$

$$\therefore \text{by subtraction, } x^2 - 2xy + y^2 = 0;$$

$$\text{and extracting the root, } x - y = 0;$$

$$\text{but } x + y = 4;$$

$$\therefore \text{by addition, } 2x = 4;$$

$$\text{and } x = 2;$$

$$\text{but by subtraction, } 2y = 4;$$

$$\therefore y = 2.$$

$$43. \quad \left. \begin{array}{l} \text{Given } (x^3 + y^3) \times (x + y) = \frac{15xy}{2} \\ \text{and } (x^4 - y^4) \cdot (x^2 + y^2) = \frac{75x^2y^2}{4} \end{array} \right\} \begin{array}{l} \text{to find the} \\ \text{values of } x \\ \text{and } y. \end{array}$$

Dividing the second equation by the first,

$$(x^3 + y^3) \cdot (x - y) = \frac{5xy}{2}.$$

Again, dividing the first equation by this last,

$$\frac{x+y}{x-y} = 3;$$

$$\therefore x + y = 3x - 3y;$$

$$\therefore \text{by transposition, } 4y = 2x,$$

$$\text{and } 2y = x;$$

$$\text{whence } (x^3 + y^3) \cdot (x - y) = 5y^3 \times y = \frac{5y \times 2y}{2},$$

$$\text{or } y = 1;$$

$$\text{and therefore } x = 2y = 2.$$

44. Given $(x^2 - y^2) \times (x - y) = 3xy$
 and $(x^2 - y^2) \times (x^2 - y^2) = 45x^2y^2$ } to find the
 values of x and y .

Dividing the second equation by the first,

$$(x^2 + y^2) \times (x + y) = 15xy,$$

$$\text{or } x^3 + x^2y + xy^2 + y^3 = 15xy;$$

$$\text{but from the first, } x^3 - x^2y - xy^2 + y^3 = 3xy;$$

$$\therefore \text{ by addition, } 2x^3 + 2y^3 = 18xy;$$

$$\text{and } x^3 + y^3 = 9xy;$$

$$\text{but by subtraction, } 2x^2y + 2xy^2 = 12xy;$$

$$\therefore \text{ dividing by } 2xy, x + y = 6;$$

$$\text{whence } x^3 + 3x^2y + 3xy^2 + y^3 = 216;$$

$$\text{but } x^3 + y^3 = 9xy;$$

$$\therefore \text{ by subtraction, } 3x^2y + 3xy^2 = 216 - 9xy,$$

$$\text{or } 3(x + y) \cdot xy = 18xy = 216 - 9xy;$$

$$\therefore 27xy = 216,$$

$$\text{and } xy = 8.$$

$$\text{Now } x^2 + 2xy + y^2 = 36,$$

$$\text{and } 4xy = 32;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 4;$$

$$\text{and extracting the root, } x - y = \pm 2;$$

$$\text{but } x + y = 6;$$

$$\therefore \text{ by addition, } 2x = 8, \text{ or } 4;$$

$$\therefore x = 4, \text{ or } 2;$$

$$\text{and by subtraction, } 2y = 4, \text{ or } 8;$$

$$\therefore y = 2, \text{ or } 4.$$

SECTION IV.

Solution of Adfected Quadratics, involving only one unknown Quantity.

(28.) LET the terms be arranged on one side of the equation, according to the dimensions of the unknown quantity, beginning with the highest; and (17) the known quantities be transposed to the other side; then, if the square of the unknown quantity has any coefficient, either positive or negative, let all the terms be divided by this coefficient (13). If the square of half the coefficient of the second term be now added (11) to both sides of the equation*, that side which involves the unknown quantity will become a complete square; and (19) extracting the square root on both sides of the equation, a simple equation will be obtained, from which the values of the unknown quantity may be determined.

* This is called completing the square; and that a complete square is thus obtained may be easily proved.

Let $x^2 \pm 2ax$ be the proposed quantity on one side, when the terms are arranged according to the form prescribed above; and suppose y^2 = the quantity requisite to complete the square. Now the square of $x \pm d = x^2 \pm 2dx + d^2$, where it is evident that four times the product of the extreme terms is equal to the square of the middle term; and therefore, in order that $x^2 \pm 2ax + y^2$ may be a square, $4x^2y^2$ must be equal to $4a^2x^2$; therefore $y^2 = a^2$ = the square of half the coefficient of the middle term.

It may be observed, that all equations may be solved as quadratics, by completing the squares, in which there are two terms involving the unknown quantity or any function of it, and the index of one is double that of the other. Thus,

$x^2 + px^2 = q$, $x^{2n} - px^{2n} = q$, $x^{\frac{n}{2}} + x^{\frac{n}{2}} = a$, $a^2 x^2 + ax = b$, $x^{2n} + ax^{\frac{3n}{2}} = b$, $p^2 x^{4n} - px^{2n} = d$, $(x^2 + px + q)^2 + (x^2 + px + q) = r$, $x^2 \cdot (x^2 + ax)^2 + bx \cdot (x^2 + ax) = d$, are of the same form as quadratics, and the value of the unknown quantity may be determined in the same manner. Many equations, also, in which more than one unknown quantity are involved, may in a similar manner be reduced to lower dimensions by completing the square, as $x^2 y^2 + pxy = q$, $(x^2 + y^2)^2 + p \cdot (x^2 + y^2) = r$. Instances of this kind occur in the following

EXAMPLES.

1. Given $x^2 + 8x = 33$, to find the values of x .

Completing the square, $x^2 + 8x + 16 = 49$;

and extracting the root, $x + 4 = \pm 7$;

whence, by transposition, $x = 3$, or -11 .

2. Given $x^2 + 6x + 4 = 59$, to find the values of x .

By transposition, $x^2 + 6x = 55$;

and completing the square, $x^2 + 6x + 9 = 64$;

$\therefore$ extracting the root, $x + 3 = \pm 8$;

whence $x = 5$, or -11 .

3. Given $x^2 - 8x + 10 = 19$, to find the values of x .

By transposition, $x^2 - 8x = 9$;

and completing the square, $x^2 - 8x + 16 = 25$;

$\therefore$ extracting the root, $x - 4 = \pm 5$,

and $x = 9$, or -1 .

4. Given $x^2 - 2px = q$, to find the values of x .

Completing the square, $x^2 - 2px + p^2 = p^2 + q$;

extracting the root, $x - p = \pm \sqrt{(p^2 + q)}$;

$$\therefore x = p \pm \sqrt{(p^2 + q)}.$$

5. Given $x^2 - px = q$, to find the values of x .

Completing the square, $x^2 - px + \frac{p^2}{4} = \frac{p^2}{4} + q$;

extracting the root, $x - \frac{p}{2} = \pm \sqrt{\frac{p^2}{4} + q}$;

$$\therefore x = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q} = \frac{p \pm \sqrt{(p^2 + 4q)}}{2}.$$

6. Given $x^2 - x + 3 = 45$, to find the values of x .

By transposition, $x^2 - x = 42$;

and completing the square, $x^2 - x + \frac{1}{4} = 42 + \frac{1}{4} = \frac{169}{4}$;

$\therefore$ extracting the root, $x - \frac{1}{2} = \pm \frac{13}{2}$,

and $x = 7$, or -6 .

7. Given $3x^2 + 2x - 9 = 76$, to find the values of x .

By transposition and division, $x^2 + \frac{2}{3}x = \frac{85}{3}$;

and completing the square, $x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{85}{3} + \frac{1}{9} = \frac{256}{9}$;

$\therefore$ extracting the root, $x + \frac{1}{3} = \pm \frac{16}{3}$;

whence $x = 5$, or $-\frac{17}{3}$.

8. Given $5x^2 - 4x + 3 = 159$, to find the values of x .

By transposition and division, $x^2 - \frac{4x}{5} = \frac{156}{5}$;

$\therefore$ completing the square, $x^2 - \frac{4x}{5} + \frac{4}{25} = \frac{156}{5} + \frac{4}{25} = \frac{784}{25}$;

and extracting the root, $x - \frac{2}{5} = \pm \frac{28}{5}$;

consequently, $x = 6$, or $-\frac{26}{5}$.

9. Given $ax^2 - bx = c$, to find the values of x .

By dividing each term by a , $x^2 - \frac{b}{a} \cdot x = \frac{c}{a}$;

completing the square, $x^2 - \frac{b}{a} \cdot x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} + \frac{c}{a}$
 $= \frac{b^2 + 4ac}{4a^2}$;

extracting the root, $x - \frac{b}{2a} = \pm \frac{\sqrt{(b^2 + 4ac)}}{2a}$;

$\therefore x = \frac{b \pm \sqrt{(b^2 + 4ac)}}{2a}$.

10. Given $6x + \frac{35 - 3x}{x} = 44$, to find the values of x .

(18. Cor. 1.) $6x^2 + 35 - 3x = 44x$;

$\therefore$ by transposition, $6x^2 - 47x = -35$;

and (18. Cor. 1.) $x^2 - \frac{47}{6} \cdot x = -\frac{35}{6}$;

therefore, completing the square,

$x^2 - \frac{47}{6} \cdot x + \left(\frac{47}{12}\right)^2 = \frac{2209}{144} - \frac{35}{6} = \frac{1369}{144}$;

$$\therefore \text{ extracting the root, } x - \frac{47}{12} = \pm \frac{37}{12},$$

$$\text{and } x = 7, \text{ or } \frac{5}{6}.$$

$$11. \text{ Given } 4x - \frac{14 - x}{x + 1} = 14, \text{ to find the values of } x.$$

Clearing the equation of fractions,

$$4x^2 + 4x - 14 + x = 14x + 14;$$

$$\text{and therefore, by transposition, } 4x^2 - 9x = 28,$$

$$\text{and (18. Cor. 1.) } x^2 - \frac{9}{4}x = 7;$$

$$\therefore \text{ completing the square, } x^2 - \frac{9}{4}x + \frac{9}{8} = 7 + \frac{81}{64} = \frac{529}{64};$$

$$\text{and extracting the root, } x - \frac{9}{8} = \pm \frac{23}{8};$$

$$\text{whence } x = 4, \text{ or } -\frac{7}{4}.$$

$$12. \text{ Given } 3x - \frac{1121 - 4x}{x} = 2, \text{ to find the values of } x.$$

$$(18. \text{ Cor. 1.}) \quad 3x^2 - 1121 + 4x = 2x;$$

$$\therefore \text{ by transposition, } 3x^2 + 2x = 1121;$$

$$\text{and (18. Cor. 1.) } x^2 + \frac{2}{3}x = \frac{1121}{3};$$

$$\therefore \text{ completing the square, } x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{1121}{3} + \frac{1}{9} = \frac{3364}{9};$$

$$\text{and extracting the root, } x + \frac{1}{3} = \pm \frac{58}{3};$$

$$\therefore x = 19, \text{ or } -\frac{59}{3}.$$

$$13. \text{ Given } \frac{8 - x}{2} - \frac{2x - 11}{x - 3} = \frac{x - 2}{6}, \text{ to find the values of } x.$$

Multiplying by 6, $24 - 3x - \frac{12x - 66}{x - 3} = x - 2$;

by transposition, $26 - 4x = \frac{12x - 66}{x - 3}$,

and (18. Cor. 2.) $13 - 2x = \frac{6x - 33}{x - 3}$;

$\therefore 19x - 39 - 2x^2 = 6x - 33$;

$\therefore$ changing signs, and transposing,

$$2x^2 - 13x = -6,$$

and $x^2 - \frac{13}{2}x = -3$;

and completing the square,

$$x^2 - \frac{13}{2}x + \left(\frac{13}{4}\right)^2 = \frac{169}{16} - 3 = \frac{121}{16};$$

$\therefore$ extracting the root, $x - \frac{13}{4} = \pm \frac{11}{4}$;

$\therefore x = 6$, or $\frac{1}{2}$.

14. Given $5x - \frac{3x - 3}{x - 3} = 2x + \frac{3x - 6}{2}$, to find the values of x .

(18. Cor. 1.) $10x^2 - 36x + 6 = 4x^2 - 12x + 3x^2 - 15x + 18$;

$\therefore$ by transposition, $3x^2 - 9x = 12$,

and $x^2 - 3x = 4$;

$\therefore$ completing the square, $x^2 - 3x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4}$;

and extracting the root, $x - \frac{3}{2} = \pm \frac{5}{2}$,

and $x = 4$, or -1 .

15. Given $\frac{16}{x} - \frac{100 - 9x}{4x^2} = 3$, to find the values of x .

$$(18. \text{ Cor. 1.}) \quad 64x - 100 + 9x = 12x^2;$$

whence, by transposition, $12x^2 - 73x = -100$,

$$\text{and } x^2 - \frac{73}{12}x = -\frac{100}{12};$$

and completing the square,

$$x^2 - \frac{73}{12}x + \left(\frac{73}{24}\right)^2 = \frac{5329}{576} - \frac{100}{12} = \frac{529}{576};$$

$$\therefore \text{ extracting the root, } x - \frac{73}{24} = \pm \frac{23}{24},$$

$$\text{and } x = 4, \text{ or } \frac{25}{12}.$$

$$16. \text{ Given } 3x - \frac{169 - 3x}{x} = 29, \text{ to find the values of } x.$$

$$\text{Here } 3x^2 - 169 + 3x = 29x;$$

$$\therefore \text{ by transposition, } 3x^2 - 26x = 169,$$

$$\text{and } x^2 - \frac{26}{3}x = \frac{169}{3};$$

therefore, completing the square,

$$x^2 - \frac{26}{3}x + \frac{169}{9} = \frac{169}{3} + \frac{169}{9} = \frac{4}{9} \cdot 169;$$

$$\text{and extracting the root, } x - \frac{13}{3} = \pm \frac{2}{3} \cdot 13;$$

$$\therefore x = 13, \text{ or } -\frac{13}{3}.$$

$$17. \text{ Given } 16 - \frac{2x^2}{3} = \frac{4x}{5} + 7\frac{3}{5}, \text{ to find the values of } x.$$

$$\text{Multiplying every term by } \frac{3}{2}, \quad 24 - x^2 = \frac{6x}{5} + \frac{57}{5};$$

$$\therefore (17. \text{ Cor. 1.}) \text{ and by transposition,}$$

$$x^2 + \frac{6x}{5} = 24 - \frac{57}{5} = \frac{63}{5};$$

and completing the square,

$$x^2 + \frac{6x}{5} + \frac{9}{25} = \frac{63}{5} + \frac{9}{25} = \frac{324}{25};$$

$$\therefore \text{extracting the root, } x + \frac{3}{5} = \pm \frac{18}{5},$$

$$\text{and } x = 3, \text{ or } -\frac{21}{5}.$$

18. Given $\frac{10}{x} - \frac{14-2x}{x^2} = \frac{22}{9}$, to find the values of x .

$$(18. \text{ Cor. 1.}) \quad 10x - 14 + 2x = \frac{22x^2}{9};$$

$$\therefore (17. \text{ Cor. 1.}) \quad \frac{22x^2}{9} - 12x = -14,$$

$$\text{and } x^2 - \frac{54}{11}x = -\frac{63}{11};$$

$\therefore$ completing the square,

$$x^2 - \frac{54}{11}x + \left(\frac{27}{11}\right)^2 = \frac{729}{121} - \frac{63}{11} = \frac{36}{121};$$

$$\text{and extracting the root, } x - \frac{27}{11} = \pm \frac{6}{11};$$

$$\therefore x = 3, \text{ or } \frac{21}{11}.$$

19. Given $\frac{3x-4}{x-4} + 1 = 10 - \frac{x-2}{2}$, to find the values of x .

Clearing the equation of fractions,

$$6x - 8 + 2x - 8 = 20x - 80 - x^2 + 6x - 8;$$

$$\therefore \text{by transposition, } x^2 - 18x = -72;$$

and completing the square, $x^2 - 18x + 81 = 81 - 72 = 9;$

$$\text{whence, extracting the root, } x - 9 = \pm 3;$$

$$\text{and therefore } x = 12, \text{ or } 6.$$

20. Given $\frac{3x+4}{5} - \frac{30-2x}{x-6} = \frac{7x-14}{10}$, to find the values of x .

$$\text{Multiplying by } 10, 6x + 8 - \frac{300 - 20x}{x-6} = 7x - 14;$$

$$\therefore \text{ by transposition, } 22 - x = \frac{300 - 20x}{x-6};$$

$$\text{and } 28x - x^2 - 132 = 300 - 20x;$$

$$\therefore \text{ by transposition, and changing signs, } x^2 - 48x = -432;$$

$$\text{completing the square, } x^2 - 48x + 576 = 576 - 432 = 144;$$

$$\therefore \text{ extracting the root, } x - 24 = \pm 12;$$

$$\therefore x = 36, \text{ or } 12.$$

21. Given $3x - \frac{3x-10}{9-2x} = 2 + \frac{6x^2-40}{2x-1}$, to find the values of x .

$$\text{Multiplying by } 2x-1,$$

$$6x^2 - 3x - \frac{6x^2 - 23x + 10}{9-2x} = 4x - 2 + 6x^2 - 40,$$

$$\text{or } 7x + \frac{6x^2 - 23x + 10}{9-2x} = 42;$$

$$\therefore 63x - 14x^2 + 6x^2 - 23x + 10 = 378 - 84x;$$

$$\text{by transposition, } 124x - 8x^2 = 368,$$

$$\text{and } x^2 - \frac{31}{2}x = -46;$$

$$\therefore \text{ completing the square,}$$

$$x^2 - \frac{31}{2}x + \frac{961}{16} = \frac{961}{16} - 46 = \frac{225}{16};$$

$$\therefore \text{ extracting the root, } x - \frac{31}{4} = \pm \frac{15}{4};$$

$$\text{and therefore } x = \frac{23}{2}, \text{ or } 4.$$

22. Given $\frac{x}{5+x} + \frac{7}{6-4x} = \frac{11x}{11x-8}$, to find the values of x .

$$\text{Multiplying by } (5+x), x + \frac{35+7x}{6-4x} = \frac{55x+11x^2}{11x-8};$$

and multiplying by $11x-8$,

$$11x^2 - 8x + \frac{329x + 77x^2 - 280}{6-4x} = 55x + 11x^2,$$

$$\text{or } \frac{329x + 77x^2 - 280}{6-4x} = 63x;$$

$$\text{or, dividing by } 7, \frac{47x + 11x^2 - 40}{6-4x} = 9x;$$

$$\therefore 47x + 11x^2 - 40 = 54x - 36x^2;$$

$$\text{by transposition, } 47x^2 - 7x = 40;$$

$$\therefore x^2 - \frac{7}{47}x = \frac{40}{47};$$

and completing the square,

$$x^2 - \frac{7}{47}x + \left(\frac{7}{94}\right)^2 = \frac{49}{8336} + \frac{40}{47} = \frac{7569}{8336};$$

$$\text{and extracting the root, } x - \frac{7}{94} = \pm \frac{87}{94};$$

$$\therefore x = 1, \text{ or } -\frac{40}{47}.$$

23. Given $\frac{90}{x} - \frac{27}{x+2} = \frac{90}{x+1}$, to find the values of x .

$$\text{Dividing every term by } 9, \frac{10}{x} - \frac{3}{x+2} = \frac{10}{x+1};$$

$$\therefore (18. \text{ Cor. } 1.) 10x^2 + 30x + 20 - 3x^2 - 3x = 10x^2 + 20x;$$

$$\therefore \text{ by transposition and } (17. \text{ Cor. } 1.) 3x^2 - 7x = 20,$$

$$\text{and } x^2 - \frac{7}{3}x = \frac{20}{3};$$

completing the square, $x^2 - \frac{7}{3}x + \frac{49}{36} = \frac{20}{3} + \frac{49}{36} = \frac{259}{36}$;

and extracting the root, $x - \frac{7}{6} = \pm \frac{17}{6}$;

$$\therefore x = 4, \text{ or } -\frac{5}{3}.$$

24. Given $\frac{1}{x^2 - 3x} + \frac{1}{x^2 + 4x} = \frac{9}{6x}$, to find the values of x .

Multiplying every term by x , $\frac{1}{x-3} + \frac{1}{x+4} = \frac{9}{8}$;

$\therefore$ (18. Cor. 1.) $8x + 32 + 8x - 24 = 9x^2 + 9x - 108$;

$\therefore$ by transposition and (17. Cor. 1.) $9x^2 - 7x = 116$,

$$\text{and } x^2 - \frac{7}{9}x = \frac{116}{9};$$

completing the square,

$$x^2 - \frac{7}{9}x + \frac{49}{324} = \frac{116}{9} + \frac{49}{324} = \frac{4225}{324};$$

$\therefore$ extracting the root, $x - \frac{7}{18} = \pm \frac{65}{18}$;

$$\therefore x = 4, \text{ or } -\frac{29}{9}.$$

25. Given $\frac{x^3 - 10x^2 + 1}{x^2 - 6x + 9} = x - 3$, to find the values of x .

(18. Cor. 1.) $x^3 - 10x^2 + 1 = x^3 - 9x^2 + 27x - 27$;

$\therefore$ by transposition and (17. Cor. 1.) $x^2 + 27x = 28$;

and completing the square,

$$x^2 + 27x + \left(\frac{27}{2}\right)^2 = 28 + \frac{729}{4} = \frac{841}{4};$$

$\therefore$ extracting the root, $x + \frac{27}{2} = \pm \frac{29}{2}$,

$$\text{and } x = 1, \text{ or } -28.$$

26. Given $\frac{x}{7-x} + \frac{7-x}{x} = 2\frac{9}{10}$, to find the values of x .

Clearing the equation of fractions,

$$20x^2 - 140x + 490 = 203x - 29x^2;$$

$\therefore$ by transposition, $49x^2 - 343x = -490$,

$$\text{and } x^2 - 7x = -10;$$

$\therefore$ completing the square, $x^2 - 7x + \frac{49}{4} = \frac{49}{4} - 10 = \frac{9}{4}$;

and extracting the root, $x - \frac{7}{2} = \pm \frac{3}{2}$;

$$\therefore x = 5, \text{ or } 2.$$

27. Given $\frac{7-12x}{x^4} = \frac{x}{\sqrt{x}} - \frac{8x+110}{\sqrt{x^3}}$, to find the values of x .

Clearing the equation of fractions,

$$7 - 12x = x^2 - 8x - 110;$$

$$\therefore 117 = x^2 + 4x;$$

and completing the square, $121 = x^2 + 4x + 4$;

$\therefore$ extracting the root, $\pm 11 = x + 2$;

and therefore $x = 9$, or -13 .

28. Given $\sqrt{(x+5)} \times \sqrt{(x+12)} = 12$, to find the values of x .

Squaring both sides, $(x+5) \cdot (x+12) = 144$;

$$\text{or } x^2 + 17x + 60 = 144;$$

$\therefore$ by transposition, $x^2 + 17x = 84$;

and completing the square,

$$x^2 + 17x + \left(\frac{17}{2}\right)^2 = \frac{289}{4} + 84 = \frac{625}{4};$$

extracting the root, $x + \frac{17}{2} = \pm \frac{25}{2}$,

and $x = 4$, or -21 .

29. Given $\sqrt[3]{(x^3 - a^3)} = x - b$, to find the values of x .

Cubing each side, $x^3 - a^3 = x^3 - 3bx^2 + 3b^2x - b^3$;

$\therefore$ by transposition, $3bx^2 - 3b^2x = a^3 - b^3$,

and $x^2 - bx = \frac{a^3 - b^3}{3b}$;

$\therefore$ completing the square,

$$x^2 - bx + \frac{b^2}{4} = \frac{a^3 - b^3}{3b} + \frac{b^2}{4} = \frac{4a^3 - b^3}{12b}$$

extracting the root, $x - \frac{b}{2} = \pm \sqrt{\left(\frac{4a^3 - b^3}{12b}\right)}$,

and $x = \frac{b}{2} \pm \sqrt{\left(\frac{4a^3 - b^3}{12b}\right)}$.

30. Given $x^{2n} - mx^n = p$, to find the values of x .

Completing the square,

$$x^{2n} - mx^n + \frac{m^2}{4} = \frac{m^2}{4} + p = \frac{m^2 + 4p}{4}$$

extracting the root, $x^n - \frac{m}{2} = \pm \frac{\sqrt{(m^2 + 4p)}}{2}$;

$$\therefore x^n = \frac{m \pm \sqrt{(m^2 + 4p)}}{2},$$

$$\text{and } x = \left(\frac{m \pm \sqrt{(m^2 + 4p)}}{2}\right)^{\frac{1}{n}}.$$

31. Given $\frac{\sqrt{4x} + 2}{4 + \sqrt{x}} = \frac{4 - \sqrt{x}}{\sqrt{x}}$, to find the values of x .

Clearing the equation of fractions, $2x + 2\sqrt{x} = 16 - x$;

$\therefore$ by transposition, $3x + 2\sqrt{x} = 16$,

$$\text{and } x + \frac{2}{3}\sqrt{x} = \frac{16}{3};$$

completing the square, $x + \frac{2}{3}\sqrt{x} + \frac{1}{9} = \frac{16}{3} + \frac{1}{9} = \frac{49}{9}$;

and extracting the root, $\sqrt{x} + \frac{1}{3} = \pm \frac{7}{3}$;

$$\therefore \sqrt{x} = 2, \text{ or } -\frac{8}{3},$$

$$\text{and } x = 4, \text{ or } \frac{64}{9}.$$

32. Given $\frac{\sqrt{a^2x+b}}{a+\sqrt{x}} = \frac{a-\sqrt{x}}{\sqrt{x}}$, to find the values of x .

Clearing the equation of fractions, $ax + b\sqrt{x} = a^2 - x$;

by transposition, $(a+1) \cdot x + b\sqrt{x} = a^2$,

$$\text{and } x + \frac{b}{a+1} \cdot \sqrt{x} = \frac{a^2}{a+1};$$

completing the square,

$$x + \frac{b}{a+1} \cdot \sqrt{x} + \frac{b^2}{4 \cdot (a+1)^2} = \frac{a^2}{a+1} + \frac{b^2}{4 \cdot (a+1)^2} = \frac{4a^2 + 4a^2 + b^2}{4 \cdot (a+1)^2},$$

and extracting the root,

$$\sqrt{x} + \frac{b}{2 \cdot (a+1)} = \pm \frac{\sqrt{(4a^2 + 4a^2 + b^2)}}{2 \cdot (a+1)};$$

$$\therefore x = \left(\frac{-b \pm \sqrt{(4a^2 + 4a^2 + b^2)}}{2 \cdot (a+1)} \right)^2.$$

83. Given $\sqrt{x^3} - 2\sqrt{x} - x = 0$, to find the values of x .

Dividing by $\sqrt{x}$, we find $x - 2 - \sqrt{x} = 0$;

$\therefore$ by transposition, $x - \sqrt{x} = 2$;

and completing the square, $x - \sqrt{x} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}$;

extracting the root, $\sqrt{x} - \frac{1}{2} = \pm \frac{3}{2}$;

$\therefore \sqrt{x} = 2$, or -1 ,

and $x = 4$, or 1 .

34. Given $\sqrt{x^3} + \sqrt{x^3} = 6\sqrt{x}$, to find the values of x .

Dividing by $\sqrt{x}$, $x^3 + x = 6$;

$\therefore$ completing the square, $x^2 + x + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$;

and extracting the root, $x + \frac{1}{2} = \pm \frac{5}{2}$,

and $x = 2$, or -3 .

35. Given $\frac{x}{2} = 22\frac{1}{2} + \frac{\sqrt{x}}{3}$, to find the values of x .

Multiplying by 2, and transposing, $x - \frac{2}{3}\sqrt{x} = \frac{133}{3}$;

and completing the square,

$$x - \frac{2}{3}\sqrt{x} + \frac{1}{9} = \frac{133}{3} + \frac{1}{9} = \frac{400}{9};$$

$\therefore$ extracting the root, $\sqrt{x} - \frac{1}{3} = \pm \frac{20}{3}$,

and $\sqrt{x} = 7$, or $-\frac{19}{3}$;

$\therefore x = 49$, or $\frac{361}{9}$.

36. Given $\frac{3\sqrt{x}}{5} - 2 - \frac{1}{20} = 0$, to find the values of x .

Clearing the equation of fractions,

$$12\sqrt{x} - 40 - x + 5 = 0;$$

$$\therefore (17. \text{ Cor. } 1.) x - 12\sqrt{x} = -35,$$

and completing the square, $x - 12\sqrt{x} + 36 = 36 - 35 = 1$;

$$\therefore \text{ extracting the root, } \sqrt{x} - 6 = \pm 1;$$

$$\therefore \sqrt{x} = 7, \text{ or } 5,$$

$$\text{and } x = 49, \text{ or } 25.$$

37. Given $x^{\frac{1}{2}} + x^{\frac{1}{2}} = 756$, to find the values of x .

Completing the square, $x^{\frac{1}{2}} + x^{\frac{1}{2}} + \frac{1}{4} = 756 + \frac{1}{4} = \frac{3025}{4}$;

and extracting the root, $x^{\frac{1}{2}} + \frac{1}{2} = \pm \frac{55}{2}$;

$$\therefore x^{\frac{1}{2}} = 27, \text{ or } -28,$$

$$\text{and } x^{\frac{1}{2}} = 3, \text{ or } \sqrt[3]{(-28)};$$

$$\therefore x = 243, \text{ or } (-28)^{\frac{2}{3}}.$$

38. Given $x^3 - x^{\frac{1}{2}} = 56$, to find the values of x .

Completing the square, $x^3 - x^{\frac{1}{2}} + \frac{1}{4} = 56 + \frac{1}{4} = \frac{225}{4}$;

$$\therefore \text{ extracting the root, } x^{\frac{1}{2}} - \frac{1}{2} = \pm \frac{15}{2};$$

$$\text{and } x^{\frac{1}{2}} = 8, \text{ or } -7;$$

$$\therefore x^{\frac{1}{2}} = 2, \text{ or } \sqrt[3]{(-7)},$$

$$\text{and } x = 4, \text{ or } (-7)^{\frac{2}{3}}.$$

39. Given $3x^{\frac{1}{2}} + x^{\frac{1}{2}} = 3104$, to find the values of x .

$$\text{Dividing by 3, } x^{\frac{1}{2}} + \frac{1}{3}x^{\frac{1}{2}} = \frac{3104}{3};$$

and completing the square,

$$x^{\frac{1}{2}} + \frac{1}{3}x^{\frac{1}{2}} + \frac{1}{36} = \frac{3104}{3} + \frac{1}{36} = \frac{37249}{36};$$

$$\therefore \text{ extracting the root, } x^{\frac{1}{2}} + \frac{1}{6} = \pm \frac{193}{6};$$

$$\text{whence } x^{\frac{1}{2}} = 32, \text{ or } -\frac{97}{3},$$

$$\text{and } x^{\frac{1}{2}} = 2, \text{ or } \left(\frac{-97}{3}\right)^{\frac{1}{2}};$$

$$\therefore x = 64, \text{ or } \left(\frac{-97}{3}\right)^{\frac{4}{3}}.$$

40. Given $ax^{\frac{1}{2}} + bx^{\frac{1}{2}} = c$, to find the values of x .

$$\text{Dividing by } a, x^{\frac{1}{2}} + \frac{b}{a}x^{\frac{1}{2}} = \frac{c}{a};$$

and completing the square,

$$x^{\frac{1}{2}} + \frac{b}{a}x^{\frac{1}{2}} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} + \frac{c}{a} = \frac{b^2 + 4ac}{4a^2};$$

$$\therefore \text{ extracting the root, } x^{\frac{1}{2}} + \frac{b}{2a} = \frac{\pm \sqrt{(b^2 + 4ac)}}{2a},$$

$$\text{and } x^{\frac{1}{2}} = \frac{\pm \sqrt{(b^2 + 4ac)} - b}{2a};$$

$$\text{whence } x = \left(\frac{\pm \sqrt{(b^2 + 4ac)} - b}{2a}\right)^{\frac{4}{3}}.$$

41. Given $3x^{\frac{1}{2}} - \frac{5x^{\frac{1}{2}}}{2} = -592$, to find the values of x .

$$(17. \text{ Cor. 1.}) \quad \frac{5x^{\frac{1}{2}}}{2} - 3x^{\frac{1}{2}} = 592;$$

$$\text{and therefore } x^{\frac{1}{2}} - \frac{6}{5}x^{\frac{1}{2}} = \frac{1184}{5}.$$

Completing the square,

$$x^{\frac{1}{2}} - \frac{6}{5}x^{\frac{1}{2}} + \frac{9}{25} = \frac{1184}{5} + \frac{9}{25} = \frac{5929}{25};$$

$$\therefore \text{ extracting the root, } x^{\frac{1}{2}} - \frac{3}{5} = \pm \frac{77}{5};$$

$$\therefore x^{\frac{1}{2}} = 16, \text{ or } -\frac{74}{5},$$

$$\text{and } x = 8, \text{ or } \left(-\frac{74}{5}\right)^2.$$

42. Given $x^a - 2ax^{\frac{n}{2}} = b$, to find the values of x .

$$\text{Completing the square, } x^a - 2ax^{\frac{n}{2}} + a^2 = a^2 + b;$$

$$\therefore \text{ extracting the root, } x^{\frac{n}{2}} - a = \pm \sqrt{a^2 + b},$$

$$\text{and } x^{\frac{n}{2}} = a \pm \sqrt{a^2 + b};$$

$$\therefore x = (a \pm \sqrt{a^2 + b})^{\frac{2}{n}}.$$

43. Given $a^2x^2 - bx = c$, to find the values of x .

$$\text{The equation is the same as } a^2x^2 - \frac{b}{a} \cdot ax = c;$$

$$\therefore \text{ completing the square, } a^2x^2 - \frac{b}{a} \cdot ax + \frac{b^2}{4a^2} = c + \frac{b^2}{4a^2};$$

$$\text{extracting the root, } ax - \frac{b}{2a} = \pm \sqrt{c + \frac{b^2}{4a^2}};$$

$$\therefore ax = \frac{b \pm \sqrt{4a^2c + b^2}}{2a},$$

$$\text{and } x = \frac{b \pm \sqrt{4a^2c + b^2}}{2a^2}.$$

44. Given $\sqrt{(2x+1)} + 2\sqrt{x} = \frac{21}{\sqrt{(2x+1)}}$, to find the values of x .

$$(18. \text{ Cor. 1.}) \quad 2x + 1 + 2\sqrt{(2x^2 + x)} = 21;$$

$$\therefore \text{ by transposition, } 2\sqrt{(2x^2 + x)} = 20 - 2x;$$

$$\text{and therefore } \sqrt{(2x^2 + x)} = 10 - x;$$

$$\therefore \text{ squaring both sides, } 2x^2 + x = 100 - 20x + x^2;$$

$$\text{and transposing, } x^2 + 21x = 100;$$

completing the square,

$$x^2 + 21x + \left(\frac{21}{2}\right)^2 = 100 + \frac{441}{4} = \frac{841}{4};$$

$$\therefore \text{ extracting the root, } x + \frac{21}{2} = \pm \frac{29}{2};$$

$$\therefore x = 4, \text{ or } -25.$$

45. Given $2\sqrt{(x-a)} + 3\sqrt{(2x)} = \frac{7a+5x}{\sqrt{(x-a)}}$, to find the values of x .

$$(18. \text{ Cor. 1.}) \quad 2x - 2a + 3\sqrt{(2x^2 - 2ax)} = 7a + 5x;$$

$$\text{by transposition, } 3\sqrt{(2x^2 - 2ax)} = 9a + 3x;$$

$$\therefore \sqrt{(2x^2 - 2ax)} = 3a + x;$$

$$\text{squaring both sides, } 2x^2 - 2ax = 9a^2 + 6ax + x^2;$$

$$\text{by transposition, } x^2 - 8ax = 9a^2;$$

$$\text{completing the square, } x^2 - 8ax + 16a^2 = 25a^2;$$

$$\text{extracting the root, } x - 4a = \pm 5a;$$

$$\therefore x = 9a, \text{ or } -a.$$

46. Given

$$\sqrt{(x+60)} + \sqrt{(x^2+9)} = \frac{2\sqrt{(x^3+60x^2+9x+540)} + 89}{\sqrt{(x+60)} + \sqrt{(x^2+9)}},$$

to find the values of x .

Clearing the equation of fractions,

$$x + 60 + x^2 + 9 + 2\sqrt{(x^2 + 60x^2 + 9x + 540)} = 2\sqrt{(x^2 + 60x^2 + 9x + 540)} + 89;$$

$$\therefore \text{ by transposition, } x^2 + x = 20;$$

$$\text{and completing the square, } x^2 + x + \frac{1}{4} = 20 + \frac{1}{4} = \frac{81}{4}$$

$$\therefore \text{ extracting the root, } x + \frac{1}{2} = \pm \frac{9}{2},$$

$$\text{and } x = 4, \text{ or } -5.$$

47. Given

$$\frac{123 + 41\sqrt{x}}{5\sqrt{x} - x} = \frac{20\sqrt{x} + 4x}{3 - \sqrt{x}} - \frac{2x^2}{(5\sqrt{x} - x) \cdot (3 - \sqrt{x})},$$

to find the values of x .

This equation is

$$\frac{41 \cdot (3 + \sqrt{x})}{5\sqrt{x} - x} = \frac{4 \cdot (5\sqrt{x} + x)}{3 - \sqrt{x}} - \frac{2x^2}{(5\sqrt{x} - x) \cdot (3 - \sqrt{x})};$$

and therefore, clearing it of fractions,

$$41 \cdot (9 - x) = 4 \cdot (25x - x^2) - 2x^2,$$

$$\text{or } 369 - 41x = 100x - 4x^2 - 2x^2;$$

$$\therefore \text{ by transposition, } 6x^2 - 141x = -369,$$

$$\text{and } x^2 - \frac{47}{2}x = -\frac{123}{2};$$

completing the square,

$$x^2 - \frac{47}{2}x + \left(\frac{47}{4}\right)^2 = \frac{2209}{16} - \frac{123}{2} = \frac{1225}{16};$$

$$\therefore \text{ extracting the root, } x - \frac{47}{4} = \pm \frac{35}{4},$$

$$\text{and } x = \frac{41}{2}, \text{ or } 3.$$

48. Given $\frac{x}{\sqrt{x} + \sqrt{(a-x)}} + \frac{x}{\sqrt{x} - \sqrt{(a-x)}} = \frac{b}{\sqrt{x}},$

to find the values of x .

Multiplying the equation by

$$(\sqrt{x} + \sqrt{a-x}) \cdot (\sqrt{x} - \sqrt{a-x});$$

$$x \cdot (\sqrt{x} - \sqrt{a-x} + \sqrt{x} + \sqrt{a-x}) = \frac{b}{\sqrt{x}} \cdot (2x - a),$$

$$\text{or } 2x\sqrt{x} = \frac{b}{\sqrt{x}} \cdot (2x - a);$$

$$\therefore 2x^2 = 2bx - ab,$$

$$\text{and } x^2 - bx = -\frac{ab}{2};$$

completing the square,

$$x^2 - bx + \frac{b^2}{4} = \frac{b^2}{4} - \frac{ab}{2} = \frac{b^2 - 2ab}{4};$$

$$\text{extracting the root, } x - \frac{b}{2} = \pm \frac{\sqrt{(b^2 - 2ab)}}{2};$$

$$\therefore x = \frac{b \pm \sqrt{(b^2 - 2ab)}}{2}.$$

49. Given $\frac{x + \sqrt{(x^2 - 9)}}{x - \sqrt{(x^2 - 9)}} = (x - 2)^2$, to find the values of x .

Multiplying the numerator and the denominator of the fraction by $x + \sqrt{(x^2 - 9)}$,

$$\frac{(x + \sqrt{x^2 - 9})^2}{9} = (x - 2)^2;$$

$$\text{extracting the square root, } \frac{x + \sqrt{(x^2 - 9)}}{3} = \pm (x - 2);$$

$$\text{taking the positive sign, } x + \sqrt{(x^2 - 9)} = 3x - 6;$$

$$\text{by transposition, } \sqrt{(x^2 - 9)} = 2x - 6,$$

$$\text{and squaring both sides, } x^2 - 9 = 4x^2 - 24x + 36;$$

$$\text{by transposition, } 3x^2 - 24x = -45,$$

$$\text{and } x^2 - 8x = -15;$$

completing the square, $x^2 - 8x + 16 = 1$;

extracting the root, $x - 4 = \pm 1$;

$\therefore x = 5$, or 3 .

But if $\frac{x + \sqrt{(x^2 - 9)}}{3} = -(x - 2)$;

$x + \sqrt{(x^2 - 9)} = -3x + 6$,

and $\sqrt{(x^2 - 9)} = -4x + 6$;

$\therefore$ squaring both sides, $x^2 - 9 = 16x^2 - 48x + 36$,

and by transposition, $15x^2 - 48x = -45$;

$\therefore x^2 - \frac{16}{5}x = -3$;

and completing the square,

$x^2 - \frac{16}{5}x + \frac{64}{25} = \frac{64}{25} - 3 = \frac{-11}{25}$;

extracting the root, $x - \frac{8}{5} = \pm \frac{\sqrt{(-11)}}{5}$;

$\therefore x = \frac{8 \pm \sqrt{(-11)}}{5}$.

50. Given $x + 5 = \sqrt{(x + 5)} + 6$, to find the values of x .

By transposition, $(x + 5) - \sqrt{(x + 5)} = 6$;

and therefore, completing the square,

$(x + 5) - \sqrt{(x + 5)} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$;

extracting the root, $\sqrt{(x + 5)} - \frac{1}{2} = \pm \frac{5}{2}$;

$\therefore \sqrt{(x + 5)} = 3$, or -2 ;

and squaring both sides, $x + 5 = 9$, or 4 ;

whence $x = 4$, or -1 .

51. Given $x + 16 - 7\sqrt{(x + 16)} = 10 - 4\sqrt{(x + 16)}$, to find the values of x .

By transposition, $(x + 16) - 3\sqrt{(x + 16)} = 10$;
and completing the square,

$$(x + 16) - 3\sqrt{(x + 16)} + \frac{9}{4} = 10 + \frac{9}{4} = \frac{49}{4};$$

$$\therefore \text{ extracting the root, } \sqrt{(x + 16)} - \frac{3}{2} = \pm \frac{7}{2};$$

$$\text{and } \sqrt{(x + 16)} = 5, \text{ or } -2;$$

$$\text{whence } x + 16 = 25, \text{ or } 4;$$

$$\text{and } x = 9, \text{ or } -12.$$

52. Given $\sqrt{(x + 12)} + \sqrt[4]{(x + 12)} = 6$, to find the values of x .

Completing the square,

$$\sqrt{(x + 12)} + \sqrt[4]{(x + 12)} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4};$$

$$\therefore \text{ extracting the root, } \sqrt[4]{(x + 12)} + \frac{1}{2} = \pm \frac{5}{2},$$

$$\text{and } \sqrt[4]{(x + 12)} = 2, \text{ or } -3;$$

$$\text{whence } x + 12 = 16, \text{ or } 81;$$

$$\text{and } x = 4, \text{ or } 62.$$

53. Given $x^2 - 2x + 6\sqrt{(x^2 - 2x + 5)} = 11$, to find the values of x .

Adding 5 to each side of the equation,

$$(x^2 - 2x + 5) + 6\sqrt{(x^2 - 2x + 5)} = 16;$$

$\therefore$ completing the square,

$$(x^2 - 2x + 5) + 6\sqrt{(x^2 - 2x + 5)} + 9 = 16 + 9 = 25;$$

$$\text{and extracting the root, } \sqrt{(x^2 - 2x + 5)} + 3 = \pm 5,$$

$$\text{and } \sqrt{(x^2 - 2x + 5)} = 2, \text{ or } -8;$$

$$\therefore \text{ squaring both sides, } x^2 - 2x + 5 = 4, \text{ or } 64;$$

$$\text{whence } x^2 - 2x + 1 = 0, \text{ or } 60;$$

and extracting the root, $x - 1 = 0$, or $\pm \sqrt{60}$; *

$$\therefore x = 1, \text{ or } 1 \pm 2\sqrt{15}.$$

54. Given $2x^2 + 3x - 5\sqrt{(2x^2 + 3x + 9)} + 3 = 0$, to find the values of x .

Adding 6 to each side, $2x^2 + 3x + 9 - 5\sqrt{(2x^2 + 3x + 9)} = 6$; and completing the square,

$$(2x^2 + 3x + 9) - 5\sqrt{(2x^2 + 3x + 9)} + \frac{25}{4} = 6 + \frac{25}{4} = \frac{49}{4};$$

$$\therefore \text{extracting the root, } \sqrt{(2x^2 + 3x + 9)} - \frac{5}{2} = \pm \frac{7}{2};$$

$$\text{and } \sqrt{(2x^2 + 3x + 9)} = 6, \text{ or } -1,$$

suppose the value to be 6,

$$\text{then } 2x^2 + 3x + 9 = 36,$$

$$\text{and } x^2 + \frac{3}{2}x = \frac{27}{2};$$

$\therefore$ completing the square,

$$x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{27}{2} + \frac{9}{16} = \frac{225}{16};$$

$$\text{and extracting the root, } x + \frac{3}{4} = \pm \frac{15}{4},$$

$$\text{or } x = 3, \text{ or } -\frac{9}{2}.$$

$$\text{But if } -1 \text{ be taken, } \sqrt{(2x^2 + 3x + 9)} = -1;$$

$$\therefore 2x^2 + 3x + 9 = 1,$$

$$\text{and } x^2 + \frac{3}{2}x = -4;$$

completing the square,

$$x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{9}{16} - 4 = \frac{-55}{16};$$

* In this example, if 0 be the value, the two roots of the equation are $+1$ and $+1$.

and extracting the root, $x + \frac{3}{4} = \frac{\pm \sqrt{(-55)}}{4}$;

$$\text{and } x = \frac{-3 \pm \sqrt{(-55)}}{4}.$$

55. Given $\frac{\sqrt{(x^2 + x + 6)}}{3} = \frac{18 - \left(\frac{4}{3}\sqrt{(x^2 + x + 6)} - 2\right)}{\sqrt{(x^2 + x + 6)}}$,

to find the values of x .

$$(18. \text{ Cor. 1.}) (x^2 + x + 6) = 54 - 4\sqrt{(x^2 + x + 6)} + 6;$$

$$\therefore \text{ by transposition, } (x^2 + x + 6) + 4\sqrt{(x^2 + x + 6)} = 60;$$

and completing the square,

$$(x^2 + x + 6) + 4\sqrt{(x^2 + x + 6)} + 4 = 64;$$

$$\text{extracting the root, } \sqrt{(x^2 + x + 6)} + 2 = \pm 8,$$

$$\text{and } \sqrt{(x^2 + x + 6)} = 6, \text{ or } -10; \text{ suppose the former,}$$

$$\text{then, squaring both sides, } x^2 + x + 6 = 36;$$

$$\text{and by transposition, } x^2 + x = 30;$$

$$\text{completing the square, } x^2 + x + \frac{1}{4} = 30 + \frac{1}{4} = \frac{121}{4};$$

$$\text{extracting the root, } x + \frac{1}{2} = \pm \frac{11}{2},$$

$$\text{and } x = 5, \text{ or } -6.$$

$$\text{But taking } \sqrt{(x^2 + x + 6)} = -10,$$

$$\text{then } x^2 + x + 6 = 100;$$

$$\text{and by transposition, } x^2 + x = 94;$$

$$\text{completing the square, } x^2 + x + \frac{1}{4} = 94 + \frac{1}{4} = \frac{377}{4};$$

$$\text{extracting the root, } x + \frac{1}{2} = \frac{\pm \sqrt{(377)}}{2},$$

$$\text{and } x = \frac{-1 \pm \sqrt{(377)}}{2}.$$

56. Given $\{(x-2)^2 - x\}^2 - (x-2)^2 = 88 - (x-2)$, to find the values of x .

By transposition, $\{(x-2)^2 - x\}^2 - \{(x-2)^2 - x\} = 90$;
and completing the square,

$$\{(x-2)^2 - x\}^2 - \{(x-2)^2 - x\} + \frac{1}{4} = 90 + \frac{1}{4} = \frac{361}{4};$$

$$\text{extracting the root, } (x-2)^2 - x - \frac{1}{2} = \pm \frac{19}{2},$$

$$\text{and } (x-2)^2 - x = 10, \text{ or } -9;$$

whence, adding 2 to each side,

$(x-2)^2 - (x-2) = 12, \text{ or } -7$; supposing the former,
and completing the square,

$$(x-2)^2 - (x-2) + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4};$$

$$\text{extracting the root, } x-2 - \frac{1}{2} = \pm \frac{7}{2};$$

$$\text{whence } x = \frac{5}{2} \pm \frac{7}{2} = 6, \text{ or } -1;$$

and in the second case, where $(x-2)^2 - (x-2) = -7$;

completing the square,

$$(x-2)^2 - (x-2) + \frac{1}{4} = \frac{1}{4} - 7 = \frac{-27}{4};$$

$$\text{extracting the root, } x-2 - \frac{1}{2} = \frac{\pm \sqrt{(-27)}}{2};$$

$$\therefore x = \frac{5 \pm 3\sqrt{(-3)}}{2}.$$

57. Given $(x+6)^2 + 2x^{\frac{1}{2}} \cdot (x+6) = 138 + x^{\frac{1}{2}}$, to find the values of x .

Completing the square,

$$(x+6)^2 + 2x^{\frac{1}{2}} \cdot (x+6) + x = 138 + x + x^{\frac{1}{2}};$$

extracting the root, $x + 6 + x^{\frac{1}{2}} = \pm \sqrt{(138 + x + x^{\frac{1}{2}})}$;
and squaring both sides,

$$(x + x^{\frac{1}{2}})^2 + 12 \cdot (x + x^{\frac{1}{2}}) + 36 = 138 + x + x^{\frac{1}{2}};$$

$\therefore$ by transposition, $(x + x^{\frac{1}{2}})^2 + 11 \cdot (x + x^{\frac{1}{2}}) = 102$;

completing the square,

$$(x + x^{\frac{1}{2}})^2 + 11 \cdot (x + x^{\frac{1}{2}}) + \frac{121}{4} = 102 + \frac{121}{4} = \frac{529}{4};$$

$$\text{extracting the root, } x + x^{\frac{1}{2}} + \frac{11}{2} = \pm \frac{23}{2},$$

and $x + x^{\frac{1}{2}} = 6$, or -17 ; supposing the former,

and completing the square, $x + x^{\frac{1}{2}} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$;

$$\text{extracting the root, } x^{\frac{1}{2}} + \frac{1}{2} = \pm \frac{5}{2},$$

and $x^{\frac{1}{2}} = 2$, or -3 ;

$\therefore x = 4$, or 9 .

But if $x + x^{\frac{1}{2}} = -17$,

completing the square, $x + x^{\frac{1}{2}} + \frac{1}{4} = \frac{1}{4} - 17 = \frac{-67}{4}$;

and extracting the root, $x^{\frac{1}{2}} + \frac{1}{2} = \frac{\pm \sqrt{(-67)}}{2}$,

$$\text{and } x^{\frac{1}{2}} = \frac{-1 \pm \sqrt{(-67)}}{2};$$

$$\text{whence } x = \frac{-33 \mp \sqrt{(-67)}}{2}.$$

58. Given $x - 1 = 2 + \frac{2}{x^{\frac{1}{2}}}$, to find the values of x .

$$\text{Since } x - 1 = (x^{\frac{1}{2}} + 1) \times (x^{\frac{1}{2}} - 1);$$

$$\therefore (x^{\frac{1}{2}} + 1) \cdot (x^{\frac{1}{2}} - 1) = 2 \cdot \frac{x^{\frac{1}{2}} + 1}{x^{\frac{1}{2}}};$$

and therefore, dividing by $x^{\frac{1}{2}} + 1$, $x^{\frac{1}{2}} - 1 = \frac{2}{x^{\frac{1}{2}}}$;

whence $x - x^{\frac{1}{2}} = 2$;

and completing the square, $x - x^{\frac{1}{2}} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}$;

$\therefore$ extracting the root, $x^{\frac{1}{2}} - \frac{1}{2} = \pm \frac{3}{2}$,

and $x^{\frac{1}{2}} = 2$, or -1 ;

whence $x = 4$, or 1 .

59. Given $x^4 - 2x^3 + x = 132$, to find the values of x .

(15) Adding and subtracting x^2 , there results

$$x^4 - 2x^3 + x^2 - (x^2 - x) = 132;$$

and completing the square,

$$(x^2 - x)^2 - (x^2 - x) + \frac{1}{4} = 132 + \frac{1}{4} = \frac{529}{4};$$

extracting the root, $x^2 - x - \frac{1}{2} = \pm \frac{23}{2}$;

whence $x^2 - x = 12$, or -11 ; supposing the former;

then, completing the square, $x^2 - x + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4}$;

and extracting the root, $x - \frac{1}{2} = \pm \frac{7}{2}$,

and $x = 4$, or -3 .

But if $x^2 - x = -11$,

completing the square, $x^2 - x + \frac{1}{4} = \frac{1}{4} - 11 = -\frac{43}{4}$;

extracting the root, $x - \frac{1}{2} = \frac{\pm \sqrt{(-43)}}{2}$,

$$\text{and } x = \frac{1 \pm \sqrt{(-43)}}{2}.$$

60. Given $(x^2 + 2x) \cdot (x + 4) = 2 - (x + 4)$, to find the values of x .

By subtraction, $(x^2 + 2x) \cdot (x + 4) = - (x + 2)$;

$\therefore$ dividing by $x + 2$, $x \cdot (x + 4) = -1$;

$$\therefore x^2 + 4x = -1;$$

and completing the square, $x^2 + 4x + 4 = 4 - 1 = 3$;

extracting the root, $x + 2 = \pm \sqrt{3}$;

$$\therefore x = -2 \pm \sqrt{3}.$$

61. Given $\frac{33\frac{24}{25}}{\sqrt{(5x^2 - x^4)}} + \frac{\sqrt{(5 - x^2)}}{25x} = \frac{34}{x}$, to find the values of x .

Multiplying every term by $25\sqrt{(5x^2 - x^4)}$;

$$\therefore 849 + 5 - x^2 = 850\sqrt{(5 - x^2)};$$

and by transposition, $(5 - x^2) - 850\sqrt{(5 - x^2)} = -849$;

completing the square,

$$(5 - x^2) - 850\sqrt{(5 - x^2)} + 425^2 = 180625 - 849 = 179776;$$

and extracting the root, $\sqrt{(5 - x^2)} - 425 = \pm 424$;

$$\text{whence } \sqrt{(5 - x^2)} = 849, \text{ or } 1.$$

$$\text{and } 5 - x^2 = 720801, \text{ or } 1,$$

$$\text{and } x^2 = -720796, \text{ or } 4;$$

$$\therefore x = \pm \sqrt{(-720796)}, \text{ or } \pm 2.$$

62. Given $\frac{a-x}{x} + \frac{x}{a-x} = \frac{b}{c}$, to find the values of x .

Multiplying every term by $\frac{a-x}{x}$;

$$\left(\frac{a-x}{x}\right)^2 + 1 = \frac{b}{c} \cdot \frac{a-x}{x};$$

by transposition, $\left(\frac{a-x}{x}\right)^2 - \frac{b}{c} \cdot \frac{a-x}{x} = -1;$

ompleting the square,

$$\left(\frac{a-x}{x}\right)^2 - \frac{b}{c} \cdot \frac{a-x}{x} + \frac{b^2}{4c^2} = \frac{b^2}{4c^2} - 1 = \frac{b^2 - 4c^2}{4c^2};$$

extracting the root, $\frac{a-x}{x} - \frac{b}{2c} = \pm \frac{\sqrt{(b^2 - 4c^2)}}{2c},$

or, $\frac{a}{x} - 1 - \frac{b}{2c} = \pm \frac{\sqrt{(b^2 - 4c^2)}}{2c};$

by transposition, $\frac{a}{x} = \frac{2c + b \pm \sqrt{(b^2 - 4c^2)}}{2c};$

$$\therefore x = \frac{2ac}{2c + b \pm \sqrt{(b^2 - 4c^2)}}.$$

63. Given $9x + \sqrt{(16x^2 + 36x^2)} = 15x^2 - 4$, to find the values of x .

By transposition, $9x + 4 + 2x\sqrt{(9x + 4)} = 15x^2;$

and completing the square,

$$(9x + 4) + 2x\sqrt{(9x + 4)} + x^2 = 16x^2;$$

and extracting the root, $\sqrt{(9x + 4)} + x = \pm 4x;$

$$\therefore \sqrt{(9x + 4)} = 3x, \text{ or } -5x,$$

and $9x + 4 = 9x^2$, or $25x^2$; supposing the former;

then, by transposition, $9x^2 - 9x = 4;$

and completing the square, $9x^2 - 9x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4};$

$$\therefore \text{extracting the root, } 3x - \frac{3}{2} = \pm \frac{5}{2};$$

and $3x = 4$, or $-1;$

whence $x = \frac{4}{3}$, or $-\frac{1}{3}.$

$$\text{But if } 9x + 4 = 25x^2,$$

$$\text{then, by transposition, } 25x^2 - 9x = 4;$$

and completing the square,

$$25x^2 - 9x + \left(\frac{9}{10}\right)^2 = \frac{81}{100} + 4 = \frac{481}{100};$$

$$\text{and extracting the root, } 5x - \frac{9}{10} = \pm \frac{\sqrt{(481)}}{10};$$

$$\therefore 5x = \frac{9 \pm \sqrt{(481)}}{10},$$

$$\text{and } x = \frac{9 \pm \sqrt{(481)}}{50}.$$

64. Given $x = \frac{12 + 8x^{\frac{1}{2}}}{x - 5}$, to find the values of x .

$$(18. \text{ Cor. 1.}) \quad x^2 - 5x = 12 + 8x^{\frac{1}{2}},$$

$$\text{or } x^2 - 4x = 12 + 8x^{\frac{1}{2}} + x;$$

$$\text{and completing the square, } x^2 - 4x + 4 = 16 + 8x^{\frac{1}{2}} + x;$$

extracting the root, $x - 2 = \pm (4 + x^{\frac{1}{2}})$, and first taking the positive value;

$$\text{then, by transposition, } x - x^{\frac{1}{2}} = 6;$$

$$\text{completing the square, } x - x^{\frac{1}{2}} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4};$$

$$\text{extracting the root, } x^{\frac{1}{2}} - \frac{1}{2} = \pm \frac{5}{2};$$

$$\therefore x^{\frac{1}{2}} = 3, \text{ or } -2,$$

$$\text{and } x = 9, \text{ or } 4.$$

$$\text{But if the negative value be used, } x - 2 = -4 - x^{\frac{1}{2}};$$

$$\therefore \text{ by transposition, } x + x^{\frac{1}{2}} = -2;$$

$$\text{and completing the square, } x + x^{\frac{1}{2}} + \frac{1}{4} = \frac{1}{4} - 2 = \frac{-7}{4};$$

extracting the root, $x^{\frac{1}{2}} + \frac{1}{2} = \frac{\pm \sqrt{(-7)}}{2}$;

$$\therefore x^{\frac{1}{2}} = \frac{-1 \pm \sqrt{(-7)}}{2},$$

$$\text{and } x = \frac{-3 \mp \sqrt{(-7)}}{2}.$$

65. Given $\frac{49x^2}{4} + \frac{48}{x^2} - 49 = 9 + \frac{6}{x}$, to find the values of x .

Adding $\frac{1}{x^2}$ to each side, in order to complete the square,

$$\text{then } \frac{49x^2}{4} - 49 + \frac{49}{x^2} = 9 + \frac{6}{x} + \frac{1}{x^2};$$

$$\text{extracting the root, } \frac{7x}{2} - \frac{7}{x} = \pm \left(3 + \frac{1}{x}\right),$$

and first taking the positive value;

$$\therefore \text{ by transposition, } \frac{7x}{2} - 3 = \frac{8}{x};$$

$$\text{and therefore, } x^2 - \frac{6x}{7} = \frac{16}{7};$$

completing the square,

$$x^2 - \frac{6x}{7} + \frac{9}{49} = \frac{16}{7} + \frac{9}{49} = \frac{121}{49};$$

$$\text{extracting the root, } x - \frac{3}{7} = \pm \frac{11}{7},$$

$$\text{and } x = 2, \text{ or } -\frac{8}{7}.$$

But if the negative value be used, $\frac{7x}{2} - \frac{7}{x} = -3 - \frac{1}{x}$;

$$\therefore \text{ by transposition, } \frac{7x}{2} + 3 = \frac{6}{x},$$

$$\text{and } x^2 + \frac{6x}{7} = \frac{12}{7};$$

$\therefore$ completing the square,

$$x^2 + \frac{6x}{7} + \frac{9}{49} = \frac{12}{7} + \frac{9}{49} = \frac{93}{49};$$

$$\text{and extracting the root, } x + \frac{3}{7} = \frac{\pm \sqrt{(93)}}{7};$$

$$\therefore x = \frac{-3 \pm \sqrt{(93)}}{7}.$$

66. Given $\frac{x^4}{2} + \frac{17x^3}{4} - 17x = 8$, to find the values of x .

$$\text{Multiplying by 2, } x^4 + \frac{17x^3}{2} - 34x = 16,$$

$$\therefore \text{ by transposition, } x^4 + \frac{17x^3}{2} = 34x + 16;$$

and completing the square,

$$x^4 + \frac{17x^3}{2} + \left(\frac{17x}{4}\right)^2 = \left(\frac{17x}{4}\right)^2 + 34x + 16;$$

$$\text{extracting the root, } x^2 + \frac{17x}{4} = \pm \left(\frac{17x}{4} + 4\right);$$

first, let the positive value be taken;

$$\text{then, by transposition, } x^2 = 4,$$

$$\text{and } x = \pm 2.$$

But if the negative value be taken,

$$x^2 + \frac{17x}{4} = -\frac{17x}{4} - 4;$$

and by transposition, $x^3 + \frac{17x}{2} = -4$;

∴ completing the square,

$$x^3 + \frac{17x}{2} + \left(\frac{17}{4}\right)^2 = \frac{289}{16} - 4 = \frac{225}{16};$$

and extracting the root, $x + \frac{17}{4} = \pm \frac{15}{4}$;

$$\therefore x = -8, \text{ or } -\frac{1}{2}.$$

67. Given $27x^3 - \frac{841}{3x^3} + \frac{17}{3} = \frac{232}{3x} - \frac{1}{3x^3} + 5$, to find the values of x .

Multiplying every term by 3,

$$81x^3 - \frac{841}{x^3} + 17 = \frac{232}{x} - \frac{1}{x^3} + 15;$$

$$\therefore \text{by transposition, } 81x^3 + 17 + \frac{1}{x^3} = \frac{841}{x^3} + \frac{232}{x} + 15.$$

Adding unity to each side, in order to complete the square;

$$\therefore 81x^3 + 18 + \frac{1}{x^3} = \frac{841}{x^3} + \frac{232}{x} + 16;$$

and extracting the root, $9x + \frac{1}{x} = \pm \left(\frac{29}{x} + 4\right).$

Let the positive value be taken;

$$\text{then, by transposition, } 9x - 4 = \frac{28}{x};$$

$$\text{and therefore } 9x^2 - 4x = 28;$$

$$\text{completing the square, } 9x^2 - 4x + \frac{4}{9} = \frac{4}{9} + 28 = \frac{256}{9};$$

$$\text{extracting the root, } 3x - \frac{2}{3} = \pm \frac{16}{3};$$

$$\therefore 3x = 6, \text{ or } -\frac{14}{3},$$

$$\text{and } x = 2, \text{ or } -\frac{14}{9}.$$

But if the negative value be taken, $9x^2 + 4x = -30$;

and completing the square,

$$9x^2 + 4x + \frac{4}{9} = \frac{4}{9} - 30 = \frac{-266}{9};$$

$$\text{extracting the root, } 3x + \frac{2}{3} = \frac{\pm \sqrt{(-266)}}{3};$$

$$\therefore 3x = \frac{-2 \pm \sqrt{(-266)}}{3};$$

$$\text{and } x = \frac{-2 \pm \sqrt{(-266)}}{9}.$$

68. Given $\left(x^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}} + \left(a^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}} = \frac{x^2}{a}$, to find the values of x .

$$\text{By transposition, } \left(x^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}} - \frac{x^2}{a} = -\left(a^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}};$$

and squaring both sides,

$$x^2 - \frac{a^4}{x^2} + \frac{x^4}{a^2} - \frac{2x^2}{a} \cdot \left(x^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}} = a^2 - \frac{a^4}{x^2};$$

$$\text{and by transposition, } x^2 - \frac{2x^2}{a} \cdot \left(x^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}} + \frac{x^4}{a^2} - a^2 = 0,$$

$$\text{or } x^2 - \frac{2x}{a} \cdot (x^2 - a^4)^{\frac{1}{2}} + \frac{x^4 - a^4}{a^2} = 0;$$

$$\text{extracting the root, } x - \frac{\sqrt{(x^4 - a^4)}}{a} = 0;$$

$$\therefore \text{ by transposition, } x = \frac{\sqrt{(x^4 - a^4)}}{a},$$

$$\text{and } ax = \sqrt{(x^4 - a^4)}$$

$$\text{Squaring both sides, } a^2 x^2 = x^4 - a^4;$$

$$\therefore \text{ by transposition, } x^4 - a^2 x^2 = a^4;$$

and completing the square,

$$x^4 - a^2 x^2 + \frac{a^4}{4} = a^4 + \frac{a^4}{4} = \frac{5a^4}{4};$$

$$\text{extracting the root, } x^2 - \frac{a^2}{2} = \frac{\pm \sqrt{5} \cdot a^2}{2};$$

$$\therefore x^2 = \frac{1 \pm \sqrt{5}}{2} \cdot a^2,$$

$$\text{and } x = \pm a \cdot \sqrt{\frac{1 \pm \sqrt{5}}{2}}.$$

SECTION V.

Solution of Adfected Quadratics, involving two unknown Quantities.

(29.) If the equations involve two unknown quantities, they may, by the preceding rules, be reduced to one containing only one of the unknown quantities, the values of which may be found by Art. 28; whence, by substitution, the values of the other may also be determined. In many cases, however, it may be convenient to solve the equations first, considering one of the quantities as known; when the rules for exterminating unknown quantities (23) may be more easily applied.

EXAMPLES.

1. Given $x - y = 15$ }
and $\frac{xy}{2} = y^2$ } , to find the values of x and y .

From the second equation, $x = 2y^2$;

Substituting this in the first, $2y^2 - y = 15$;

$$\therefore y^2 - \frac{1}{2}y = \frac{15}{2};$$

and completing the square,

$$y^2 - \frac{1}{2}y + \frac{1}{16} = \frac{15}{2} + \frac{1}{16} = \frac{121}{16};$$

$$\therefore \text{ extracting the root, } y - \frac{1}{4} = \pm \frac{11}{4},$$

$$\text{and } y = 3, \text{ or } -\frac{5}{2};$$

$$\text{whence } x = 2y^2 = 18, \text{ or } \frac{25}{2}.$$

$$2. \quad \text{Given } \left. \begin{array}{l} \frac{10x+y}{xy} = 3 \\ \text{and } 9y - 9x = 18 \end{array} \right\}, \text{ to find the values of } x \text{ and } y.$$

$$\text{From the second equation, } y - x = 2;$$

$$\text{and therefore } y = x + 2;$$

$$\text{but from the first, } 10x + y = 3xy.$$

Substituting in this the value of y found above,

$$10x + x + 2 = 3x \cdot (x + 2),$$

$$\text{or } 11x + 2 = 3x^2 + 6x;$$

$$\therefore \text{ by transposition, } 3x^2 - 5x = 2,$$

$$\text{and (18) } x^2 - \frac{5}{3}x = \frac{2}{3};$$

$\therefore$ completing the square,

$$x^2 - \frac{5}{3}x + \frac{25}{36} = \frac{25}{36} + \frac{2}{3} = \frac{49}{36};$$

$$\therefore \text{ extracting the root, } x - \frac{5}{6} = \pm \frac{7}{6},$$

$$\text{and } x = 2, \text{ or } -\frac{1}{3};$$

$$\text{whence } y = x + 2 = 4, \text{ or } \frac{5}{3}.$$

$$3. \quad \text{Given } \left. \begin{array}{l} x + y : x - y :: 13 : 5 \\ \text{and } y^2 + x = 25 \end{array} \right\} \text{ to find the values of } x \text{ and } y.$$

$$(Alg. 112.) \quad 2x : 2y :: 18 : 8 ;$$

$$\therefore (Alg. 184.) \quad x : y :: 9 : 4 ;$$

$$\therefore (21) \quad 4x = 9y, \text{ and } x = \frac{9y}{4}.$$

Substituting this value in the second equation,

$$y^2 + \frac{9y}{4} = 25 ;$$

completing the square,

$$y^2 + \frac{9y}{4} + \frac{81}{64} = \frac{81}{64} + 25 = \frac{1681}{64} ;$$

$$\therefore \text{ extracting the root, } y + \frac{9}{8} = \pm \frac{41}{8} ;$$

$$\text{whence } y = 4, \text{ or } -\frac{25}{4} ;$$

$$\therefore x = \frac{9y}{4} = 9, \text{ or } -\frac{225}{16}.$$

4. Given $4xy = 96 - x^2y^2$ } , to find the values of x and y .
and $x + y = 6$

From the first equation, $x^2y^2 + 4xy = 96$;

$\therefore$ completing the square, $x^2y^2 + 4xy + 4 = 100$;

and extracting the root, $xy + 2 = \pm 10$;

$$\therefore xy = 8, \text{ or } -12.$$

Now, squaring the second equation,

$$x^2 + 2xy + y^2 = 36 ;$$

$$\text{but } 4xy = 32, \text{ or } -48 ;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 4, \text{ or } 84 ;$$

whence, extracting the root, $x - y = \pm 2$ or $\pm \sqrt{84}$;

$$\text{but } x + y = 6 ;$$

$$\therefore \text{ by addition, } 2x = 8, \text{ or } 4, \text{ or } 6 \pm 2\sqrt{21} ;$$

whence $x = 4$, or 2 , or $3 \pm \sqrt{21}$;
 and by subtraction, $2y = 4$, or 8 , or $6 \mp 2\sqrt{21}$;
 $\therefore y = 2$, or 4 , or $3 \mp \sqrt{21}$.

5. Given $x^n + y^n = 2a^n$ }
 and $xy = c^2$ } to find the values of x and y .

From the second equation, $y = \frac{c^2}{x}$;

and substituting this value in the first equation,

$$x^n + \frac{c^{2n}}{x^n} = 2a^n;$$

$$\therefore x^{2n} + c^{2n} = 2a^n x^n;$$

by transposition, $x^{2n} - 2a^n x^n = -c^{2n}$;

completing the square, $x^{2n} - 2a^n x^n + a^{2n} = a^{2n} - c^{2n}$;

$\therefore$ extracting the square root, $x^n - a^n = \pm \sqrt{a^{2n} - c^{2n}}$,

and $x^n = a^n \pm \sqrt{a^{2n} - c^{2n}}$;

$$\therefore x = (a^n \pm \sqrt{a^{2n} - c^{2n}})^{\frac{1}{n}},$$

$$\text{and } y = \frac{c^2}{x} = \frac{c^2}{\{a^n \pm \sqrt{a^{2n} - c^{2n}}\}^{\frac{1}{n}}}.$$

6. Given $x^3 + x + y = 18 - y^3$ }
 and $xy = 6$ } to find the values of x
 and y .

By transposition, $x^3 + y^3 + x + y = 18$;

and from the second equation, $2xy = 12$;

$$\therefore \text{ by addition, } x^3 + 2xy + y^3 + x + y = 30;$$

and completing the square,

$$(x + y)^3 + (x + y) + \frac{1}{4} = 30 + \frac{1}{4} = \frac{121}{4};$$

$$\therefore \text{ extracting the root, } x + y + \frac{1}{2} = \pm \frac{11}{2},$$

$$\text{and } x + y = 5, \text{ or } -6;$$

$$\text{whence, from the first equation, } x^2 + y^2 = 13, \text{ or } 24;$$

$$\text{but } 2xy = 12;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 1, \text{ or } 12;$$

$$\therefore x - y = \pm 1, \text{ or } \pm 2\sqrt{3}.$$

$$\text{Now } x + y = 5, \text{ or } -6;$$

$$\therefore \text{ by addition, } 2x = 6, \text{ or } 4, \text{ or } -6 \pm 2\sqrt{3};$$

$$\therefore x = 3, \text{ or } 2, \text{ or } -3 \pm \sqrt{3};$$

$$\text{and by subtraction, } 2y = 4, \text{ or } 6, \text{ or } -6 \mp 2\sqrt{3};$$

$$\therefore y = 2, \text{ or } 3, \text{ or } -3 \mp \sqrt{3}.$$

7. Given $x^2 + 2xy + y^2 + 2x = 120 - 2y$ }
 and $xy - y^2 = 8$ } , to find the
 values of x and y .

$$\text{By transposition, } (x + y)^2 + 2 \cdot (x + y) = 120;$$

$$\therefore \text{ completing the square, } (x + y)^2 + 2 \cdot (x + y) + 1 = 121;$$

$$\therefore \text{ extracting the root, } (x + y) + 1 = \pm 11,$$

$$\text{and } x + y = 10, \text{ or } -12; \text{ and first let } x + y = 10;$$

$$\text{from the second equation, } x - y = \frac{8}{y};$$

$$\therefore \text{ by subtraction, } 2y = 10 - \frac{8}{y};$$

$$\therefore y^2 = 5y - 4;$$

$$\text{and by transposition, } y^2 - 5y = -4;$$

$$\text{completing the square, } y^2 - 5y + \frac{25}{4} = \frac{25}{4} - 4 = \frac{9}{4};$$

$$\text{and extracting the root, } y - \frac{5}{2} = \pm \frac{3}{2};$$

$$\therefore y = 4, \text{ or } 1;$$

$$\text{and } x = 10 - y = 6, \text{ or } 9.$$

$$\text{But if } x + y = -12,$$

$$\text{and } x - y = \frac{8}{y};$$

$$\text{then } 2y = -12 - \frac{8}{y},$$

$$\text{and } y^2 + 6y = -4;$$

$$\therefore \text{ completing the square, } y^2 + 6y + 9 = 9 - 4 = 5;$$

$$\text{extracting the root, } y + 3 = \pm \sqrt{5};$$

$$\therefore y = -3 \pm \sqrt{5},$$

$$\text{and } x = -12 - y = -9 \mp \sqrt{5}.$$

8. Given $x^2 + y^2 - x - y = 78$ } to find the values of x
 and $xy + x + y = 39$ } and y .

$$\text{Since } x^2 + y^2 - (x + y) = 78;$$

$$\text{and from the second, } 2xy + 2 \cdot (x + y) = 78;$$

$$\therefore \text{ by addition, } x^2 + 2xy + y^2 + x + y = 156;$$

and completing the square,

$$(x + y)^2 + (x + y) + \frac{1}{4} = 156 + \frac{1}{4} = \frac{625}{4};$$

$$\text{extracting the root, } x + y + \frac{1}{2} = \pm \frac{25}{2};$$

$$\therefore x + y = 12, \text{ or } -13; \text{ supposing the former;}$$

$$\text{then } xy = 39 - (x + y) = 39 - 12 = 27,$$

$$\text{and } x^2 + y^2 = 78 + (x + y) = 78 + 12 = 90;$$

$$\text{but } 2xy = \quad \quad \quad 54;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 36;$$

and extracting the root, $x - y = \pm 6$;

$$\text{but } x + y = 12;$$

$$\therefore \text{ by addition, } 2x = 18, \text{ or } 6,$$

$$\text{and } x = 9, \text{ or } 3;$$

$$\text{and by subtraction, } 2y = 6, \text{ or } 18,$$

$$\text{and } y = 3, \text{ or } 9.$$

But if $x + y = -13$,

$$\text{then } xy = 39 + 13 = 52,$$

$$\text{and } x^2 + y^2 = 78 - 13 = 65;$$

$$\text{but } 2xy = 104;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = -39;$$

and extracting the root, $x - y = \pm \sqrt{-39}$;

$$\text{but } x + y = -13;$$

$$\therefore \text{ by addition, } 2x = -13 \pm \sqrt{-39};$$

$$\text{and } x = \frac{-13 \pm \sqrt{-39}}{2};$$

$$\text{but by subtraction, } 2y = -13 \mp \sqrt{-39};$$

$$\therefore y = \frac{-13 \mp \sqrt{-39}}{2}.$$

9. Given $x^2y^4 - 7xy^2 - 945 = 765$ } to find the values of
and $xy - y = 12$ } x and y .

From the first equation, by transposition,

$$x^2y^4 - 7xy^2 = 1710;$$

and completing the square,

$$x^2y^4 - 7xy^2 + \frac{49}{4} = 1710 + \frac{49}{4} = \frac{6889}{4};$$

$$\text{extracting the root, } xy^2 - \frac{7}{2} = \pm \frac{83}{2};$$

$$\therefore xy^2 = 45, \text{ or } -38.$$

Multiplying the second equation by y , $xy^2 - y^3 = 12y$.

Substituting in this the value of xy^2 found above,

$$45 - y^3 = 12y, \text{ in one case ;}$$

$$\text{and by transposition, } y^3 + 12y = 45 ;$$

$$\text{completing the square, } y^3 + 12y + 36 = 45 + 36 = 81 ;$$

$$\text{extracting the root, } y + 6 = \pm 9,$$

$$\text{and } y = 3, \text{ or } -15 ;$$

$$\text{whence } x = \frac{45}{y^2} = 5, \text{ or } \frac{1}{5} ;$$

$$\text{and in the other case, } -38 - y^3 = 12y ;$$

$$\text{whence } y^3 + 12y = -38 ;$$

$$\text{completing the square, } y^3 + 12y + 36 = 36 - 38 = -2 ;$$

$$\text{extracting the root, } y + 6 = \pm \sqrt{-2} ;$$

$$\therefore y = -6 \pm \sqrt{-2},$$

$$\text{and } x = \frac{-38}{y^2} = \frac{-38}{34 \mp 12\sqrt{-2}} = \frac{-19}{17 \mp 6\sqrt{-2}}.$$

10. Given $x - 2\sqrt{xy} + y - \sqrt{x} + \sqrt{y} = 0$ } to find the
and $\sqrt{x} + \sqrt{y} = 5$ }
values of x and y .

Completing the square in the first equation,

$$(\sqrt{x} - \sqrt{y})^2 - (\sqrt{x} - \sqrt{y}) + \frac{1}{4} = \frac{1}{4} ;$$

$$\text{and extracting the root, } \sqrt{x} - \sqrt{y} - \frac{1}{2} = \pm \frac{1}{2} ;$$

$$\sqrt{x} - \sqrt{y} = 1, \text{ or } 0 ;$$

$$\text{but from the second equation, } \sqrt{x} + \sqrt{y} = 5 ;$$

$$\text{by addition, } 2\sqrt{x} = 6, \text{ or } 5,$$

$$\text{and } \sqrt{x} = 3, \text{ or } \frac{5}{2} ;$$

$$\therefore x = 9, \text{ or } \frac{25}{4} ;$$

but by subtraction, $2\sqrt{y} = 4$, or 5 ,

$$\text{or } \sqrt{y} = 2, \text{ or } \frac{5}{2};$$

$$\text{and } y = 4, \text{ or } \frac{25}{4}.$$

11. Given $\left. \begin{array}{l} \frac{x^2}{y^2} + \frac{4x}{y} = \frac{85}{9} \\ \text{and } x - y = 2 \end{array} \right\}$, to find the values of x and y .

Completing the square in the first equation,

$$\frac{x^2}{y^2} + \frac{4x}{y} + 4 = \frac{85}{9} + 4 = \frac{121}{9};$$

$$\text{and extracting the root, } \frac{x}{y} + 2 = \pm \frac{11}{3};$$

$$\therefore \frac{x}{y} = \frac{5}{3}, \text{ or } -\frac{17}{3},$$

$$\text{and } x = \frac{5y}{3} \text{ or } -\frac{17y}{3}; \text{ supposing the former;}$$

$$\text{then, from the second equation, } \frac{5y}{3} - y = 2,$$

$$\text{or } 2y = 6,$$

$$\text{and } y = 3;$$

$$\therefore x = \frac{5y}{3} = 5.$$

$$\text{And if the second value be taken, } -\frac{17y}{3} - y = 2,$$

$$\text{or } -20y = 6;$$

$$\therefore y = \frac{-3}{10};$$

$$\text{and } x = -\frac{17y}{3} = \frac{17}{10}.$$

$$12. \quad \left. \begin{array}{l} \text{Given } \sqrt{\left(\frac{3x}{x+y}\right)} + \sqrt{\left(\frac{x+y}{3x}\right)} = 2 \\ \text{and } xy - (x+y) = 54 \end{array} \right\}, \quad \begin{array}{l} \text{to find the} \\ \text{values of} \\ x \text{ and } y. \end{array}$$

(18. Cor. 1.) and by transposition,

$$3x - 2\sqrt{(3x)} \cdot \sqrt{(x+y)} + (x+y) = 0;$$

$$\therefore \text{ extracting the root, } \sqrt{(3x)} - \sqrt{(x+y)} = 0;$$

$$\text{by transposition, } \sqrt{(3x)} = \sqrt{(x+y)};$$

$$\text{and squaring both sides, } 3x = x + y,$$

$$\text{and } \therefore 2x = y;$$

substituting this value in the second equation, $2x^2 - 3x = 54$,

$$\text{or } x^2 - \frac{3}{2}x = 27;$$

whence completing the square,

$$x^2 - \frac{3}{2}x + \frac{9}{16} = 27 + \frac{9}{16} = \frac{441}{16};$$

$$\text{extracting the root, } x - \frac{3}{4} = \pm \frac{21}{4},$$

$$\text{and } x = 6, \text{ or } -\frac{9}{2};$$

$$\text{whence } y = 2x = 12, \text{ or } -9.$$

$$13. \quad \left. \begin{array}{l} \text{Given } x^4 - 2x^2y + y^2 = 49, \\ \text{and } x^4 - 2x^2y^3 + y^4 - x^2 + y^2 = 20 \end{array} \right\}, \quad \begin{array}{l} \text{to find the} \\ \text{values of } x \text{ and } y. \end{array}$$

Completing the square in the second equation,

$$(x^2 - y^2)^2 - (x^2 - y^2) + \frac{1}{4} = 20 + \frac{1}{4} = \frac{81}{4};$$

$$\text{extracting the root, } x^2 - y^2 - \frac{1}{2} = \pm \frac{9}{2};$$

$$\therefore x^2 - y^2 = 5, \text{ or } -4;$$

$$\left. \begin{array}{l} \text{but extracting the root} \\ \text{of the first equation,} \end{array} \right\} x^2 - y = \pm 7;$$

$$\therefore \text{ by subtraction, } y^2 - y = 2, \text{ or } -12, \text{ or } 11, \text{ or } -3.$$

• Taking the first value, and completing the square,

$$y^2 - y + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4};$$

$$\text{extracting the root, } y - \frac{1}{2} = \pm \frac{3}{2},$$

$$\text{and } y = 2, \text{ or } -1;$$

$$\therefore x = \pm \sqrt{7 + y} = \pm 3, \text{ or } \pm \sqrt{6}.$$

$$\text{Taking the second value, } y^2 - y = -12;$$

$$\text{completing the square, } y^2 - y + \frac{1}{4} = \frac{1}{4} - 12 = \frac{-47}{4};$$

$$\text{extracting the root, } y - \frac{1}{2} = \frac{\pm \sqrt{(-47)}}{2},$$

$$\text{and } y = \frac{1 \pm \sqrt{(-47)}}{2};$$

$$\begin{aligned} \text{whence } x &= \pm \sqrt{y - 7} = \pm \sqrt{\left(\frac{1 \pm \sqrt{(-47)}}{2} - 7\right)} \\ &= \pm \sqrt{\left(\frac{-13 \pm \sqrt{(-47)}}{2}\right)}. \end{aligned}$$

$$\text{Taking the third value, } y^2 - y = 11;$$

$$\text{completing the square, } y^2 - y + \frac{1}{4} = 11 + \frac{1}{4} = \frac{45}{4};$$

$$\text{extracting the root, } y - \frac{1}{2} = \frac{\pm \sqrt{45}}{2};$$

$$\therefore y = \frac{1 \pm 3\sqrt{5}}{2};$$

and

$$x = \pm \sqrt{7 + y} = \pm \sqrt{\left(7 + \frac{1 \pm 3\sqrt{5}}{2}\right)} = \pm \sqrt{\left(\frac{15 \pm 3\sqrt{5}}{2}\right)}.$$

Taking the fourth value, $y^2 - y = -3$;

completing the square, $y^2 - y + \frac{1}{4} = \frac{1}{4} - 3 = \frac{-11}{4}$;

extracting the root, $y - \frac{1}{2} = \frac{\pm \sqrt{(-11)}}{2}$,

and $y = \frac{1 \pm \sqrt{(-11)}}{2}$;

$\therefore x = \pm \sqrt{(y - 7)}$

$$= \pm \sqrt{\left(\frac{1 \pm \sqrt{(-11)}}{2} - 7\right)} = \pm \sqrt{\left(\frac{-13 \pm \sqrt{(-11)}}{2}\right)}.$$

14. Given $xy + xy^2 = 12$ }
and $x^2 + xy^2 = 18$ }, to find the values of x and y .

From the first equation, $x = \frac{12}{y \cdot (1 + y)}$,

and from the second, $x = \frac{18}{1 + y^2}$;

whence $\frac{12}{y \cdot (1 + y)} = \frac{18}{1 + y^2}$, and dividing by $\frac{6}{1 + y}$,

$$\frac{2}{y} = \frac{3}{y^2 - y + 1};$$

$$\therefore 2y^2 - 2y + 2 = 3y,$$

and by transposition, $2y^2 - 5y = -2$,

$$\text{or } y^2 - \frac{5}{2} \cdot y = -1;$$

completing the square, $y^2 - \frac{5}{2} \cdot y + \frac{25}{16} = \frac{25}{16} - 1 = \frac{9}{16}$;

extracting the root, $y - \frac{5}{4} = \pm \frac{3}{4}$;

$$\therefore y = 2, \text{ or } \frac{1}{2};$$

hence $x = \frac{12}{y + y^2} = 2, \text{ or } 16.$

15. Given $x - x^{\frac{1}{2}} = 3 - y$ } to find the values of x and y .
 and $4 - x = y - y^{\frac{1}{2}}$ }

Adding the two equations together, $4 - x^{\frac{1}{2}} = 3 - y^{\frac{1}{2}}$;

$\therefore$ by transposition, $y^{\frac{1}{2}} = x^{\frac{1}{2}} - 1$.

$$\text{and } y = (x^{\frac{1}{2}} - 1)^2.$$

Substituting this value in the first equation,

$$\therefore x - x^{\frac{1}{2}} = 3 - x + 2x^{\frac{1}{2}} - 1;$$

$\therefore$ by transposition, $2x - 3x^{\frac{1}{2}} = 2$,

$$\text{and } x - \frac{3}{2} \cdot x^{\frac{1}{2}} = 1;$$

completing the square, $x - \frac{3}{2}x^{\frac{1}{2}} + \frac{9}{16} = 1 + \frac{9}{16} = \frac{25}{16}$;

extracting the root, $x^{\frac{1}{2}} - \frac{3}{4} = \pm \frac{5}{4}$,

$$\text{and } x^{\frac{1}{2}} = 2, \text{ or } -\frac{1}{2},$$

$$x = 4, \text{ or } \frac{1}{4},$$

$$\text{and } y = (x^{\frac{1}{2}} - 1)^2 = 1, \text{ or } \frac{9}{4}.$$

16. Given $(x^2 + 1) \cdot y = xy + 126$ } to find the values
 and $(x^2 + 1) \cdot y = x^2y^2 - 744$ } of x and y .

Since quantities which are equal to the same, are equal to each other,

$$x^2y^2 - 744 = xy + 126;$$

$\therefore$ by transposition, $x^2y^2 - xy = 870$;

completing the square, $x^2y^2 - xy + \frac{1}{4} = 870 + \frac{1}{4} = \frac{3481}{4}$;

extracting the root, $xy - \frac{1}{2} = \pm \frac{59}{2}$,

and $xy = 30$, or -29 ; let the former value be taken,

then from the first equation $(x^2 + 1) \cdot \frac{30}{x} = 150$;

$$\therefore x^2 + 1 = \frac{156x}{30} = \frac{26x}{5};$$

and by transposition, $x^2 - \frac{26x}{5} = -1$;

completing the square, $x^2 - \frac{26}{5} \cdot x + \frac{169}{25} = \frac{169}{25} - 1 = \frac{144}{25}$;

extracting the root, $x - \frac{13}{5} = \pm \frac{12}{5}$;

whence $x = 5$, or $\frac{1}{5}$;

and $\therefore y = \frac{30}{x} = 6$, or 150 .

In the second case, $(x^2 + 1) \times -\frac{29}{x} = -29 + 126 = 97$,

and $x^2 + 1 = -\frac{97x}{29}$;

by transposition, $x^2 + \frac{97x}{29} = -1$;

completing the square, $x^2 + \frac{97}{29} \cdot x + \left(\frac{97}{58}\right)^2 = \frac{9409}{3364} - 1 = \frac{6045}{3364}$;

extracting the root, $x + \frac{97}{58} = \pm \frac{\sqrt{(6045)}}{58}$;

$$\therefore x = \frac{-97 \pm \sqrt{(6045)}}{58};$$

$$\therefore y = -\frac{29}{x} = \frac{1682}{97 \mp \sqrt{(6045)}}.$$

17. Given $x + y + \sqrt{(x + y)} = 12$ } to find the values
 and $x^2 + y^2 = 189$ } of x and y .

Completing the square in the first equation,

$$x + y + \sqrt{(x + y)} + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4};$$

extracting the root, $\sqrt{(x + y)} + \frac{1}{2} = \pm \frac{7}{2};$

$$\therefore \sqrt{(x + y)} = 3, \text{ or } -4,$$

$$\text{and } x + y = 9, \text{ or } 16;$$

$$\therefore x^2 + 3x^2y + 3xy^2 + y^2 = 729, \text{ or } 4096;$$

$$\text{but } x^2 + y^2 = 189;$$

$$\therefore \text{ by subtraction, } 3x^2y + 3xy^2 = 540, \text{ or } 3907;$$

$$\therefore (x + y) \cdot xy = 180, \text{ or } \frac{3907}{3};$$

$$\therefore 9xy = 180 \text{ in one case, and } 16xy = \frac{3907}{3} \text{ in the other,}$$

whence in the first case $xy = 20$.

$$\text{Now } x^2 + 2xy + y^2 = 81;$$

$$\text{but } \quad \quad \quad 4xy = 80;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 1;$$

$$\text{and extracting the root, } x - y = \pm 1;$$

$$\text{but } x + y = 9;$$

$$\therefore \text{ by addition, } 2x = 10, \text{ or } 8,$$

$$\text{and } x = 5, \text{ or } 4;$$

$$\text{but by subtraction, } 2y = 8, \text{ or } 10;$$

$$\text{and } y = 4, \text{ or } 5.$$

$$\text{Now in the second case, } 16xy = \frac{3907}{3};$$

$$\therefore 4xy = \frac{3907}{12};$$

and since $x^2 + 2xy + y^2 = 256$,

$$\text{and } 4xy = \frac{3907}{12};$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = -\frac{835}{12};$$

and extracting the root, $x - y = \pm \sqrt{\left(-\frac{835}{12}\right)}$,

$$\text{but } x + y = 16;$$

$$\therefore \text{ by addition, } 2x = 16 \pm \sqrt{\left(-\frac{835}{12}\right)},$$

$$\text{and } x = 8 \pm \frac{1}{4} \sqrt{\left(-\frac{835}{3}\right)};$$

$$\text{but by subtraction, } 2y = 16 \mp \sqrt{\left(-\frac{835}{12}\right)},$$

$$\text{and } y = 8 \mp \frac{1}{4} \sqrt{\left(-\frac{835}{3}\right)}.$$

18. Given $x^2 + y^2 + x - y = 132$ to find the values of
and $(x^2 + y^2) \cdot (x - y) = 1220$ x and y .

From the first equation, $x^2 + y^2 = 132 - (x - y)$;

$$\text{and from the second, } x^2 + y^2 = \frac{1220}{x - y};$$

$$\text{whence } 132 - (x - y) = \frac{1220}{x - y};$$

$$\text{and, } \therefore 132 \cdot (x - y) - (x - y)^2 = 1220;$$

$$\text{and (17. Cor. 1.) } (x - y)^2 - 132 \cdot (x - y) = -1220;$$

completing the square,

$$(x - y)^2 - 132 \cdot (x - y) + 66^2 = 4356 - 1220 = 3136;$$

$$\text{extracting the root, } x - y - 66 = \pm 56;$$

and $x - y = 10$, or 122 , supposing the former,

$$x^2 + y^2 = 122;$$

$$\therefore 2x^2 + 2y^2 = 244;$$

$$\text{but } x^2 - 2xy + y^2 = 100;$$

$\therefore$ by subtraction, $x^2 + 2xy + y^2 = 144$,

and extracting the root, $x + y = \pm 12$;

$$\text{but } x - y = 10;$$

$\therefore$ by addition, $2x = 22$, or -2 ,

and $x = 11$, or -1 ;

by subtraction, $2y = 2$, or -22 ,

and $y = 1$, or -11 .

But if $x - y = 122$,

then $x^2 + y^2 = 10$,

$$\text{and } x^2 - 2xy + y^2 = 122^2;$$

$$\text{but } 2x^2 + 2y^2 = 20;$$

$\therefore$ by subtraction, $x^2 + 2xy + y^2 = 20 - 122^2$,

and extracting the root, $x + y = \pm \sqrt{20 - 122^2}$

$$\text{but } x - y = 122;$$

$\therefore$ by addition, $2x = 122 \pm 2\sqrt{5 - 61^2}$

$$\text{and } x = 61 \pm \sqrt{-3716};$$

and by subtraction, $2y = -122 \pm 2\sqrt{-3716}$;

$$\therefore y = -61 \pm \sqrt{-3716}.$$

19. Given $\left. \begin{array}{l} x^{\frac{1}{2}}y^{\frac{1}{2}} = 2y^{\frac{1}{2}} \\ \text{and } 8x^{\frac{1}{2}} - y^{\frac{1}{2}} = 14 \end{array} \right\}$, to find the values of x and y .

From the first equation, $x^{\frac{1}{2}} = 2y^{\frac{1}{2}}$;

$$\text{and, } \therefore \frac{1}{2}x^{\frac{1}{2}} = y^{\frac{1}{2}};$$

substituting this value in the second equation,

$$8x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} = 14,$$

$$\text{and } 16x^{\frac{1}{2}} - x^{\frac{1}{2}} = 28;$$

$$\text{or (17. Cor. 1.) } x^{\frac{1}{2}} - 16x^{\frac{1}{2}} = -28;$$

$$\text{completing the square, } x^{\frac{1}{2}} - 16x^{\frac{1}{2}} + 64 = 64 - 28 = 36;$$

$$\text{and extracting the root, } x^{\frac{1}{2}} - 8 = \pm 6;$$

$$\therefore x^{\frac{1}{2}} = 14, \text{ or } 2,$$

$$\text{and } x = \overline{14}^2 \text{ or } 8;$$

$$\text{but } y^{\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}} = 98, \text{ or } 2;$$

$$\therefore y = \overline{98}^2, \text{ or } 4.$$

✕ 20. Given $\left. \begin{array}{l} x^{\frac{1}{2}} + y^{\frac{1}{2}} = 3x \\ \text{and } x^{\frac{1}{2}} + y^{\frac{1}{2}} = x \end{array} \right\}$, to find the values of x and y .

$$\text{Squaring the second equation, } \therefore x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{1}{2}} = x^2;$$

$$\text{but } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 3x;$$

$$\therefore \text{ by subtraction, } x - x^{\frac{1}{2}} + 2x^{\frac{1}{2}}y^{\frac{1}{2}} = x^2 - 3x;$$

$$\text{but from the second equation, } y^{\frac{1}{2}} = x - x^{\frac{1}{2}}.$$

Let this value be substituted in the preceding equation,

$$\text{then } x - x^{\frac{1}{2}} + 2x^{\frac{1}{2}} - 2x = x^2 - 3x;$$

$$\text{and by transposition, } 2x = x^2 - x^{\frac{1}{2}};$$

$$\text{and dividing by } x, 2 = x - x^{\frac{1}{2}};$$

$$\text{completing the square, } x - x^{\frac{1}{2}} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4};$$

$$\text{extracting the root, } x^{\frac{1}{2}} - \frac{1}{2} = \pm \frac{3}{2},$$

$$\text{and } x^{\frac{1}{2}} = 2, \text{ or } -1;$$

$$\therefore x = 4, \text{ or } 1,$$

$$\text{and } y\frac{1}{2} = x - x\frac{1}{2} = 2;$$

$$\therefore y = 8.$$

$$21. \text{ Given } x + x\frac{1}{2} = \frac{y^2 + y + 2}{x\frac{1}{2}} + 4 \left. \vphantom{\frac{y^2 + y + 2}{x\frac{1}{2}}} \right\} \text{ to find the values of } x \text{ and } y.$$

$$\text{and } y + xy = y^2 + 4y$$

$$\text{From the first equation, } x\frac{1}{2} + x - 4x\frac{1}{2} = y^2 + y + 2,$$

$$\text{and from the second, } x = y + 3.$$

Substituting this value for x in the former,

$$x\frac{1}{2} + y + 3 - 4x\frac{1}{2} = y^2 + y + 2,$$

$$\text{and by transposition, } x\frac{1}{2} - 4x\frac{1}{2} = y^2 - 1.$$

$$\text{But since } x = y + 3; \therefore x - 4 = y - 1,$$

by which equation let the preceding one be divided;

$$\therefore x\frac{1}{2} = y + 1;$$

$$\text{squaring both sides of this equation, } x = y^2 + 2y + 1.$$

Equating therefore the two values of x ,

$$y^2 + 2y + 1 = y + 3;$$

$$\therefore \text{ by transposition, } y^2 + y = 2;$$

$$\text{completing the square, } y^2 + y + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4};$$

$$\text{extracting the root, } y + \frac{1}{2} = \pm \frac{3}{2};$$

$$\therefore y = 1, \text{ or } -2;$$

$$\text{whence } x\frac{1}{2} = y + 1 = 2, \text{ or } -1,$$

$$\text{and therefore } x = 4, \text{ or } 1.$$

$$22. \text{ Given } \frac{x^2}{y^2} + \frac{y}{x} + \frac{x}{y} = 6\frac{3}{4} - \frac{y^2}{x^2} \left. \vphantom{\frac{y^2}{x^2}} \right\} \text{ to find the values of } x \text{ and } y.$$

$$\text{and } x - y = 2$$

$$\text{By transposition, } \frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x} = \frac{27}{4};$$

$$\therefore \text{adding 2 to each side, } \frac{x^2}{y^2} + 2 + \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x} = \frac{35}{4};$$

$$\text{completing the square, } \left(\frac{x}{y} + \frac{y}{x}\right)^2 + \left(\frac{x}{y} + \frac{y}{x}\right) + \frac{1}{4} = \frac{36}{4};$$

$$\text{extracting the root, } \frac{x}{y} + \frac{y}{x} + \frac{1}{2} = \pm \frac{6}{2};$$

$$\therefore \frac{x}{y} + \frac{y}{x} = \frac{5}{2}, \text{ or } -\frac{7}{2};$$

$$\therefore x^2 + y^2 = \frac{5xy}{2}, \text{ or } -\frac{7xy}{2};$$

now from the second equation squared,

$$x^2 + y^2 = 2xy + 4;$$

$$\therefore 2xy + 4 = \frac{5xy}{2}, \text{ or } -\frac{7xy}{2};$$

whence by multiplication and transposition,

$$xy = 8, \text{ or } -\frac{8}{11};$$

$$\text{and since } x^2 - 2xy + y^2 = 4,$$

$$\text{and } 4xy = 32, \text{ or } -\frac{32}{11};$$

$$\therefore \text{by addition, } x^2 + 2xy + y^2 = 36, \text{ or } \frac{12}{11};$$

$$\text{and extracting the root, } x + y = \pm 6, \text{ or } \pm \frac{2\sqrt{3}}{\sqrt{11}};$$

$$\text{but } x - y = 2;$$

$$\therefore \text{by addition, } 2x = 8, \text{ or } -4, \text{ or } 2 \pm \frac{2\sqrt{3}}{\sqrt{11}},$$

$$\text{and } x = 4, \text{ or } -2, \text{ or } 1 \pm \frac{\sqrt{3}}{\sqrt{11}};$$

$$\therefore \text{ by subtraction, } 2y = 4, \text{ or } -8, \text{ or } -2 \pm \frac{2\sqrt{3}}{\sqrt{11}},$$

$$\therefore y = 2, \text{ or } -4, \text{ or } -1 \pm \frac{\sqrt{3}}{\sqrt{11}}.$$

$$\left. \begin{array}{l} 32. \text{ Given } 2x + y = 26 - 7\sqrt{2x + y + 4} \\ \text{and } \frac{2x + \sqrt{y}}{2x - \sqrt{y}} = \frac{16}{15} + \frac{2x - \sqrt{y}}{2x + \sqrt{y}} \end{array} \right\} \begin{array}{l} \text{to find the} \\ \text{values of } x \\ \text{and } y. \end{array}$$

Adding 4 to each side of the first equation,

$$2x + y + 4 = 30 - 7\sqrt{2x + y + 4};$$

$$\therefore \text{ by transposition, } 2x + y + 4 + 7\sqrt{2x + y + 4} = 30;$$

completing the square,

$$(2x + y + 4) + 7\sqrt{2x + y + 4} + \frac{49}{4} = 30 + \frac{49}{4} = \frac{169}{4};$$

$$\text{extracting the root, } \sqrt{2x + y + 4} + \frac{7}{2} = \pm \frac{13}{2};$$

$$\therefore \sqrt{2x + y + 4} = 3, \text{ or } -10,$$

$$\text{and } 2x + y + 4 = 9, \text{ or } 100;$$

$$\therefore 2x + y = 5, \text{ or } 96.$$

Multiplying every term of the second equation by $\frac{2x + \sqrt{y}}{2x - \sqrt{y}}$,

$$\left(\frac{2x + \sqrt{y}}{2x - \sqrt{y}}\right)^2 = \frac{16}{15} \cdot \frac{2x + \sqrt{y}}{2x - \sqrt{y}} + 1;$$

$$\therefore \text{ by transposition, } \left(\frac{2x + \sqrt{y}}{2x - \sqrt{y}}\right)^2 - \frac{16}{15} \cdot \frac{2x + \sqrt{y}}{2x - \sqrt{y}} = 1;$$

completing the square,

$$\left(\frac{2x + \sqrt{y}}{2x - \sqrt{y}}\right)^2 - \frac{16}{15} \cdot \frac{2x + \sqrt{y}}{2x - \sqrt{y}} + \frac{64}{225} = 1 + \frac{64}{225} = \frac{289}{225};$$

$$\text{extracting the root, } \frac{2x + \sqrt{y}}{2x - \sqrt{y}} - \frac{8}{15} = \pm \frac{17}{15};$$

$$\therefore \frac{2x + \sqrt{y}}{2x - \sqrt{y}} = \frac{5}{3}, \text{ or } -\frac{3}{5}.$$

Let the former value be taken, then

$$6x + 3\sqrt{y} = 10x - 5\sqrt{y},$$

$$\text{and by transposition, } 8\sqrt{y} = 4x,$$

$$\text{and } 2\sqrt{y} = x;$$

$$\text{but if the second value be taken, } \frac{2x + \sqrt{y}}{2x - \sqrt{y}} = -\frac{3}{5};$$

$$\therefore 10x + 5\sqrt{y} = -6x + 3\sqrt{y},$$

$$\text{and } 16x = -2\sqrt{y};$$

$$\text{or } 8x = -\sqrt{y}.$$

Now $2x + y = 5$, or 96; supposing the former,

$$\text{and taking the first value of } 2x = 4\sqrt{y},$$

$$y + 4\sqrt{y} = 5;$$

$$\text{completing the square, } y + 4\sqrt{y} + 4 = 9;$$

$$\text{extracting the root, } \sqrt{y} + 2 = \pm 3;$$

$$\therefore \sqrt{y} = 1, \text{ or } -5,$$

$$\text{and } y = 1, \text{ or } 25;$$

$$\text{but } x = 2\sqrt{y} = 2, \text{ or } -10.$$

$$\text{Again, taking the value, } 2x = -\frac{1}{4}\sqrt{y},$$

$$y - \frac{1}{4}\sqrt{y} = 5;$$

completing the square,

$$y - \frac{1}{4}\sqrt{y} + \frac{1}{64} = 5 + \frac{1}{64} = \frac{321}{64};$$

$$\therefore \text{extracting the root, } \sqrt{y} - \frac{1}{8} = \frac{\pm \sqrt{(321)}}{8},$$

$$\text{and } \sqrt{y} = \frac{1 \pm \sqrt{(321)}}{8};$$

$$\therefore y = \frac{161 \pm \sqrt{(321)}}{32},$$

$$\text{and } x = -\frac{1}{8}\sqrt{y} = \frac{-1 \mp \sqrt{(321)}}{64}.$$

Now taking the equation $2x + y = 96$, and the first value $2x = 4\sqrt{y}$;

$$\text{then } y + 4\sqrt{y} = 96;$$

completing the square, $y + 4\sqrt{y} + 4 = 100$,

$$\text{and } \sqrt{y} + 2 = \pm 10;$$

$$\therefore \sqrt{y} = 8, \text{ or } -12;$$

$$\text{and } \therefore y = 64, \text{ or } 144;$$

$$\text{whence } x = 2\sqrt{y} = 16, \text{ or } -24.$$

Again, taking the value, $2x = -\frac{1}{4}\sqrt{y}$;

$$\text{then } y - \frac{1}{4}\sqrt{y} = 96;$$

completing the square,

$$y - \frac{1}{4}\sqrt{y} + \frac{1}{64} = 96 + \frac{1}{64} = \frac{6145}{64};$$

$$\text{extracting the root, } \sqrt{y} - \frac{1}{8} = \frac{\pm\sqrt{(6145)}}{8};$$

$$\text{and } \sqrt{y} = \frac{1 \pm \sqrt{(6145)}}{8};$$

$$\therefore y = \frac{3073 \pm \sqrt{(6145)}}{32};$$

$$\text{and } x = -\frac{1}{8}\sqrt{y} = \frac{-1 \mp \sqrt{(6145)}}{64}.$$

$$\left. \begin{aligned} 24. \quad & \sqrt{y} + \sqrt{x} : \sqrt{y} - \sqrt{x} :: \sqrt{x} + 2 : 1 \\ & \text{and } \frac{\sqrt{y} + 2}{\sqrt{x}} - 1 = \frac{3\sqrt{x} + 1 + \frac{\sqrt{y}}{\sqrt{x}}}{\sqrt{y}} \end{aligned} \right\},$$

to find the values of x and y .

From the first equation,

$$\sqrt{y} : \sqrt{x} :: \sqrt{x} + 3 : \sqrt{x} + 1;$$

$$\therefore \sqrt{xy} + \sqrt{y} = x + 3\sqrt{x};$$

and from the second,

$$y + 2\sqrt{y} - \sqrt{xy} = 3x + \sqrt{x} + \sqrt{y};$$

$$\therefore \text{ by transposition, } y + \sqrt{y} - \sqrt{xy} = 3x + \sqrt{x};$$

$$\text{but } \sqrt{y} + \sqrt{xy} = x + 3\sqrt{x};$$

$$\therefore \text{ by addition, } y + 2\sqrt{y} = 4x + 4\sqrt{x};$$

$$\text{completing the square, } y + 2\sqrt{y} + 1 = 4x + 4\sqrt{x} + 1,$$

$$\text{and extracting the root, } \sqrt{y} + 1 = \pm (2\sqrt{x} + 1);$$

$$\text{and } \therefore \text{ if the positive value be taken, } \sqrt{y} = 2\sqrt{x};$$

$$\therefore x + 3\sqrt{x} = 2\sqrt{x} + 2x,$$

$$\text{and } x = \sqrt{x};$$

$$\therefore \sqrt{x} = 1,$$

$$\text{and } x = 1;$$

$$\therefore \sqrt{y} = 2\sqrt{x} = 2,$$

$$\text{and } y = 4.$$

But if the negative value be taken,

$$\sqrt{y} + 1 = -2\sqrt{x} - 1;$$

$$\text{by transposition, } \sqrt{y} = -2\sqrt{x} - 2;$$

and if this value be substituted in the first equation,

$$-\sqrt{x} - 2 : -3\sqrt{x} - 2 :: \sqrt{x} + 2 : 1,$$

$$\text{or } \sqrt{x} + 2 : 3\sqrt{x} + 2 :: \sqrt{x} + 2 : 1;$$

and since the first term in this proportion is equal to the third, the second will be equal to the fourth;

$$\therefore 3\sqrt{x} + 2 = 1;$$

$$\text{by transposition, } 3\sqrt{x} = -1;$$

$$\therefore \sqrt{x} = -\frac{1}{3},$$

$$\text{and } x = \frac{1}{9};$$

whence $\sqrt{y} = -2 \cdot (\sqrt{x} + 1) = -\frac{4}{3},$

and $y = \frac{16}{9}.$

25. Given $y + \sqrt{\frac{y}{x}} = \frac{42}{x}$
and $\frac{x^2}{3} + \frac{x}{2\sqrt{y}} = \frac{54}{y}$ } to find the values of x and y .

Completing the square in the first equation,

$$y + \sqrt{\frac{y}{x}} + \frac{1}{4x} = \frac{42}{x} + \frac{1}{4x} = \frac{169}{4x},$$

and extracting the root, $\sqrt{y} + \frac{1}{2\sqrt{x}} = \frac{\pm 13}{2\sqrt{x}};$

$$\therefore \sqrt{y} = \frac{6}{\sqrt{x}}, \text{ or } \frac{-7}{\sqrt{x}};$$

$$\therefore y = \frac{36}{x}, \text{ or } \frac{49}{x}.$$

Again, from the second equation,

$$x^2 + \frac{3x}{2\sqrt{y}} = \frac{162}{y};$$

completing the square,

$$x^2 + \frac{3x}{2\sqrt{y}} + \frac{9}{16y} = \frac{162}{y} + \frac{9}{16y} = \frac{2601}{16y};$$

and extracting the root, $x + \frac{3}{4\sqrt{y}} = \pm \frac{51}{4\sqrt{y}};$

$$\therefore x = \frac{12}{\sqrt{y}}, \text{ or } \frac{-27}{2\sqrt{y}}.$$

But $\sqrt{y} = \frac{6}{\sqrt{x}}, \text{ or } \frac{-7}{\sqrt{x}};$

whence, $x = \frac{12}{\sqrt{y}}, \text{ or } \frac{-27}{2\sqrt{y}} = 2\sqrt{x}, \text{ or}$

$$\frac{-12\sqrt{x}}{7}, \text{ or } \frac{-9\sqrt{x}}{4}, \text{ or } \frac{27\sqrt{x}}{14};$$

$$\therefore \sqrt{x} = 2, \text{ or } \frac{-12}{7}, \text{ or } \frac{-9}{4}, \text{ or } \frac{27}{14};$$

$$\therefore x = 4, \text{ or } \frac{144}{49}, \text{ or } \frac{81}{16}, \text{ or } \frac{729}{196},$$

$$\text{and } y = \frac{36}{x}, \text{ or } \frac{49}{x} = 9, \text{ or } \frac{49}{4}, \text{ or } \frac{784}{81}, \text{ or } \frac{49 \times 196}{729}.$$

26. Given $\frac{x^2}{y^2} + \frac{2x+y}{\sqrt{y}} = 20 - \frac{y^2+x}{y}$ } , to find the values
and $x + 8 = 4y$ } of x and y .

From the first equation, by transposition,

$$\frac{x^2}{y^2} + \frac{2x}{\sqrt{y}} + y + \frac{x}{y} + \sqrt{y} = 20;$$

completing the square,

$$\left(\frac{x}{y} + \sqrt{y}\right)^2 + \left(\frac{x}{y} + \sqrt{y}\right) + \frac{1}{4} = 20 + \frac{1}{4} = \frac{81}{4};$$

$$\text{extracting the root, } \frac{x}{y} + \sqrt{y} + \frac{1}{2} = \pm \frac{9}{2};$$

$$\therefore \frac{x}{y} + \sqrt{y} = 4, \text{ or } -5,$$

$$\text{and } x + y^{\frac{1}{2}} = 4y, \text{ or } -5y.$$

Let the former be taken.

$$\text{Now from the second equation, } x + 8 = 4y;$$

$$\therefore \text{ by subtraction, } y^{\frac{1}{2}} - 8 = 0,$$

$$\text{or } y^{\frac{1}{2}} = 8;$$

$$\therefore y = 4;$$

$$\therefore x = 4y - 8 = 16 - 8 = 8.$$

$$\text{But if } x + y^{\frac{1}{2}} = -5y,$$

$$\text{and } x + 8 = 4y;$$

$$\therefore \text{ by subtraction, } 8 - y^{\frac{1}{2}} = 9y;$$

$$\therefore \text{ by transposition, } 8 - 9y = y^{\frac{1}{2}},$$

$$\text{or } 8 \cdot (1 - y) = y \cdot (1 + y^{\frac{1}{2}}).$$

$$\text{Dividing by } (1 + y^{\frac{1}{2}}), \therefore 8 \cdot (1 - y^{\frac{1}{2}}) = y;$$

$$\therefore \text{ by transposition, } y + 8y^{\frac{1}{2}} = 8;$$

$$\text{completing the square, } y + 8y^{\frac{1}{2}} + 16 = 24,$$

$$\text{and extracting the root, } y^{\frac{1}{2}} + 4 = \pm 2\sqrt{6};$$

$$\therefore y^{\frac{1}{2}} = -4 \pm 2\sqrt{6},$$

$$\text{and } y = 40 \mp 16\sqrt{6};$$

$$\therefore x = 4y - 8 = 152 \mp 64\sqrt{6}.$$

27. Given $8x + 23y = 2x^2 + 2y^2$
 and $34y + 6x^2 - 5y^2 = 13xy + 24$ } , to find the
 values of x and y .

From the second equation,

$$6x^2 - 13xy = 5y^2 - 34y + 24,$$

$$\text{and } x^2 - \frac{13}{6}xy = \frac{5}{6}y^2 - \frac{34}{6}y + 4;$$

completing the square,

$$\begin{aligned} x^2 - \frac{13}{6}xy + \frac{169}{144}y^2 &= \frac{169}{144}y^2 + \frac{5}{6}y^2 - \frac{34}{6}y + 4 \\ &= \frac{289}{144}y^2 - \frac{34}{6}y + 4; \end{aligned}$$

$$\text{extracting the root, } x - \frac{13}{12}y = \pm \left(\frac{17}{12}y - 2 \right);$$

$$\text{and first, taking the positive value, } x = \frac{5}{2}y - 2.$$

Let this value of x be substituted in the first equation,

$$\therefore 20y - 16 + 23y = \frac{125y^3}{4} - 75y^3 + 60y - 16 + 2y^3;$$

$$\therefore \text{by transposition, } \frac{133y^3}{4} - 75y^3 + 17y = 0,$$

$$\text{and dividing by } \frac{133y}{4}, y^2 - \frac{300}{133}y + \frac{68}{133} = 0;$$

$$\text{by transposition, } y^2 - \frac{300}{133}y = -\frac{68}{133};$$

completing the square,

$$y^2 - \frac{300}{133} \cdot y + \left(\frac{150}{133}\right)^2 = \frac{22500}{(133)^2} - \frac{68}{133} = \frac{13456}{(133)^2};$$

$$\text{extracting the root, } y - \frac{150}{133} = \pm \frac{116}{133};$$

$$\therefore y = 2, \text{ or } \frac{34}{133};$$

$$\therefore x = \frac{5y}{2} - 2 = 3, \text{ or } -\frac{171}{133}.$$

But if the negative value be taken,

$$x = 2 - \frac{1}{3}y.$$

Let this value be substituted in the first equation,

$$\therefore 16 - \frac{8}{3}y + 23y = 16 - 6y + \frac{4}{3}y^3 - \frac{2y^3}{27} + 2y^3;$$

$$\therefore \text{by transposition, } \frac{52y^3}{27} + \frac{4}{3}y^3 = \frac{85}{3}y,$$

$$\text{and dividing by } \frac{52y}{27}, y^2 + \frac{9}{13}y = \frac{765}{52};$$

completing the square,

$$y^2 + \frac{9}{13}y + \left(\frac{9}{26}\right)^2 = \left(\frac{9}{26}\right)^2 + \frac{765}{52} = \frac{10026}{(26)^2};$$

$$\text{extracting the root, } y + \frac{9}{26} = \frac{\pm \sqrt{(10026)}}{26};$$

$$\therefore y^{\frac{1}{2}} = \frac{5x - 4}{25} = \frac{1}{25}, \text{ or } -\frac{1}{4};$$

$$\therefore y = \pm \frac{1}{5}, \text{ or } \pm \frac{1}{2} \sqrt{-1}.$$

29. Given $16x - y^{\frac{1}{2}} = 6y^{\frac{1}{2}}x^{\frac{1}{2}}$
 and $\frac{x^{\frac{1}{2}}}{y} - \frac{12}{x^{\frac{1}{2}}} = \frac{x}{\sqrt{y}}$ } , to find the values of x and y .

From the first equation, by transposition,

$$16x = y^{\frac{1}{2}} + 6y^{\frac{1}{2}}x^{\frac{1}{2}};$$

$$\text{completing the square, } 25x = y^{\frac{1}{2}} + 6y^{\frac{1}{2}}x^{\frac{1}{2}} + 9x;$$

$$\text{extracting the root, } \pm 5x^{\frac{1}{2}} = y^{\frac{1}{4}} + 3x^{\frac{1}{4}};$$

$$\therefore 2x^{\frac{1}{2}}, \text{ or } -8x^{\frac{1}{2}} = y^{\frac{1}{4}},$$

$$\text{and therefore } 4x, \text{ or } 64x = y^{\frac{1}{2}}.$$

$$\text{Now from the second equation, } \frac{x^{\frac{1}{2}}}{y} - \frac{x}{\sqrt{y}} = \frac{12}{x^{\frac{1}{2}}};$$

$$\text{completing the square, } \frac{x^{\frac{1}{2}}}{y} - \frac{x}{\sqrt{y}} + \frac{1}{4x^{\frac{1}{2}}} = \frac{12}{x^{\frac{1}{2}}} + \frac{1}{4x^{\frac{1}{2}}} = \frac{49}{4x^{\frac{1}{2}}};$$

$$\text{extracting the root, } \frac{x^{\frac{1}{4}}}{\sqrt{y}} - \frac{1}{2x} = \pm \frac{7}{2x};$$

$$\therefore \frac{x^{\frac{1}{4}}}{\sqrt{y}} = \frac{4}{x^{\frac{1}{2}}} \text{ or } -\frac{3}{x^{\frac{1}{2}}},$$

$$\text{and } \frac{x^{\frac{1}{4}}}{\sqrt{y}} = 4, \text{ or } -3.$$

$$\text{Now } \sqrt{y} = 4x, \text{ or } 64x;$$

which, substituted in the last equation, gives,

$$\frac{x^{\frac{1}{4}}}{4}, \text{ or } \frac{x^{\frac{1}{4}}}{64} = 4, \text{ or } -3;$$

$$\therefore \frac{x}{2}, \text{ or } \frac{x}{8} = \pm 2, \text{ or } \pm \sqrt{-3};$$

whence $x = \pm 4$, or ± 16 , or $\pm 2\sqrt{-3}$, or $\pm 8\sqrt{-3}$,
and $y = 256$, or $\overline{256}^{\frac{1}{2}}$, or -192 , or $-3 \times \overline{64}^{\frac{1}{4}}$.

30. Given $y^2 - 64 = 8x^{\frac{1}{2}}y$
and $y - 4 = 2y^{\frac{1}{2}}x^{\frac{1}{2}}$ }, to find the values of x and y .

From the first equation by transposition,

$$y^2 - 8x^{\frac{1}{2}}y = 64;$$

completing the square, $y^2 - 8x^{\frac{1}{2}}y + 16x = 16x + 64$;

extracting the root, $y - 4x^{\frac{1}{2}} = \pm 4\sqrt{x + 4}$,

$$\text{and } y = 4x^{\frac{1}{2}} \pm 4\sqrt{x + 4}.$$

Also from the second equation, $y - 2y^{\frac{1}{2}}x^{\frac{1}{2}} = 4$;

completing the square, $y - 2y^{\frac{1}{2}}x^{\frac{1}{2}} + x = x + 4$;

extracting the root, $y^{\frac{1}{2}} - x^{\frac{1}{2}} = \pm \sqrt{x + 4}$;

$\therefore 4y^{\frac{1}{2}} = 4x^{\frac{1}{2}} \pm 4\sqrt{x + 4} = y$, from the last equation;

$$\therefore 4 = y^{\frac{1}{2}}, \text{ and } 16 = y;$$

And from the second equation, $x^{\frac{1}{2}} = \frac{y - 4}{2y^{\frac{1}{2}}} = \frac{12}{8} = \frac{3}{2}$;

$$\therefore x = \frac{9}{4}.$$

31. Given $\sqrt{5\sqrt{x} + 5\sqrt{y}} + \sqrt{y} = 10 - \sqrt{x}$
and $\sqrt{x^3} + \sqrt{y^3} = 275$ }, to find
the values of x and y .

From the first equation,

$$\sqrt{x} + \sqrt{y} + \sqrt{5} \cdot \sqrt{(\sqrt{x} + \sqrt{y})} = 10;$$

completing the square,

$$\sqrt{x} + \sqrt{y} + \sqrt{5} \cdot \sqrt{(\sqrt{x} + \sqrt{y})} + \frac{5}{4} = 10 + \frac{5}{4} = \frac{45}{4};$$

extracting the root, $\sqrt{(\sqrt{x} + \sqrt{y})} + \frac{\sqrt{5}}{2} = \pm \frac{3\sqrt{5}}{2},$

$$\text{and } \sqrt{(\sqrt{x} + \sqrt{y})} = \sqrt{5}, \text{ or } -2\sqrt{5};$$

$$\therefore \sqrt{x} + \sqrt{y} = 5, \text{ or } 20, \text{ supposing the former,}$$

$\therefore$ by involution,

$$x^{\frac{1}{2}} + 5x^{\frac{1}{2}}y^{\frac{1}{2}} + 10x^{\frac{1}{2}}y + 10xy^{\frac{1}{2}} + 5x^{\frac{1}{2}}y^2 + y^{\frac{1}{2}} = 3125;$$

$$\text{but } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 275;$$

$\therefore$ by subtraction,

$$5x^{\frac{1}{2}}y^{\frac{1}{2}} + 10x^{\frac{1}{2}}y + 10xy^{\frac{1}{2}} + 5x^{\frac{1}{2}}y^2 = 2850;$$

$$\text{or } 5x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} + 2xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y + y^{\frac{1}{2}}) = 2850;$$

$$\text{and } x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} + 2xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y + y^{\frac{1}{2}}) = 570;$$

$$\text{but } x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} + 3xy^{\frac{1}{2}} + 3x^{\frac{1}{2}}y + y^{\frac{1}{2}}) = 125x^{\frac{1}{2}}y^{\frac{1}{2}};$$

$$\therefore \text{ by subtraction, } x^{\frac{1}{2}}y^{\frac{1}{2}} \times (xy^{\frac{1}{2}} + x^{\frac{1}{2}}y) = 125x^{\frac{1}{2}}y^{\frac{1}{2}} - 570;$$

$$\text{or } x^{\frac{1}{2}}y^{\frac{1}{2}} \times x^{\frac{1}{2}}y^{\frac{1}{2}} \times (x^{\frac{1}{2}} + y^{\frac{1}{2}}) = 125x^{\frac{1}{2}}y^{\frac{1}{2}} - 570;$$

$$\text{or } 5xy = 125x^{\frac{1}{2}}y^{\frac{1}{2}} - 570;$$

$$\therefore xy - 25x^{\frac{1}{2}}y^{\frac{1}{2}} = -114;$$

completing the square,

$$xy - 25x^{\frac{1}{2}}y^{\frac{1}{2}} + \left(\frac{25}{2}\right)^2 = \frac{625}{4} - 114 = \frac{169}{4};$$

$$\text{extracting the root, } x^{\frac{1}{2}}y^{\frac{1}{2}} - \frac{25}{2} = \pm \frac{13}{2};$$

$$\therefore x^{\frac{1}{2}}y^{\frac{1}{2}} = 19, \text{ or } 6;$$

$$\text{but } x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 25,$$

$$\text{and } 4x^{\frac{1}{2}}y^{\frac{1}{2}} = 24, \text{ or } 76;$$

$$\therefore \text{ by subtraction, } x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 1, \text{ or } -51;$$

$$\text{extracting the root, } x^{\frac{1}{2}} - y^{\frac{1}{2}} = \pm 1, \text{ or } \pm \sqrt{-51};$$

$$\text{but } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5;$$

$$\therefore \text{ by addition, } 2x^{\frac{1}{2}} = 6, \text{ or } 4, \text{ or } 5 \pm \sqrt{-51};$$

$$\therefore x^{\frac{1}{2}} = 3, \text{ or } 2, \text{ or } \frac{5 \pm \sqrt{(-51)}}{2},$$

$$\text{and } x = 9, \text{ or } 4, \text{ or } \frac{-13 \pm \sqrt{(-51)}}{2};$$

by subtraction, $2y^{\frac{1}{2}} = 4, \text{ or } 6, \text{ or } 5 \mp \sqrt{(-51)};$

$$\therefore y^{\frac{1}{2}} = 2, \text{ or } 3, \text{ or } \frac{5 \mp \sqrt{(-51)}}{2};$$

$$\therefore y = 4, \text{ or } 9, \text{ or } \frac{-13 \mp \sqrt{(-51)}}{2}.*$$

* The other case, where $\sqrt{x} + \sqrt{y} = 20$, is solved in the same manner.

SECTION VI.

On the Solution of Problems which involve Simple Equations.

(32.) THE solution of a problem, or method of discovering by analysis quantities which will answer its several conditions, is performed by assuming algebraic symbols to represent the quantities sought, and by deducing equations from the application of these, in the same manner as if they were known quantities, to the conditions of the problem. The independent equations derived from this process, if the conditions be properly limited, will equal in number the unknown quantities assumed; and from the solution of these several equations by the rules already given (23. 27. 29), the values of the algebraic symbols will be determined. Whether these values are correct, may be determined synthetically, by applying them instead of their respective symbols to the several conditions of the problem.

If the conditions of the problem are not properly limited, that is, are not sufficient in number, or not sufficiently independent of each other, the resulting equations will either exceed in number the unknown quantities, and will therefore some of them be identical or inconsistent, or will be fewer in number than the unknown quantities, and consequently will admit of an indefinite number of answers.

In many cases, instead of assuming a symbol to represent each of the required quantities, it is convenient to assume one only, and from the conditions of the problem to deduce

expressions for the others in terms of that one and known quantities. And as the number of conditions ought to be one more than the number of quantities thus expressed, there will remain one to be stated in an equation; from which the value of the unknown quantity may be determined (22. 27. 28): and this being substituted in the other expressions, their value also may be discovered.

Examples of the Solution of Problems producing Simple Equations involving only one unknown quantity.

1. What number is that, to the double of which if 18 be added the sum will be 82?

Let x = the number required.

Then by the problem, $2x + 18 = 82$;

$\therefore$ by transposition, $2x = 64$.

and $x = 32$.

2. What number is that, to the double of which if 44 be added, the sum is equal to four times the required number?

Let x = the number.

Then $2x + 44 = 4x$, by supposition;

$\therefore$ by transposition, $44 = 2x$

and $22 = x$.

3. What number is that, the double of which exceeds its half by 6?

Let x = the number.

Then by the problem, $2x - \frac{x}{2} = 6$,

$$\therefore 4x - x = 12,$$

$$\text{or } 3x = 12,$$

$$\therefore x = 4.$$

4. From two towns which are 187 miles distant, two travellers set out at the same time with an intention of meeting. One of them goes 8 miles, and the other 9 miles a day. In how many days will they meet?

Let x = the number of days required;

then $8x$ = the number of miles one travelled,

and $9x$ = the number the other travelled;

and since they meet, they must together have travelled the whole distance,

$$\text{consequently } 8x + 9x = 187,$$

$$\text{or } 17x = 187,$$

$$\therefore x = 11.$$

5. A Gentleman meeting 4 poor persons distributed five shillings amongst them: to the second he gave twice, to the third thrice, and to the fourth four times as much as to the first. What did he give to each?

Let x = the pence he gave to the first,

$\therefore 2x$ = the pence given to the second,

and $3x$ = ----- to the third,

$4x$ = ----- to the fourth.

$$\therefore x + 2x + 3x + 4x = 60,$$

$$\text{or } 10x = 60,$$

$$\therefore x = 6,$$

and $\therefore$ he gave 6, 12, 18, 24 pence respectively to them.

6. A Bookseller sold 10 books at a certain price; and afterwards 15 more at the same rate. Now at the latter time he received 35 shillings more than at the former. What did he receive for each book?

Let x = the price of a book.
 Then $10x$ = price of the first set,
 and $15x$ = price of the second set.
 But by the problem $15x = 10x + 35$;
 $\therefore$ by transposition, $5x = 35$,
 and $x = 7$.

7. A Gentleman dying bequeathed a legacy of £140. to three servants. A was to have twice as much as B ; and B three times as much as C . What were their respective shares?

Let x = C 's share,
 $\therefore 3x$ = B 's share,
 and $6x$ = A 's share;
 whence $(6x + 3x + x =) 10x = 140$,
 $\therefore x = 14$,
 $A \therefore$ received £84; B , £42; and C £14.

8. Four Merchants entered into a speculation, for which they subscribed £4755; of which B paid three times as much as A ; C paid as much as A and B ; and D paid as much as C and B . What did each pay?

Let x = number of pounds A paid;
 $\therefore 3x$ = number B paid,
 $4x$ = number C paid,
 and $7x$ = number D paid;
 $\therefore (x + 3x + 4x + 7x =) 15x = 4755$,
 and $\therefore x = 317$.

$\therefore$ they contributed 317, 951, 1268, and 2219 pounds respectively.

9. A Draper bought three pieces of cloth, which together measured 159 yards. The second piece was 15 yards longer than the first, and the third 24 yards longer than the second. What was the length of each?

Let x = the number of yards in the first piece,
 $\therefore x + 15$ = the number in the second,
 and $x + 39$ = the number in the third.
 $\therefore x + x + 15 + x + 39 = 159$,
 and by transposition, $3x = 105$,
 $\therefore x = 35$,
 $\therefore$ the lengths are 35, 50, and 74 yards respectively.

10. A cask which held 146 gallons, was filled with a mixture of brandy, wine, and water. In it there were 15 gallons of wine more than there were of brandy, and as much water as both wine and brandy. What quantity was there of each?

Let x = the number of gallons of brandy,
 $\therefore x + 15$ = number of gallons of wine,
 and $2x + 15$ = number of gallons of water.
 $\therefore x + x + 15 + 2x + 15 = 146$,
 $\therefore$ by transposition, $4x = 116$,
 and $x = 29$.
 $\therefore$ there were 29, 44, and 73 gallons respectively of brandy, wine, and water.

11. A person employed 4 workmen; to the first of whom he gave 2 shillings more than to the second; to the second 3 shillings more than to the third; and to the third 4 shillings more than to the fourth. Their wages amounted to 32 shillings. What did each receive?

Let x = the sum received by the fourth,
 $\therefore x + 4 =$ ----- third,
 $x + 7 =$ ----- second,
 and $x + 9 =$ ----- first.
 $\therefore x + x + 4 + x + 7 + x + 9 = 32$,
 and by transposition, $4x = 12$,
 $\therefore x = 3$.

$\therefore$ they received 12, 10, 7, and 3 shillings respectively.

12. A Father taking his 4 sons to school, divided a certain sum amongst them. Now the third had 9 shillings more than the youngest; the second 12 shillings more than the third; and the eldest 18 shillings more than the second; and the whole sum was 6 shillings more than 7 times the sum which the youngest received. How much had each?

Suppose the youngest received x shillings,
 then the third received $x + 9$ - - - -
 the second - - - - $x + 21$ - - - -
 and the eldest - - - - $x + 39$ - - - -
 $\therefore x + x + 9 + x + 21 + x + 39 = 7x + 6$,
 $\therefore$ by transposition, $63 = 3x$,
 and $\therefore 21 = x$,

consequently they received 21, 30, 42, and 60 shillings respectively.

13. A sum of money was to be divided amongst six poor persons; the second received 10*d.* the third 14*d.* the fourth 25*d.* the fifth 28*d.* and the sixth 33*d.* less than the first. Now the sum distributed was 10*d.* more than the treble of what the first received. What did each receive?

Let x = what the first received,

$$\therefore x - 10 = \text{--- second ---}$$

$$x - 14 = \text{--- third ---}$$

$$x - 25 = \text{--- fourth ---}$$

$$x - 28 = \text{--- fifth ---}$$

$$x - 33 = \text{--- sixth ---}$$

The sum of which = $6x - 110 = 3x + 10$ by supposition.

$$\therefore \text{ by transposition, } 3x = 120,$$

$$\text{and } x = 40.$$

$\therefore$ they received 40, 30, 26, 15, 12, 7 pence respectively.

14. It is required to divide the number 99 into five such parts, that the first may exceed the second by 3; be less than the third by 10; greater than the fourth by 9; and less than the fifth by 16.

Let x = the first part,

$$\therefore x - 3 = \text{second,}$$

$$x + 10 = \text{third,}$$

$$x - 9 = \text{fourth,}$$

$$x + 16 = \text{fifth.}$$

$$\therefore x + x - 3 + x + 10 + x - 9 + x + 16 = 99,$$

$$\text{or } 5x + 14 = 99,$$

$$\therefore \text{ by transposition, } 5x = 85,$$

$$\text{and } x = 17.$$

$\therefore$ the parts are 17, 14, 27, 8, and 33.

15. What two numbers are those whose sum is 59, and difference 17?

Let x = the less,

$$\therefore x + 17 = \text{the greater,}$$

and $\therefore x + x + 17 = 59$,
 by transposition, $2x = 42$.
 and $x = 21$ the less,
 $\therefore$ the greater $= 38$.

16. What number is that, the treble of which increased by 12, shall as much exceed 54 as that treble is below 144?

Let $x =$ the number.
 $\therefore 3x + 12 - 54 = 144 - 3x$ by supposition;
 $\therefore$ by transposition, $6x = 186$,
 and $x = 31$.

17. Two persons began to play with equal sums of money: the first lost 14 shillings, the other won 24 shillings; and then the second had twice as many shillings as the first. What sum had each at first?

Let $x =$ the sum;
 Then $x - 14$ } = the sums each had after playing;
 and $x + 24$ }
 $\therefore$ by the problem $2x - 28 = x + 24$;
 $\therefore x = 52$.

18. At a certain election 943 men voted, and the candidate chosen had a majority of 65. How many voted for each?

Let $x =$ the number of votes the unsuccessful candidate had;
 $\therefore x + 65 =$ the number the successful one had.
 $\therefore x + x + 65 = 943$;
 by transposition, $2x = 878$,
 and $x = 439$.
 $\therefore$ the numbers were 439 and 504.

19. Two Robbers after plundering a house found that they had 35 guineas between them; and that if one of them had had 4 guineas more, he should have had twice as many as the other. How many had each?

Let x = the number one had,
 $\therefore 35 - x$ = the number the other had,
 and $35 - x + 4 = 2x$.
 by transposition, $39 = 3x$,
 and $13 = x$.
 $\therefore$ they had 13 and 22 guineas respectively.

20. A Mercer having cut 19 yards from each of three equal pieces of silk, and 17 from another of the same length, found that the remnants taken together were 142 yards. What was the length of each piece?

Let x = the length,
 $\therefore x - 19$ = the length of each of the 3 remnants,
 and $x - 17$ = the length of the other.
 then $3 \cdot (x - 19) + x - 17 = 142$,
 or $3x - 57 + x - 17 = 142$.
 by transposition, $4x = 216$;
 $\therefore x = 54$.

21. A Farmer has two flocks of sheep, each containing the same number. From one of these he sells 39, and from the other 93; and finds just twice as many remaining in one as in the other. How many did each flock originally contain?

Let x = the number required.
 Then $x - 39$, and $x - 93$, are the numbers remaining;
 $\therefore x - 39 = 2x - 186$;
 and by transposition, $147 = x$.

22. Bought 12 yards of cloth for £10. 14s. For part of it I gave 19 shillings a yard, and for the rest 17 shillings a yard. How many yards of each were bought?

Let x = the number of yards at 19s. per yard;

$\therefore 12 - x$ = the number at 17s.

and $19x$ = the price of the former,

and $17 \cdot (12 - x)$ = the price of the latter.

$$\therefore 19x + 204 - 17x = 214.$$

and by transposition, $2x = 10$,

$$\text{and } x = 5.$$

$\therefore$ there were 5 yards at 19 shillings, and 7 at 17 shillings.

23. Divide the number 197 into two such parts that four times the greater may exceed five times the less by 50.

Let x = the less,

and $\therefore 197 - x$ = the greater.

$$\text{Then } 788 - 4x = 5x + 50;$$

and by transposition, $738 = 9x$,

$$\text{and } 82 = x;$$

$$\therefore \text{the greater} = 115.$$

24. A Courier, who travels 60 miles a day, had been dispatched 5 days, when a second was sent to overtake him; in order to which, he must go 75 miles a day: In what time will he overtake the former?

Let x = the number of days the second courier travels;

then $x + 5$ = the number the first travels;

$\therefore 75x$ = the number of miles the second travels,

and $60 \cdot (x + 5)$ = the number the first travels.

But by the supposition they both travelled the same number of miles;

$$\begin{aligned}\therefore 75x &= 60x + 300, \\ \text{by transposition, } 15x &= 300, \\ \text{and } x &= 20.\end{aligned}$$

25. After A had lost 10 guineas to B , he wanted only 8 guineas in order to have as much money as B ; and together they had 60 guineas. What money had each at first?

Let x = the number of guineas A had;
 $\therefore 60 - x$ = the number - - - - B had.
 Then after playing A had $x - 10$, and B had $70 - x$;
 $\therefore x - 10 + 8 = 70 - x$,
 by transposition, $2x = 72$,
 and $x = 36$.
 $\therefore$ they had 36 and 24 guineas respectively.

26. A and B began trade with equal stocks. In the first year A tripled his stock, and had £27 to spare; B doubled his stock, and had £153 to spare. Now the amount of both their gains was five times the stock of either. What was that stock?

Let x = the stock;
 then $3x + 27$ = A 's stock at the end of the year,
 $\therefore 2x + 27$ = his gain,
 and $2x + 153$ = B 's stock at the end of the year;
 $\therefore x + 153$ = B 's gain;
 $\therefore 5x = 2x + 27 + x + 153$.
 by transposition, $2x = 180$,
 and $x = 90$.

27. Two workmen *A* and *B* were employed together for 50 days, at 5 shillings per day each. *A* spent sixpence a day less than *B* did, and at the end of the fifty days he found he had saved twice as much as *B*, and the expense of two days over. What did each spend per day?

Let x = what *A* spent per day (in pence);

$\therefore 60 - x$ = what he saved per day,

and $54 - x$ = what *B* saved per day.

and $\therefore 3000 - 50x = 5400 - 100x + 2x$.

by transposition, $48x = 2400$,

and $x = 50$;

$\therefore$ *A* spent 50 pence, and *B* 56 pence a day.

28. *A* and *B* began to trade with equal sums of money. In the first year *A* gained 40 pounds and *B* lost 40; but in the second *A* lost one-third of what he then had, and *B* gained a sum less by 40 pounds than twice the sum that *A* had lost; when it appeared that *B* had twice as much money as *A*. What money did each begin with?

Let x = the number of pounds each had at first.

then $x + 40$ = the sum *A* had after the first year,

and $x - 40$ = the sum *B* had,

also $\frac{2}{3} \cdot (x + 40)$ = the sum *A* had after the 2^d year,

and $x - 40 + \frac{2}{3} (x + 40) - 40$ = the sum *B* had;

$\therefore \frac{2}{3} \cdot (x + 40) = x - 40 + \frac{2}{3} \cdot (x + 40) - 40$,

and $\frac{2}{3} \cdot (x + 40) = x - 80$;

$\therefore 2x + 80 = 3x - 240$,

and by transposition, $320 = x$.

29. Divide the number 68 into two such parts, that the difference between the greater and 84 may equal three times the difference between the less and 40.

Let x = the less,
 then $68 - x$ = the greater;
 $\therefore 84 - (68 - x) = 3 \cdot (40 - x)$,
 or $16 + x = 120 - 3x$.
 by transposition, $4x = 104$,
 and $x = 26$;
 and $\therefore$ the greater = 42.

30. A and B being at play severally cut packs of cards so as to take off more than they left. Now it happened that A cut off twice as many as B left, and B cut off seven times as many as A left. How were the cards cut by each?

Suppose A cut off $2x$ cards,
 then $52 - 2x$ = the number he left,
 and x = the number B left;
 $\therefore 52 - x$ = the number he cut off;
 whence $52 - x = 364 - 14x$;
 by transposition, $13x = 312$,
 and $x = 24$;
 $\therefore A$ cut off 48, and B cut off 28 cards.

31. What number is that whose one-third part exceeds its one-fourth part by 16?

Let $12x$ = the number;
 $\therefore 4x - 3x = 16$,
 or $x = 16$;
 and $\therefore$ the number = $12 \times 16 = 192$.

32. Upon measuring the corn produced by a field, being 48 quarters; it appeared that it yielded only one-third part more than was sown. How much was that?

Let $3x$ = the number of quarters sown,

$$\text{then } 3x + x = 48,$$

$$\text{or } 4x = 48,$$

$$\text{and } x = 12;$$

$\therefore$ the quantity sown was 36 quarters.

33. A Farmer sold 96 loads of hay to two persons. To the first one-half, and to the second one-fourth of what his stack contained. How many loads did that stack contain?

Let $4x$ = the number of loads,

then $2x$ = the number the first bought,

and x = the number the second had.

$$\therefore (2x + x =) 3x = 96,$$

$$\text{and } x = 32.$$

whence, the stack contained 128 loads.

34. A Gentleman bequeathed £210 to two servants; to one he left half as much as to the other. What did he leave to each?

Let $2x$ = the sum one received;

$\therefore x$ = the sum left to the other.

$$\therefore (2x + x =) 3x = 210,$$

$$\text{and } x = 70;$$

$\therefore$ they had 140 and 70 pounds respectively.

35. A prize of £2329 was divided between two persons *A* and *B*, whose shares therein were in proportion of 5 to 12. What was the share of each?

Let $5x = A$'s share,
 then $12x = B$'s share ;
 $\therefore (5x + 12x) 17x = 2329$,
 and $x = 137$;
 $\therefore$ their shares were 685 and 1644 pounds respectively.

36. A sum of money is to be shared between two persons A and B , so that as often as A receives 9 pounds, B takes 4. Now it happens that A receives 15 pounds more than B . What are their respective shares ?

Since for every £9 that A receives, B receives £4,

Let $9x =$ the whole sum A receives ;

$\therefore 4x =$ the whole sum B receives ;

$$\therefore 9x = 4x + 15 ;$$

and by transposition, $5x = 15$;

$$\therefore x = 3 ;$$

$\therefore A$ receives £27, and B , £12.

37. A Gentleman gave to 3 persons 98 pounds. The second received five-eighths of the sum given to the first, and the third one-fifth of what the second had. What did each receive ?

Let $8x =$ the number of pounds the first received ;

$\therefore 5x =$ - - - - - second - - -

and $x =$ - - - - - third - - -

$$\therefore (8x + 5x + x) 14x = 98,$$

$$\text{and } x = 7 ;$$

$\therefore$ they received 56, 35, and 7 pounds, respectively.

38. A person bought two casks of beer, one of which held exactly three times as much as the other. From each of these he drew four gallons, and then found that there were four times as many gallons remaining in the larger, as in the other. How many were there in each at first?

Let $3x$ = the number of gallons in the larger ;

and $\therefore x$ = the number in the smaller ;

$$\therefore 4 \cdot (x - 4) = 3x - 4 ;$$

by transposition, $x = 12$;

$\therefore$ they held 36 and 12 gallons, respectively.

39. A man at a party at cards betted three shillings to two upon every deal. After twenty deals he won five shillings. How many deals did he win?

Let x = the number of deals he won ;

$\therefore 20 - x$ = the number he lost ;

$\therefore 2x$ = the money won,

and $3 \cdot (20 - x)$ = the money lost ;

$$\text{whence } 2x - 3 \cdot (20 - x) = 5 ;$$

$\therefore$ by transposition, $5x = 65$,

and $x = 13$.

40. What two numbers are as 2 to 3 ; to each of which if 4 be added, the sums will be as 5 to 7 ?

Let $2x$ and $3x$ be the number ;

$$\therefore 2x + 4 : 3x + 4 :: 5 : 7,$$

$$\text{and } (21) \ 14x + 28 = 15x + 20 ;$$

by transposition, $8 = x$;

and $\therefore$ the numbers are 16 and 24.

41. A sum of money was divided between two persons A and B , so that the share of A was to that of B as 5 to 3; and exceeded five-ninths of the whole sum by 50 pounds. What was the share of each person?

Let $5x = A$'s share;

$\therefore 3x = B$'s share,

and $8x =$ the whole sum;

$\therefore 5x = \frac{5}{8} \cdot 8x + 50$,

or $x = \frac{8}{9} \cdot x + 10$,

and $9x = 8x + 90$;

$\therefore$ by transposition, $x = 90$,

and the sums were 450 and 270 pounds.

42. Being sent to market to buy a certain quantity of meat, I found that if I bought beef, which was then 4 pence a pound, I should lay out all the money I was entrusted with; but if I bought mutton which was then threepence halfpenny a pound, I should have two shillings left. How much meat was sent for?

Let $2x =$ the number of pounds;

$\therefore 8x =$ the price of $2x$ lbs. of beef,

and $7x =$ the price of $2x$ lbs. of mutton,

and $8x = 7x + 24$;

$\therefore x = 24$;

whence 48 lbs. were sent for.

43. A Fish was caught, whose tail weighed 9 lbs.; his head weighed as much as his tail, and half his body; and his body weighed as much as his head and tail. What did the fish weigh?

Let $2x =$ the number of lbs. the body weighed;

then $9 + x =$ the weight of the head;

$$\therefore 9 + 9 + x = 2x;$$

by transposition, $18 = x$;

$$\therefore \text{the fish weighed } 36 + 27 + 9 = 72 \text{ lbs.}$$

44. The joint stock of two partners whose particular shares differed by 40 pounds was to the share of the lesser as 14 to 5. Required the shares.

Suppose $14x =$ the joint stock ;

$\therefore 5x =$ the less,

and $9x =$ the greater ;

$$\therefore 9x = 5x + 40;$$

by transposition, $4x = 40$,

$$\text{and } x = 10;$$

$\therefore$ the shares are 90 and 50 pounds, respectively.

45. A Bankrupt owed to two creditors 140 pounds; the difference of the debts was to the greater as 4 to 9. What were the debts ?

Let $4x =$ the difference of the debts ;

and $\therefore 9x =$ the greater,

and $5x =$ the less ;

$$\therefore (9x + 5x =) 14x = 140,$$

$$\text{and } x = 10;$$

$\therefore$ the debts are 90 and 50 pounds.

46. A Gentleman employed two labourers at different times, one for 3 shillings, and the other for 5 shillings a day. Now the number of days added together was 40; and they each received the same sum. How many days was each employed?

Let x = the number of days the second was employed ;
 $\therefore 40 - x$ = the number the first was employed ;
 $\therefore 5x$ = the sum received by the second,
 and $3 \cdot (40 - x)$ = the sum received by the first ;
 $\therefore 5x = 3 \cdot (40 - x)$;
 by transposition, $8x = 120$,
 and $x = 15$;
 $\therefore$ the second was employed 15, and the first 25 days.

47. Some persons agreed to give sixpence each to a waterman for carrying them from London to Gravesend ; but with this condition, that for every other person taken in by the way, three pence should be abated in their joint fare. Now the waterman took in three more than a fourth part of the number of the first passengers, in consideration of which he took of them but five pence each. How many persons were there at first ?

Let $4x$ = the number of passengers at first ;
 then $x + 3$ = the number taken in,
 and $3x + 9$ = the sum deducted from their joint fare ;
 $\therefore 24x - (3x + 9) = 20x$;
 by transposition, $x = 9$;
 consequently there were 36 passengers.

48. In a mixture of wine and cyder, half of the whole + 25 gallons was wine, and one-third of the whole - 5 gallons was cyder. How many gallons were there of each ?

Let $6x$ = the number of gallons in all ;
 $\therefore 3x + 25$ = the number of gallons of wine,
 and $2x - 5$ = the number of gallons of cyder ;
 $\therefore 6x = 3x + 25 + 2x - 5$;
 by transposition, $x = 20$;
 consequently there were 85 gallons of wine, and 35 of cyder.

49. *A* and *B* engaged in trade, *A* with £240, and *B* with £96. *A* lost twice as much as *B*; and upon settling their accounts it appeared that *A* had three times as much remaining as *B*. How much did each lose?

Let x = what *B* lost;

$\therefore 96 - x$ = what he had remaining;

then $2x$ = what *A* lost,

and $240 - 2x$ = what he had remaining;

$\therefore 240 - 2x = 3 \cdot (96 - x)$

by transposition, $x = 48$;

$\therefore$ *A* lost £96, and *B* lost £48.

50. Four places are situated in the order of the four letters *A*, *B*, *C*, *D*. The distance from *A* to *D* is 34 miles, the distance from *A* to *B* : distance from *C* to *D* :: 2 : 3, and one-fourth of the distance from *A* to *B* added to half the distance from *C* to *D* is three times the distance from *B* to *C*. What are the respective distances?

Let $2x$ = the distance from *A* to *B*;

$\therefore 3x$ = the distance from *C* to *D*.

and $\left(\frac{x}{2} + \frac{3x}{2}\right) 2x = 3 \cdot BC$;

$\therefore BC = \frac{2}{3}x$,

and $\left(2x + 3x + \frac{2}{3}x\right) \frac{17x}{3} = 34$;

$\therefore \frac{x}{3} = 2$, and $x = 6$;

whence $AB = 12$, $BC = 4$, and $CD = 18$.

51. A Field of wheat and oats which contained 20 acres was put out to a labourer to reap for six guineas, the wheat at 7 shillings an acre, and the oats at 5 shillings. Now the labourer falling ill, reaped only the wheat. How much money ought he to receive according to the bargain?

Let x = the number of acres of wheat ;
 then $20 - x$ = the number of acres of oats ;
 and $7x$ = the price of reaping the wheat (in shillings),
 and $100 - 5x$ = the price of reaping the oats ;
 $\therefore 7x + 100 - 5x = 126$;
 by transposition, $2x = 26$,
 and $x = 13$;
 $\therefore$ he ought to receive £4. 11s.

52. A General having lost a battle, found that he had only half his army + 3600 men left, fit for action ; one-eighth of his men + 600 being wounded, and the rest, which were one-fifth of the whole army, either slain, taken prisoners, or missing. Of how many men did his army consist?

Let x = the number required ;
 $\therefore \frac{x}{2} + 3600$ = the number fit for service ;
 $\frac{x}{8} + 600$ = the number wounded,
 and $\frac{x}{5}$ = the number missing ;
 $\therefore x = \frac{x}{2} + 3600 + \frac{x}{8} + 600 + \frac{x}{5}$;
 by transposition, $\frac{3x}{8} - \frac{x}{5} = 4200$,
 and $(15x - 8x) = 7x = 168000$,
 and $x = 24000$.

53. Three men, A , B , and C , entered into partnership; A paid in as much as B , and one-third of C ; B paid in as much as C , and one-third of A ; and C paid in £10, and one-third of A . What did each man contribute to the stock?

Let $3x =$ the sum A contributed;

$$\therefore 10 + x = \text{---} C \text{---}$$

and $10 + 2x = \text{---} B \text{---}$

$$\therefore 3x = 10 + 2x + \frac{10 + x}{3};$$

by transposition, $\frac{2x}{3} = 10 + \frac{10}{3}$,

and $2x = 40$;

$$\therefore x = 20.$$

and the sums contributed were £60, £50, and £30, by A , B , C , respectively.

54. It is required to divide the number 91 into two such parts that the greater being divided by their difference, the quotient may be 7.

Let $x =$ the greater ;

$\therefore 91 - x = \text{the less,}$

and $\frac{x}{2x-91} = 7$;

$$\therefore x = 14x - 637;$$

by transposition, $637 = 13x$;

and $\therefore 49 = x$;

$\therefore$ the parts are 49 and 42.

55. From each of 16 coins an artist filed the worth of half a crown, and then offered them in payment for their original value: but being detected, the pieces were found to be really worth no more than 8 guineas. What was their original value?

Let x = the number of sixpences each was worth ;
 $\therefore x - 5$ = the number each was worth after filing ;
 $\therefore 16 \cdot (x - 5) = 336$.
 by transposition, $16x = 416$,
 and $x = 26 = 13$ shillings.

56. A and B made a joint stock of £833, which, after a successful speculation, produced a clear gain of £153. Of this B had £15 more than A . What did each person contribute to the stock ?

Let x = the sum brought in by B ;
 then $833 : x :: 153 : B$'s gain $= \frac{9x}{49}$;
 $\therefore A$'s gain $= \frac{9x}{49} - 45$,
 and $\frac{9x}{49} + \frac{9x}{49} - 45 = 153$;
 by transposition, $\frac{18x}{49} = 198$;
 $\therefore x = \frac{49 \times 198}{18} = 49 \times 11 = 539$,
 whence, B brought in £539, and A £294.

57. Sold a quantity of tobacco for 19 shillings, part at 1 shilling a pound, and the rest at 15 pence. Now the first part was to the latter $:: \frac{3}{4} : \frac{3}{5}$. How much was sold of each ?

Since $\frac{3}{4} : \frac{3}{5} :: 9 : 8$,
 Let $9x$ = the number of lbs. of the former ;
 $\therefore 8x$ = the number of lbs. of the latter ;
 $\therefore 9x$ = the number of shillings the first sold for,
 and $8x \times \frac{3}{4} = 6x = 10x$ = the number of shillings the second sold for.

$$\therefore (10x + 9x =) 19x = 19,$$

$$\text{and } x = 1;$$

$\therefore$ there were 9 lbs. at 1 shilling, and 8 lbs. at 15 pence.

58. A Gentleman gave in charity £46; a part thereof in equal portions to 5 poor men, and the rest in equal portions to 7 poor women. Now a man and a woman had between them £8. What was given to the men, and what to the women?

Let $5x$ = the number of pounds the men received;

$\therefore 46 - 5x$ = the number the women received;

$\therefore x$ = the sum one man received,

and $8 - x$ = the sum one woman received;

$$\therefore 56 - 7x = 46 - 5x;$$

by transposition, $2x = 10$,

$$\text{and } x = 5;$$

$\therefore$ the men received £25, and the women £21.

59. Suppose that for every 10 sheep a farmer kept, he should plough an acre of land, and be allowed one acre of pasture for every 4 sheep. How many sheep may that person keep who farms 700 acres?

Let x = the number of sheep required;

then $10 : x :: 1$: the number of acres ploughed = $\frac{x}{10}$,

and $4 : x :: 1$: the number of acres of pasture = $\frac{x}{4}$;

$$\therefore \frac{x}{10} + \frac{x}{4} = 700,$$

and $(2x + 5x =) 7x = 20 \times 700$;

$$\therefore x = 20 \times 100 = 2000.$$

60. A person being asked the hour, answered that it was between five and six; and the hour and minute-hands were together. What was the time?

Let x = the time past 5;

then since the minute-hand goes 12 times round, whilst the hour-hand goes once, we have this proportion,

$$12 : 1 :: 5 + x : x,$$

$$\text{and (Alg. 180.) } 11 : 1 :: 5 : x;$$

$$\therefore 11x = 5,$$

$$\text{and } x = \frac{5}{11} = 27' . 16\frac{4}{11}''.$$

61. Divide the number 49 into two such parts that the greater increased by 6 may be to the less diminished by 11 as 9 to 2.

Let x = the greater;

$$\therefore 49 - x = \text{the less,}$$

$$\text{and } x + 6 : 38 - x :: 9 : 2;$$

$$\therefore (\text{Alg. 177. 179.}) x + 6 : 44 :: 9 : 11,$$

$$\text{and (Alg. 186.) } x + 6 : 4 :: 9 : 1;$$

$$\therefore x + 6 = 36,$$

$$\text{and } x = 30;$$

$$\therefore \text{ the parts are 30 and 19.}$$

62. A , B , and C make a joint stock; A puts in £60 less than B , and £68 more than C ; and the sum of the shares of A and B is to the sum of the shares of B and C as 5 to 4. What did each put in?

Let x = what A put in;

$$\therefore x + 60 = \text{what } B \text{ put in,}$$

$$\text{and } x - 68 = \text{what } C \text{ put in;}$$

$$\text{then } 2x + 60 : 2x - 8 :: 5 : 4,$$

$$\text{and (Alg. 184.) } x + 30 : x - 4 :: 5 : 4;$$

$$\therefore (\text{Alg. 180.}) \quad 34 : x - 4 :: 1 : 4;$$

$$\therefore 136 = x - 4,$$

$$\text{and } x = 140;$$

$\therefore$ they put in £140, £200, and £72 respectively.

63. It is required to divide the number 34 into two such parts, that the difference between the greater and 18, shall be to the difference between 18 and the less $:: 2 : 3$.

Let x = the greater;

$$\therefore 34 - x = \text{the less,}$$

$$\text{and } x - 18 : x - 16 :: 2 : 3;$$

$$\therefore (\text{Alg. 177. 180.}) \quad x - 18 : 2 :: 2 : 1;$$

$$\therefore x - 18 = 4,$$

$$\text{and } x = 22;$$

$\therefore$ the parts are 22 and 12.

64. A Bookseller sells two books, one containing 100 sheets for 10 shillings, the other containing 50 sheets, for 6 shillings, each being bound at the same price. What was that price?

Let x = the price;

$$\text{then } 10 - x : 6 - x :: 100 : 50 :: 2 : 1;$$

$$\therefore (\text{Alg. 180.}) \quad 4 : 6 - x :: 1 : 1;$$

$$\therefore 4 = 6 - x;$$

by transposition, $x = 2$.

65. A man wished to inclose a piece of ground with palisadoes, and found that if he set them a foot asunder, he should have too few by 150; but if he set them a yard asunder, he should have too many by 70. How many had he?

Let x = the number ;
 then $x - 70 : x + 150 :: 1 : 3$,
 and (*Alg.* 181.) $x - 70 : 220 :: 1 : 2$,
 or $x - 70 : 110 :: 1 : 1$;
 $\therefore x - 70 = 110$,
 and $x = 180$.

66. A Footman, who contracted for £8 a year, and a livery suit, was turned away at the end of 7 months, and received only £2. 3s. 4d. and his livery. What was its value?

Let x = its value, in pounds ;
 then $12 : 7 :: \left(8 + x : \frac{13}{6} + x :: \right) 48 + 6x : 13 + 6x$;
 $\therefore$ (*Alg.* 180.) $5 : 7 :: 35 : 13 + 6x$,
 and $1 : 7 :: 7 : 13 + 6x$;
 $\therefore 13 + 6x = 49$;
 by transposition, $6x = 36$,
 and $x = 6$.

67. What number is that to which if 1, 5, and 13, be severally added, the first sum shall be to the second, as the second to the third?

Let x = the number required ;
 then $x + 1 : x + 5 :: x + 5 : x + 13$;
 $\therefore$ (*Alg.* 181.) $x + 1 : 4 :: x + 5 : 8$,
 and alt^{do}. $x + 1 : x + 5 :: 4 : 8 :: 1 : 2$;
 $\therefore x + 1 : 4 :: 1 : 1$,
 $x + 1 = 4$,
 and $x = 3$.

68. A Landlord let his farm for £10 a year in money and a corn-rent. When corn sold at 10s. a bushel, he received at the rate of 10 shillings an acre for his land; but when it sold at 13s. 6d. a bushel, 13 shillings an acre. Of how many bushels did the corn-rent consist?

Let x = the number of bushels ;
 then $10x + 200$ = the annual income (in shillings)
 and $\therefore x + 20$ = the number of acres ;
 also in the second case $\frac{27x + 400}{26}$ = the number of acres ;
 $\therefore \frac{27x + 400}{26} = x + 20,$
 and $27x + 400 = 26x + 520,$
 by transposition, $x = 120.$

69. When the price of a bushel of barley wanted but 3d. to be to the price of a bushel of oats as 8 to 5, nine bushels of oats were received as an equivalent for four bushels of barley, and 7s. 6d. in money. What was the price of a bushel of each?

Let $5x$ = the price of a bushel of oats ;
 $\therefore 8x - 3$ = the price of a bushel of barley ;
 $45x = 32x - 12 + 90 ;$
 by transposition, $13x = 78,$
 and $x = 6 ;$
 $\therefore$ the price of a bushel of oats = 30d.
 and the price of a bushel of barley = 45d.

70. A Countryman had two flocks of sheep, the smaller consisting entirely of ewes, each of which brought him 2 lambs. On counting them he found that the number

of lambs was equal to the difference between the two flocks. If all his sheep had been ewes, and brought forth 3 lambs apiece, his stock would have been 432. Required the number in each flock.

Let x = the number in the less;

$\therefore 2x$ = the number of lambs this flock produced =
the difference of the flocks,

and $3x$ = the number in the larger flock;

$$\therefore 4x + 3 \times 4x = 4 \times 4x = 432,$$

$$\text{and } x = 27;$$

$\therefore 27$ and 81 , are the numbers required.

71. A Market-woman bought a certain number of eggs at two a penny, and as many at three a penny, and sold them out at the rate of five for two-pence; after which she found that instead of making her money again, as she expected, she lost four-pence by them. How many eggs of each sort had she?

Let x = the number required;

then $2 : x :: 1$: the price of x eggs at 2 a penny = $\frac{x}{2}$;

in the same way, $\frac{x}{3}$ = the price of x eggs at 3 a penny.

and $5 : 2x :: 2$: the price at which she sold all, = $\frac{4x}{5}$;

$$\therefore \frac{4x}{5} + 4 = \frac{x}{2} + \frac{x}{3},$$

$$\text{and } 24x + 120 = (15x + 10x) = 25x;$$

by transposition, $x = 120$.

72. A man and his wife did usually drink out a vessel of beer in 12 days: but when the man was out, the vessel lasted the woman 30 days. In how many days would the man alone drink it out?

Let x = the number of days required ;
 then $x : 12 :: 1 : \text{the part drank by the man}$

$$\text{in 12 days} = \frac{12}{x},$$

and $30 : 12 :: 1 : \text{the part drank by the woman}$

$$\text{in 12 days} = \frac{12}{30} = \frac{2}{5};$$

$$\therefore \frac{12}{x} + \frac{2}{5} = 1,$$

$$\text{and } 60 + 2x = 5x,$$

$$\text{by transposition, } 60 = 3x,$$

$$\text{and } 20 = x.$$

73. A cistern into which water was let by two cocks A and B , will be filled by them both running together in 12 hours, and by the cock A alone in 20 hours. In what time will it be filled by the cock B alone?

Let x = the number of hours ;
 then $x : 12 :: 1 : \text{the quantity supplied by } B$

$$\text{in 12 hours} = \frac{12}{x}.$$

In the same way, the quantity supplied by A

$$\text{in 12 hours} = \frac{3}{5};$$

$$\therefore \frac{12}{x} + \frac{3}{5} = 1,$$

$$\text{and } 60 + 3x = 5x;$$

$$\text{by transposition, } 60 = 2x,$$

$$\text{and } 30 = x.$$

74. The hold of a ship contained 442 gallons of water. This was emptied out by two buckets, the greater of which, holding twice as much as the other, was emptied twice in three minutes, but the less three times in two minutes; and the whole time of emptying was 12 minutes. Required the size of each.

Let x = the number of gallons the less held;

$\therefore 2x$ = the number the greater held;

and $4x$ = the quantity thrown out by the greater in 3 minutes;

$\therefore (3 : 12 ::) 1 : 4 :: 4x : \text{the quantity thrown out in}$

12 minutes = $16x$.

In the same manner the quantity thrown out by the less in

12 minutes = $18x$;

$\therefore (18x + 16x =) 34x = 442,$

and $x = 13$;

$\therefore$ the less held 13, and the greater 26 gallons.

75. A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's three; but two of the greyhound's leaps are as much as three of the hare's. How many leaps must the greyhound take to catch the hare?

Let $3x$ = the number of leaps the greyhound must take;

$\therefore 4x$ = the number the hare takes in the same time;

$\therefore 4x + 50$ = the whole number she takes,

and $2 : 3 :: 3x : 4x + 50$;

$\therefore 9x = 8x + 100$;

by transposition, $x = 100$,

and the greyhound must take 300 leaps.

76. If 10 apples cost a penny, and 25 pears cost two-pence, and I buy 100 apples and pears for nine-pence halfpenny, how many of each shall I have?

Let x = the number of apples ;

$\therefore 100 - x$ = the number of pears ;

and $10 : x :: 1 : \text{the price of } x \text{ apples} = \frac{x}{10}$.

In the same manner the price of the pears = $\frac{200 - 2x}{25}$;

$$\therefore \frac{x}{10} + \frac{200 - 2x}{25} = \frac{19}{2} ,$$

$$\text{and } 5x + 400 - 4x = 475 ;$$

by transposition, $x = 75$;

$\therefore$ the number of apples is 75, and pears 25.

77. A person has two sorts of wine, one worth 20 pence a quart, and the other 12 pence ; from which he would mix a quart to be worth 14 pence. How much of each must he take ?

Let x = the quantity of the first, the whole quart
being represented by unity ;

$\therefore 1 - x$ = the quantity of the second ;

also $20x$ = the value of the first,

and $12 - 12x$ = the value of the second ;

$$\therefore 20x + 12 - 12x = 14 ;$$

and by transposition, $8x = 2$;

$$\therefore x = \frac{1}{4} ;$$

$\therefore$ he must take $\frac{1}{4}$ of the first, and $\frac{3}{4}$ of the second.

78. A person engaged to reap a field of 35 acres, consisting partly of wheat, and partly of rye. For every acre of rye he received 5 shillings ; and what he received for an acre of wheat augmented by one shilling, is to what he received for an acre of rye as 7 to 3. For his whole labour he received £13. Required the number of acres of each sort.

Let x = the number of acres of wheat;
 $\therefore 35 - x$ = the number of acres of rye;
 and $175 - 5x$ = the price of reaping them.
 Now $3 : 7 :: 5 : 1$ + the price of reaping an acre of
 wheat = $\frac{35}{3}$;

$\therefore$ the price of reaping an acre of wheat = $\frac{32}{3}$,
 and the price of reaping all the wheat = $\frac{32x}{3}$;

$$\therefore \frac{32x}{3} + 175 - 5x = 260,$$

$$\text{and } 32x + 525 - 15x = 780;$$

$$\text{by transposition, } 17x = 255,$$

$$\text{and } x = 15;$$

$\therefore$ there were 15 acres of wheat, and 20 of rye.

79. Two pieces of cloth of equal goodness, but of different lengths, were bought, the one for £5, the other for £6. 10s. Now if the lengths of both pieces were increased by 10, the numbers resulting would be in the proportion of 5 to 6. How long was each piece, and how much did they cost a yard?

Let x = the number of 10 shillings that each yard cost,
 then $\frac{10}{x}$ = the length of the least,

and $\frac{13}{x}$ = the length of the longest;

$$\text{also } \frac{10}{x} + 10 : \frac{13}{x} + 10 :: 5 : 6;$$

$$\therefore (\text{Alg. 181.}) 10 \cdot \left(\frac{1}{x} + 1\right) : \frac{3}{x} :: 5 : 1,$$

$$\text{and } (\text{Alg. 186.}) 2 \cdot \left(\frac{1}{x} + 1\right) : \frac{3}{x} :: 1 : 1;$$

$$\therefore 2 \cdot \frac{x+1}{x} = \frac{3}{x},$$

$$\text{and } 2x + 2 = 3;$$

$$\text{by transposition, } 2x = 1,$$

$$\text{and } x = \frac{1}{2};$$

$\therefore$ the price is 5s. and the lengths are 20 and 26 yards.

80. A General, whose horse was $\frac{1}{3}$ of his foot, after a defeat found, that before the battle $\frac{1}{4} - 120$ of his foot, and $\frac{1}{4} + 120$ of his horse had deserted; $\frac{1}{4}$ of his whole army was in garrison; and $\frac{3}{4}$ remained, the rest being either taken prisoners or slain. Now $300 +$ the number slain $= \frac{1}{4}$ the foot he had at first. Of how many did his whole army consist?

Let $x =$ the number of horse;

$\therefore 3x =$ the number of foot,

and $4x =$ the whole army;

also $\frac{3x}{2} - 300 =$ the number slain;

$$\therefore \frac{x}{4} - 120 + \frac{x}{12} + 120 + x + \frac{3x}{2} + \frac{3x}{2} - 300 = 4x,$$

$$\text{and } 3x + x + 48x - 3600 = 48x;$$

$$\text{by transposition, } 4x = 3600,$$

$$\text{and } x = 900;$$

$\therefore$ the whole army consisted of 3600 men; viz. 900 horse, and 2700 foot.

81. Two persons A and B have both the same annual income. A lays by $\frac{1}{3}$ th of his; but B by spending £80 per annum more than A , at the end of 4 years finds himself £220 in debt. What did each receive and expend annually?

Let $5x =$ their annual income;

$\therefore 4x = A$'s annual expenditure,

and $4x + 80 = B$'s annual expenditure;

$\therefore (4x + 80 - 5x =) 80 - x =$ the debt B annually incurs;

$$\therefore 320 - 4x = 220;$$

by transposition, $4x = 100$,

$$\text{and } x = 25;$$

$\therefore$ their annual income is £125;

A 's annual expenditure is £100, and B 's £180.

82. A person at play won twice as much as he began with, and then lost 16 shillings. After this he lost four-fifths of what remained, and then won as much as he began with, and counting his money, found he had 80 shillings. What sum did he begin with?

Let $x =$ the number of shillings he began with;

then $3x =$ the sum he had, after winning $2x$,

and $3x - 16 =$ the sum remaining after the next loss.

Now since he lost $\frac{4}{5}$ of this, $\frac{3x - 16}{5} =$ the sum remaining;

$$\therefore \frac{3x - 16}{5} + x = 80,$$

$$\text{and } 3x - 16 + 5x = 400;$$

by transposition, $8x = 416$,

$$\text{and } x = 52.$$

83. Having lost one-third of my money at play, I won 3 times as much as I had left, half as much money as I began with, and £50; and then found I had as much above £100, as the sum I began with was below £100. What sum did I begin with?

Let $6x$ = the number of pounds required;

then $4x$ = the sum remaining after $\frac{1}{3}$ was lost,

and $12x + 3x + 50$ = the sum afterwards won;

$\therefore (12x + 3x + 50 + 4x =) 19x + 50$ = the whole sum he had,

and $19x + 50 - 100 = 100 - 6x$;

by transposition, $25x = 150$,

and $x = 6$;

$\therefore$ he began with 36 pounds.

84. A and B began to pay their debts. A 's money was at first two-thirds of B 's; but after A had paid £1 less than two-thirds of his money, and B £1 more than seven-eighths of his, it was found that B had only half as much as A had left. What sum had each at first?

Let $2x$ and $3x$ = the sums A and B had respectively,

then after payment, A had $\frac{2x}{3} + 1$ } remaining;
and B had $\frac{3x}{8} - 1$ }

$$\therefore \frac{2x}{3} + 1 = \frac{3x}{4} - 2;$$

$$\therefore 8x + 12 = 9x - 24;$$

by transposition, $36 = x$;

$\therefore A$ had £72, and B had £108.

85. It is required to divide the number 36 into three such parts, that one-half of the first, one-third of the second, and one-fourth of the third may be equal to each other.

Let $2x =$ the first part ;

$\therefore x = \frac{1}{3}$ part of the second, and $3x =$ the second ;

also $x = \frac{1}{4}$ part of the third, and $\therefore 4x =$ the third ;

$$\therefore (2x + 3x + 4x =) 9x = 36.$$

$$\text{and } x = 4 ;$$

$\therefore$ the parts are 8, 12, and 16.

86. Divide the number 116 into four such parts, that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the result in each case shall be the same.

Let $x =$ the third ;

$\therefore 3x =$ half the fourth,

and $6x =$ the fourth ;

whence $3x + 4 =$ the second,

and $3x - 5 =$ the first ;

$$\therefore 3x - 5 + 3x + 4 + x + 6x = 116 ;$$

by transposition, $13x = 117,$

$$\text{and } x = 9 ;$$

$\therefore$ the parts are 22, 31, 9, and 54.

87. A Gentleman had some of his horses at grass at 3 shillings each a week, and the rest at livery stables at 10 shillings each a week. The horses in the stables cost him twice as much a week as the horses at grass. But he finds that if he had sent 3 horses to grass out of the stables, the expence of the stables would have been only 6 shillings a week more than the grass. How many horses had he?

Let x = the number of horses at grass ;

$\therefore 3x$ = the weekly expense of these,

and $6x$ = - - - - - of horses in stables ;

$$\therefore \frac{6x}{10} = \text{their number ;}$$

also $6x - 30$ = the expense of the stables after 3 horses were sent out to grass,

and $3x + 9$ = the expense of the horses at grass, when 3 more were added ;

$$\therefore 6x - 30 (= 3x + 9 + 6) = 3x + 15 ;$$

by transposition, $3x = 45$;

$$\text{and } x = 15 ;$$

$\therefore$ there were 15 at grass, and 9 in the stables.

88. A Silversmith received in payment for a certain weight of wrought plate, the price of which was £10, the same weight of unwrought plate, and £3. 15s. besides. At another time he exchanged 12 oz. of wrought plate of the same workmanship as before for 8 oz. of unwrought (for which he allowed the same price as before), and £2. 16s. in money. What was the price of wrought plate per ounce, and the weight of the first sold ?

Let x = the number of ounces ;

$$\therefore \frac{200}{x} = \text{the price of an oz. wrought,}$$

$$\text{and } \frac{125}{x} = \text{the price of an oz. unwrought ;}$$

$$\therefore \frac{2400}{x} = \frac{1000}{x} + 56 ;$$

$$\text{by transposition, } \frac{1400}{x} = 56.$$

$$\text{and } \frac{25}{x} = 1;$$

$$\therefore x = 25;$$

whence there were 25 ounces; and the price was 8 shillings per oz.

89. In changing a bill of £85 into guineas and shillings (the number of shillings being $\frac{1}{4}$ number of guineas) on examination they all proved adulterated below the standard value, to the amount in the whole of £8. 5s. To make up the deficiency, nine more such guineas were paid; and four such shillings and three good ones returned. Required the number and average value of the guineas and shillings paid at first.

$$\begin{aligned} £85 = 85s. \times 20 &= \text{the N}^{\circ} \text{ of shillings} + 4 \times 21 \times \text{N}^{\circ} \text{ of shillings} \\ &= 85 \times \text{the number}; \end{aligned}$$

$$\therefore \text{the number of shillings} = 20,$$

$$\text{and the number of guineas} = 80;$$

Let x = the value of an adulterated guinea;

$$\therefore (1700 - 165 =) 1535 = 80x + 20 \times \text{value of an adulterated shilling},$$

$$\text{and } \therefore \text{the value of an adulterated shilling} = \frac{1535 - 80x}{20} =$$

$$\frac{307 - 16x}{4};$$

$$\therefore 165 = 9x - 3 - 307 + 16x = 25x - 310;$$

$$\text{by transposition, } 475 = 25x,$$

$$\text{and } 19 = x;$$

$\therefore$ the value of an adulterated guinea = 19s. and the value of

$$\text{an adulterated shilling} = \frac{307 - 16x}{4} = \frac{7}{4} = 9d.$$

90. Before noon, a clock which is too fast, and points to afternoon time, is put back five hours and forty minutes; and it is observed that the time before shown is to the true time as 29 to 105. Required the true time.

Let x = the time the clock pointed to;

$$\text{then } x : x + 6\frac{1}{3} :: 29 : 105;$$

$$(\text{Alg. 181.}) \quad x : \frac{19}{3} :: 29 : 76,$$

$$\text{and } x : \frac{1}{3} :: 29 : 4;$$

$$\text{whence } x = \frac{29}{12} = 2^{\text{h}} 25';$$

if $\therefore$ this be added to $6^{\text{h}} 20'$, the true time is $8^{\text{h}} 45'$, or $15'$ before 9.

91. The crew of a ship consisted of her complement of sailors and a number of soldiers. Now there were 22 seamen to every 3 guns and 10 over. Also the whole number of hands was 5 times the number of soldiers and guns together. But after an engagement, in which the slain were one-fourth of the survivors, there wanted 5, to be 13 men to every 2 guns. Required the number of guns, soldiers, and sailors.

Let $3x$ = the number of guns;

then $22x + 10$ = the number of seamen,

and since, seamen + soldiers = 5 . soldiers + 15 x ;

$$\therefore \text{ the number of soldiers} = \frac{1}{4} (22x + 10 - 15x) = \frac{7x + 10}{4},$$

$$\text{and the complement} = \frac{7x + 10}{4} + 22x + 10 = \frac{95x + 50}{4},$$

$$\text{and the survivors were } \frac{3}{4} \left(\frac{95x + 50}{4} \right) = 19x + 10;$$

$$\therefore \frac{3x}{2} \cdot 13 - 5 = 19x + 10;$$

$$\text{by transposition, } \frac{x}{2} = 15,$$

$$\text{and } x = 30;$$

$$\therefore \text{the number of guns} = 90;$$

$$\text{the number of seamen} = 30 \times 22 + 10 = 670,$$

$$\text{and the number of soldiers} = \frac{7 \times 30 + 10}{4} = 55.$$

92. A Shepherd, in time of war, was plundered by a party of soldiers, who took $\frac{1}{4}$ of his flock, and $\frac{1}{4}$ of a sheep; another party took from him $\frac{1}{3}$ of what he had left, and $\frac{1}{3}$ of a sheep; then a third party took $\frac{1}{2}$ of what now remained, and $\frac{1}{2}$ of a sheep. After which he had but 25 sheep left. How many had he at first?

Let x = the number he had at first;

$$\text{then } \frac{x+1}{4} = \text{the number the first party took away,}$$

$$\text{and } \therefore \frac{3x-1}{4} = \text{the number remaining.}$$

Now the second party took away $\frac{1}{3}$ of these + $\frac{1}{3}$ of a sheep;
 $\therefore$ there remained

$$\frac{2}{3} \left(\frac{3x-1}{4} \right) - \frac{1}{3} = \frac{3x-1}{6} - \frac{1}{3} = \frac{3x-3}{6} = \frac{x-1}{2};$$

then the third party took away half of these + $\frac{1}{2}$ of a sheep;

$$\therefore \text{there remained } \frac{x-1}{4} - \frac{1}{2} = \frac{x-3}{4};$$

$$\text{whence } \frac{x-3}{4} = 25,$$

$$\text{and } x-3 = 100;$$

$$\text{by transposition, } x = 103.$$

93. A man being at play lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had, and won 2 shillings; lastly he lost $\frac{1}{7}$ of what he then had; this done he had but 12 shillings left. What had he at first?

Let $4x$ = the number of shillings required;
 then after the first loss he had $3x$, and afterwards $3x + 3$;
 after the second loss he had $\frac{2}{3} \cdot (3x + 3) = 2x + 2$, and
 afterwards $2x + 4$.

Having lost $\frac{1}{7}$ of this, he had $\frac{6}{7} \cdot (2x + 4) = \frac{12x + 24}{7}$,

$$\text{and } \therefore \frac{12x + 24}{7} = 12,$$

$$\text{and } 12x + 24 = 7 \times 12;$$

$$\text{by transposition, } 12x = 5 \times 12,$$

$$\text{and } x = 5;$$

$\therefore$ he had at first 20 shillings.

4. A Trader maintained himself for 3 years at the expence of £50 a year; and in each of those years augmented that part of his stock which was not so expended by $\frac{1}{3}$ thereof. At the end of the third year his original stock was doubled. What was that stock?

Let x = the number of pounds required;
 then $x - 50$ = the sum not expended; and with this he
 traded;

$$\therefore \frac{x - 50}{3} = \text{his gain the first year,}$$

and $\frac{4}{3} \cdot (x - 50)$ = the sum he had at the end of the first
 year;

$$\therefore \frac{4x-200}{3} - 50 = \frac{4x-350}{3} = \text{the sum he traded with the second year;}$$

$$\therefore \frac{4}{3} \cdot \frac{4x-350}{3} = \frac{16x-1400}{9} = \text{the sum he had at the end of the second year;}$$

$$\text{and } \frac{16x-1400}{9} - 50 = \frac{16x-1850}{9} = \text{the sum he traded with the third year;}$$

$$\therefore \frac{4}{3} \cdot \frac{16x-1850}{9} = \text{the sum he had at the end of the third year;}$$

$$\text{whence } \frac{4}{3} \cdot \frac{16x-1850}{9} = 2x,$$

$$\text{and } 32x - 3700 = 27x;$$

$$\text{by transposition, } 5x = 3700,$$

$$\text{and } x = 740.$$

95. A Merchant buys a cask of brandy for £48, and sells a quantity exceeding three-fourths of the whole by 2 gallons at a profit of £25 per cent. He afterwards sells the remainder at such a price as to clear £60 per cent. by the whole transaction; and, had he sold the whole quantity at the latter price, he would have gained £175 per cent. Required the number of gallons contained in the cask.

Let $4x$ = the number of gallons;

then $\frac{12}{x}$ = the original price per gallon (in pounds),

and $100 : 125 :: \frac{12}{x} : \text{the first price of sale} = \frac{15}{x},$

and $100 : 275 :: \frac{12}{x} : \text{the latter price of sale} = \frac{33}{x};$

$$\therefore (3x + 2) \cdot \frac{15}{x} + (x - 2) \cdot \frac{33}{x} - 48 = \text{whole gain} = 30 - \frac{36}{x};$$

$$\text{hence } 100 : 60 :: 48 : 30 - \frac{36}{x},$$

$$\text{or } 5 : 3 :: 8 : 5 - \frac{6}{x},$$

$$\therefore 24 = 25 - \frac{30}{x};$$

$$\text{by transposition, } \frac{30}{x} = 1,$$

$$\text{and } \therefore x = 30,$$

and the number required $= 4x = 120$ gallons.

96. Water flows uniformly into a cistern, capable of containing 720 gallons, through a pipe; and at the same time is discharged by a pump, worked by three men, who take four strokes in a minute; but this not being sufficient, the cistern becomes full in 6 hours; they therefore now put in another pump, of such power that the quantity discharged at one stroke by this pump is to the quantity discharged at one stroke by the former $:: 2 : 3$; but being obliged to detach one of their number to work the pump, the former pump makes only 10 strokes in 3 minutes, and the latter 5 strokes in two minutes; by which means the cistern is emptied in 12 hours. How much water was discharged by each pump at one stroke? and how much flowed in through the pipe in one minute?

Let $3x$ = the number of gallons discharged by the first pump at one stroke;

$\therefore 2x$ = the number discharged by the second,

and $12x$ = the quantity discharged by the first in one minute, when 3 men worked;

$\therefore 6 \times 60 \times 12x =$ the quantity discharged in six hours ;

$\therefore 6 \times 60 \times 12x + 720 = 720 \cdot (6x + 1) =$ the quantity introduced through the pipe in that time.

Now when the additional pump is worked,

$10x =$ quantity discharged by the first in one minute ;

and $5x =$ the quantity discharged by the second in one minute ;

$\therefore 15x \times 12 \times 60 =$ the whole quantity discharged in 12 hours ;

$\therefore 15x \times 720 = 720 \times 2 \cdot (6x + 1) + 720 ;$

or $15x = 12x + 3 ;$

by transposition, $3x = 3,$

and $x = 1 ;$

$\therefore$ the first discharged 3 gallons, and the second 2, at one stroke ; and the quantity introduced by the pipe in one

minute $= \frac{720 \times (6x + 1)}{6 \times 60} = 2 \times 7 = 14$ gallons.

97. A poor man with a wife and seven children, found during a scarcity that he could only earn sufficient to procure $\frac{1}{4}$ of a white loaf of bread *per* day for each of his family, himself included. He therefore applied to the parish-officers for assistance, by whom being allowed a daily sum $= \frac{1}{2}$ his earnings, and mixed bread being made by order of Parliament, which was cheaper than white in the proportion of 4 to 5, he was now enabled to procure $\frac{1}{4}$ of a mixed loaf *per* day for each of the family (himself still included) and had 1s. $7\frac{1}{2}d.$ over. Required the sum allowed him by the parish.

Let $x =$ the price of a white loaf (in pence) ;

$\therefore \frac{9x}{4} =$ what he earned,

and $\frac{9x}{8} =$ what the parish allowed him ;

also $\frac{4x}{5}$ = the price of a mixed loaf;

$$\therefore \frac{12x}{5} + \frac{39}{2} = \left(\frac{9x}{4} + \frac{9x}{8} = \right) \frac{27x}{8},$$

$$\text{and } 96x + 780 = 135x;$$

by transposition, $780 = 39x$,

$$\text{and } 20 = x;$$

whence it appears that he earned 45 pence, and had $22\frac{1}{2}d.$
allowed him by the parish.

SECTION VII.

Examples of the Solution of Problems producing Simple Equations involving two unknown Quantities.

1. AFTER *A* had won four shillings of *B*, he had only half as many shillings as *B* had left. But had *B* won six shillings of *A*, then he would have had three times as many as *A* would have had left. How many had each?

Let x = the number of shillings *A* had,

and y = the number *B* had;

$$\text{then } y - 4 = 2x + 8,$$

$$\text{and } y + 6 = 3x - 18;$$

$$\therefore \text{ by subtraction, } 10 = x - 26,$$

$$\text{and by transposition, } 36 = x,$$

$$\text{and } y - 4 = 80; \therefore y = 84; \therefore A \text{ had } 36, \text{ and } B \text{ } 84.$$

2. A person bought a quantity of brandy and rum for £19. 4s. and gave for the brandy 9 shillings, and rum 6 shillings *per* bottle. He found however that he could have bought as many bottles of rum as he now had of brandy, and as many of brandy as he now had of rum for £1. 13s. less. How much was bought?

Let x = the number of bottles of brandy,

and y = the number of bottles of rum;

$$\text{then } 9x + 6y = 384,$$

$$\text{and } 6x + 9y = 351;$$

$$\therefore \text{ by addition, } 15x + 15y = 735,$$

$$\text{and } x + y = 49.$$

$$\text{But since } 3x + 2y = 128,$$

$$\text{and } 2x + 2y = 98;$$

$$\therefore \text{ by subtraction, } x = 30,$$

$$\text{and } y = 49 - x = 19;$$

$\therefore$ he bought 30 bottles of brandy, and 19 of rum.

3. What fraction is that, to the numerator of which if 4 be added, the value is one-half, but if 7 be added to the denominator, its value is one-fifth?

$$\text{Let } \frac{x}{y} = \text{the fraction required};$$

$$\text{then } \frac{x+4}{y} = \frac{1}{2}, \text{ and } \therefore 2x + 8 = y;$$

$$\text{also } \frac{x}{y+7} = \frac{1}{5}, \text{ and } \therefore 5x = y + 7;$$

$$\text{by subtraction, } 3x - 8 = 7;$$

$$\text{by transposition, } 3x = 15;$$

$$\text{and } x = 5;$$

$$\therefore y = 2x + 8 = 18,$$

$$\text{and the fraction is } \frac{5}{18}.$$

4. Find two numbers, the greater of which shall be to the less as their sum to 42, and as their difference to 6.

$$\text{Let } x \text{ and } y = \text{the numbers};$$

$$\text{then } x : y :: x + y : 42,$$

$$\text{and } x : y :: x - y : 6.$$

But ratios which are equal to the same ratio are equal to each other;

$$\begin{aligned}
 &\therefore x + y : 42 :: x - y : 6, \\
 \text{alt}^{\text{do}}. &x + y : x - y :: (42 : 6 ::) 7 : 1; \\
 &\therefore (\text{Alg. 182.}) 2x : 2y :: 8 : 6, \\
 &\text{and } x : y :: 4 : 3; \\
 &\therefore x = \frac{4y}{3}, \\
 &\text{and } 4 : 3 :: \frac{y}{3} : 6; \\
 &\therefore y = 24, \\
 &\text{and } x = \frac{4y}{3} = 32; \\
 &\therefore \text{the numbers are 32 and 24.}
 \end{aligned}$$

5. What two numbers are those, whose difference, sum, and product, are as the numbers 2, 3, and 5, respectively?

$$\begin{aligned}
 &\text{Let } x \text{ and } y = \text{the numbers;} \\
 &\text{then } x - y : x + y :: 2 : 3; \\
 &\therefore 2x : 2y :: 5 : 1, \\
 &\text{or } x : y :: 5 : 1; \\
 &\text{also } x + y : xy :: 3 : 5, \\
 &\text{or } 6y : xy :: 3 : 5; \\
 &\therefore 2 : x :: 1 : 5, \\
 &\text{and } \therefore x = 10, \\
 &\text{and } y = \frac{x}{5} = 2; \\
 &\therefore \text{the numbers are 10 and 2.}
 \end{aligned}$$

6. *A* and *B* playing at bowls, says *A* to *B*, If you will give me a guinea, I will bet you half a crown to eighteen

pence on each game, and will play 36 games together. *B* won his guinea back again, and £1. 17s. besides. How many games did each win?

Let x = the number of games *A* won,
and y = the number *B* won;

$$\therefore y + x = 36,$$

$$\text{and } 3y + 3x = 108;$$

$$\text{but } 5y - 3x = 116;$$

$$\therefore \text{ by addition, } 8y = 224,$$

$$\text{and } y = 28;$$

$$\therefore x = 36 - y = 8;$$

$$\therefore A \text{ won 8, and } B \text{ 28 games.}$$

7. A person exchanged 12 bushels of wheat for 8 bushels of barley, and £2. 16s.; offering at the same time to sell a certain quantity of wheat for an equal quantity of barley, and £3. 15s. in money, or for £10 in money. Required the prices of the wheat and barley per bushel.

Let x = the price of wheat per bushel, in shillings,
and y = the price of barley;

then $\frac{200}{x}$ = the number of bushels in the second offer;

$$\therefore 12x = 8y + 56;$$

$$\text{and } \frac{200}{x} \times y = 125;$$

$$\therefore 8y = 5x,$$

$$\text{and } \therefore 12x = 5x + 56;$$

$$\text{by transposition, } 7x = 56,$$

$$\text{and } x = 8;$$

$$\therefore y = 5;$$

$\therefore$ the prices of wheat and barley *per* bushel were 8 and 5 shillings, respectively.

8. A Vintner sold at one time 20 dozen of port wine, and 30 of sherry, and for the whole received £120; and at another time sold 30 dozen of port, and 25 of sherry at the same prices as before, and for the whole received £140. What was the price of a dozen of each sort of wine?

Let x = the price of a dozen of port,
and y = - - - - - of sherry;

$$\therefore 20x + 30y = 120, \text{ or } 2x + 3y = 12,$$

$$\text{and } 30x + 25y = 140, \text{ or } 6x + 5y = 28.$$

Multiplying the first equation by 3,

$$6x + 9y = 36,$$

$$\text{but } 6x + 5y = 28;$$

$$\therefore \text{ by subtraction, } 4y = 8,$$

$$\text{and } y = 2;$$

$$\text{whence } 2x = 12 - 3y = 12 - 6 = 6,$$

$$\text{and } x = 3;$$

- $\therefore$ the prices of port and sherry *per* dozen were £3 and £2 respectively.

9. A and B severally cut packs of cards, so as to cut off less than they left. Now the number of cards left by A added to the number cut off by B make 50; also the number of cards left by both exceed the number cut off, by 64. How many did each cut off?

Let x = the number cut off by A ;

$$\therefore 52 - x = \text{the number left by him.}$$

Let y = the number cut off by B ;

$$\therefore 52 - y = \text{the number left by him;}$$

$$\therefore x + y = \text{the whole number cut off,}$$

$$\text{and } 104 - (x + y) = \text{the whole number left,}$$

whence $104 - 2 \cdot (x + y) = 64$;

by transposition, $2 \cdot (x + y) = 40$,

and $x + y = 20$.

Now $52 - x + y = 50$;

$\therefore$ by transposition, $x - y = 2$,

but $x + y = 20$;

$\therefore$ by addition, $2x = 22$,

and $x = 11$;

by subtraction, $2y = 18$,

and $y = 9$;

$\therefore$ A cut off 11, and B 9.

-
10. A countryman, being employed by a poulterer to drive a flock of geese and turkeys to London, in order to distinguish his own from any he might meet on the road, pulled 3 feathers out of the tail of each turkey, and one out of the tail of each goose, and upon counting them, found that the number of turkeys' feathers exceeded twice those of the geese by 15. Having bought 10 geese and sold 15 turkeys by the way, he was surprised to find, as he drove them into the poulterer's yard, that the number of geese exceeded the number of turkeys in the proportion of 7 to 3. Required the number of each at first.

Let x = the number of turkeys;

$\therefore 3x$ = the number of feathers from their tails;

let y = the number of geese; and $\therefore$ of the feathers,

and $3x - 2y = 15$.

Also $y + 10 : x - 15 :: 7 : 3$;

$\therefore 3y + 30 = 7x - 105$;

by transposition, $7x - 3y = 135$,

$$\text{and } 14x - 6y = 270;$$

but from the first equation, $9x - 6y = 45$;

$$\therefore \text{ by subtraction, } 5x = 225,$$

$$\text{and } x = 45;$$

$$\therefore 2y = 3x - 15 = 135 - 15 = 120,$$

$$\text{and } y = 60;$$

$\therefore$ there were 45 turkeys, and 60 geese.

11. A Farmer with 28 bushels of barley at *2s. 4d.* a bushel, would mix rye at 3 shillings *per* bushel, and wheat at 4 shillings *per* bushel, so that the whole mixture may consist of 100 bushels, and be worth *3s. 4d. per* bushel. How many bushels of rye, and how many of wheat, must he mix with the barley?

Let x = the number of bushels of rye,

and y = the number of wheat;

then the value of the barley = 196 (four-pences),

of the wheat = $12y$,

of the rye = $9x$;

$$\therefore 196 + 9x + 12y = 1000;$$

$$\text{by transposition, } 9x + 12y = 804,$$

$$\text{and } 3x + 4y = 268.$$

$$\text{Now } x + y + 28 = 100,$$

$$\text{and by transposition, } x + y = 72;$$

$$\therefore 3x + 3y = 216,$$

$$\text{but } 3x + 4y = 268;$$

$$\therefore \text{ by subtraction, } y = 52,$$

$$\text{and } x = 72 - y = 20.$$

Hence he must mix 20 bushels of rye, and 52 of wheat.

12. *A* and *B* speculate with different sums; *A* gains £150, *B* loses £50, and now *A*'s stock is to *B*'s as 3 to 2. But had *A* lost £50, and *B* gained £100, then *A*'s stock would have been to *B*'s as 5 to 9. What was the stock of each?

Let $x = A$'s stock,

and $y = B$'s;

then $x + 150 : y - 50 :: 3 : 2$;

$\therefore 2x + 300 = 3y - 150$,

and by transposition, $3y - 2x = 450$;

also $x - 50 : y + 100 :: 5 : 9$;

$\therefore 9x - 450 = 5y + 500$;

by transposition, $9x - 5y = 950$;

multiplying this equation by 3, and that found above by 5,

$27x - 15y = 2850$,

and $15y - 10x = 2250$;

$\therefore$ by addition, $17x = 5100$,

and $x = 300$;

$\therefore 3y = 2x + 450 = 1050$,

and $y = 350$;

$\therefore A$'s was £300, and B 's 350.

13. A Merchant having mixed a certain number of gallons of brandy and water, found that if he had mixed 6 gallons more of each, he would have put into the mixture 7 gallons of brandy for every 6 of water; but if he had mixed 6 less of each, he would have put in 6 gallons of brandy for every 5 of water. How many of each did he mix?

Let $x =$ the number of gallons of brandy,

and $y =$ the number of gallons of water;

then $x + 6 : y + 6 :: 7 : 6$;

$\therefore$ (Alg. 181.) $x + 6 : x - y :: 7 : 1$,

but $x - 6 : y - 6 :: 6 : 5$,

and $\therefore x - y : x - 6 :: 1 : 6$,

and since $x + 6 : x - y :: 7 : 1$;

$\therefore$ *ex æquali*, $x + 6 : x - 6 :: 7 : 6$,

and $2x : 12 :: 13 : 1$,

or $x : 6 :: 13 : 1$;

$\therefore x = 78$;

whence $84 : y + 6 :: 7 : 6$,

or $12 : y + 6 :: 1 : 6$;

$\therefore y + 6 = 72$,

and $y = 66$;

$\therefore$ he mixed 78 gallons of brandy with 66 of water.

14. A person had a bag of money worth £93; but a servant having robbed him of one-sixth of his moidores, and three-fifths of his guineas, left him only £54. 15s. How many moidores and guineas had he at first?

Let x = the number of guineas,

and y = the number of moidores;

then $\frac{45y}{6} + \frac{14x}{5} = 365$,

and $\therefore 75y + 28x = 3650$;

also $7x + 9y = 620$;

$\therefore 36y + 28x = 2480$;

but since $75y + 28x = 3650$;

$\therefore$ by subtraction, $39y = 1170$,

and $y = 30$;

also $7x = 620 - 9y = 620 - 270 = 350$;

$\therefore x = 50$;

$\therefore$ he had 50 guineas, and 30 moidores.

15. A Vintner bought 6 dozen of port wine and 3 dozen of white for 12 guineas; but the price of each afterwards falling a shilling *per* bottle, he had 20 bottles of port, and 3 dozen and 8 bottles of white more, for the same sum. What was the price of each at first?

Let x = the price of the port } *per* bottle (in shillings),
 y = - - - - - white }

then $72x + 36y = 252 = 92x + 80y - 172$,

and $\therefore$ by transposition, $20x + 44y = 172$,

or $5x + 11y = 43$.

Now, since $72x + 36y = 252$;

$\therefore 2x + y = 7$,

and $22x + 11y = 77$,

but $5x + 11y = 43$;

$\therefore$ by subtraction, $17x = 34$,

and $x = 2$;

whence $y = 7 - 2x = 7 - 4 = 3$;

$\therefore$ the price of port was 3*s.* and of white 2*s.* *per* bottle.

16. A rectangular bowling-green having been measured, it was observed, that if it were 5 feet broader, and 4 feet longer, it would contain 116 feet more: but if it were 4 feet broader, and 5 feet longer, it would contain 113 feet more. Required the length and breadth.

Let x = the number of feet in length,

and y = the number of feet in breadth;

then $(x + 4) \cdot (y + 5) = xy + 5x + 4y + 20 = 116 + xy$,

and $\therefore 5x + 4y = 96$;

also $(x + 5) \cdot (y + 4) = xy + 4x + 5y + 20 = 113 + xy$;

$\therefore 4x + 5y = 93$;

multiplying the former equation by 4, and the latter by 5,

$$20x + 16y = 384,$$

$$\text{and } 20x + 25y = 465;$$

by subtraction, $9y = 81,$

$$\text{and } y = 9;$$

$$\therefore 4x = 93 - 5y = 93 - 45 = 48,$$

$$\text{and } x = 12;$$

$\therefore$ the length was 12, and the breadth 9 feet.

17. Find two numbers in the proportion of 5 to 7, to which two other required numbers in the proportion of 3 to 5 being respectively added, the sums shall be in the proportion of 9 to 13; and the difference of those sums = 16.

Let $5x$ and $7x$ = the two first numbers,

and $3y$ and $5y$ = the others;

$$\text{then } 5x + 3y : 7x + 5y :: 9 : 13;$$

$$\therefore 5x + 3y : 2x + 2y :: 9 : 4;$$

$$\text{or } 5x + 3y : x + y :: 9 : 2,$$

$$\text{and } 10x + 6y = 9x + 9y;$$

$$\text{by transposition, } x = 3y;$$

$$\text{but } 2x + 2y = 16;$$

$$\therefore (6y + 2y) = 8y = 16;$$

$$\therefore y = 2,$$

$$\text{and } x = 6;$$

whence, the two first numbers are 30 and 42; the two others, 6 and 10.

18. A Merchant finds that if he mixes sherry and brandy in quantities which are in the proportion of 2 to 1, he can sell the mixture at 78 shillings a dozen; but if the proportion be as 7 to 2, he must sell it at 79 shillings a dozen. Required the price of each liquor.

Let x = the price of the sherry } per dozen ;
 y = the price of the brandy }

$$\text{then } 2x + y = 3 \times 78 = 234,$$

$$\text{and } 7x + 2y = 9 \times 79 = 711 ;$$

$$\begin{array}{rcl} \text{but the first equation} & \} & \\ \text{being multiplied by 2,} & \} & 4x + 2y = 468 ; \end{array}$$

$$\therefore \text{ by subtraction, } 3x = 243,$$

$$\text{and } x = 81 ;$$

$$\text{whence, } y = 234 - 2x = 234 - 162 = 72 ;$$

$\therefore$ the price of the sherry was 81s., and of the brandy 72s.

19. A Corn-factor mixes wheat-flour, which costs him 10 shillings a bushel, with barley-flour, which costs him 4 shillings a bushel, in such proportion, as to gain $43\frac{3}{4}$ per cent., by selling the mixture at 11 shillings a bushel. Required the proportion.

Let the proportion be $x : y$;

then $10x + 4y$ = the cost of $x + y$ bushels,

and $11x + 11y$ = the selling price ;

$$\therefore x + 7y = \text{the gain ;}$$

whence,

$$10x + 4y : x + 7y :: 100 : 43\frac{3}{4} :: 400 : 175 :: 16 : 7,$$

$$\text{and } 5x + 2y : x + 7y :: 8 : 7 ;$$

$$\therefore 35x + 14y = 8x + 56y ;$$

$$\text{by transposition, } 27x = 42y,$$

$$\text{and } 9x = 14y ;$$

$$\therefore x : y :: 14 : 9,$$

and $\therefore$ he must mix 14 bushels of wheat-flour with 9 of barley.

20. A number consisting of 2 digits when divided by 4, gives a certain quotient and a remainder of 3 ; when divided

by 9 gives another quotient and a remainder of 8. Now the *value* of the digit on the left hand is equal the quotient which was got when the number was divided by 9; and the other digit is equal $\frac{1}{7}$ th of the quotient got when the number was divided by 4. Required the number.

Let x and y = the digits in order;

then $10x + y$ = the number,

$$\text{and } \frac{10x + y}{9} = x + \frac{8}{9};$$

$$\therefore 10x + y = 9x + 8;$$

by transposition, $x + y = 8$;

$$\text{also } \frac{10x + y}{4} = \frac{3}{4} + 17y;$$

$$\therefore 10x + y = 3 + 68y;$$

by transposition, $10x - 67y = 3$;

but from the preceding equation, $10x + 10y = 80$;

$$\therefore \text{by subtraction, } 77y = 77,$$

$$\text{and } y = 1;$$

$$\therefore x = 8 - y = 7,$$

and the number required is 71.

21. A man and his wife could drink a barrel of beer in 15 days. After drinking together 6 days, the woman alone drank the remainder in 30 days. In what time would either alone drink a barrel?

Let x = the number of days in which the man could drink it,

and y = the number in which the woman could drink it;

$$\text{then } \frac{15}{x} + \frac{15}{y} = 1,$$

$$\text{and } \frac{6}{x} + \frac{6}{y} + \frac{30}{y} = 1,$$

$$\text{or } \frac{6}{x} + \frac{36}{y} = 1;$$

$$\text{hence from the first equation, } \frac{1}{x} + \frac{1}{y} = \frac{1}{15},$$

$$\text{from the last, } \frac{1}{x} + \frac{6}{y} = \frac{1}{6},$$

$$\therefore \text{ by subtraction, } \frac{5}{y} = \frac{1}{6} - \frac{1}{15} = \frac{3}{30} = \frac{1}{10},$$

$$\text{and } \therefore y = 50;$$

$$\text{also } \frac{1}{x} = \frac{1}{15} - \frac{1}{y} = \frac{1}{15} - \frac{1}{50} = \frac{7}{150};$$

$$\text{and } \therefore x = 21\frac{3}{5};$$

$\therefore$ the man would drink it in $21\frac{3}{5}$ days, and the woman in 50 days.

22. A purse holds 19 crowns and 6 guineas. Now 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many will it hold of each?

Let x = the number of crowns,

and y = the number of guineas;

then $x : 1 :: 4 : \text{the space occupied by 4 crowns} = \frac{4}{x}$.

In the same way, the space occupied by 5 guineas = $\frac{5}{y}$;

$$\therefore \frac{4}{x} + \frac{5}{y} = \frac{17}{63},$$

$$\text{and } \frac{19}{x} + \frac{6}{y} = 1.$$

The first equation being multiplied by 6, and the second by 5,

$$\frac{24}{x} + \frac{30}{y} = \frac{34}{21},$$

$$\text{and } \frac{95}{x} + \frac{30}{y} = 5;$$

$$\therefore \text{ by subtraction, } \frac{71}{x} = 5 - \frac{34}{21} = \frac{71}{21};$$

$$\therefore x = 21,$$

$$\text{and } \frac{5}{y} = \frac{17}{63} - \frac{4}{21} = \frac{5}{63};$$

$$\therefore y = 63;$$

$\therefore$ the purse would hold 21 crowns, or 63 guineas.

23. Some smugglers discovered a cave, which would exactly hold the cargo of their boat, viz. 13 bales of cotton, and 33 casks of rum. Whilst they were unloading, a custom-house cutter coming in sight, they sailed away with 9 casks and five bales, leaving the cave two-thirds full. How many bales or casks would it hold?

Let x = the number of bales,

and y = the number of casks;

$$\text{then } \frac{13}{x} + \frac{33}{y} = 1,$$

$$\text{and } \frac{5}{x} + \frac{9}{y} = \frac{1}{3}.$$

Multiplying the first equation by 3, and the second by 11,

$$\text{then } \frac{39}{x} + \frac{99}{y} = 3,$$

$$\text{and } \frac{55}{x} + \frac{99}{y} = \frac{11}{3};$$

$$\therefore \text{ by subtraction, } \frac{16}{x} = \frac{2}{3};$$

$$\therefore 2x = 48,$$

$$\text{and } x = 24;$$

$$\text{consequently } \frac{9}{y} = \frac{1}{3} - \frac{5}{x} = \frac{1}{3} - \frac{5}{24} = \frac{3}{24} = \frac{1}{8};$$

$$\text{and } y = 72;$$

and $\therefore$ the cave would hold 24 bales, or 72 casks.

24. Round two wheels, whose circumferences are as 5 to 3, two ropes are wrapped, whose difference exceeds the difference of the circumferences by 280 yards. Now the larger rope applied to the larger wheel wraps round it a certain number of times, greater by 12 than the smaller round the smaller wheel; and if the larger wheel turns round 3 times as quick as the other, the ropes will be discharged at the same time. Required the lengths of the ropes and the circumferences of the wheels.

Let $5x$ and $3x$ = the circumferences of the wheels (in yards);

then $2x + 280$ = the difference of the ropes,

and $(15x : 3x ::) 5 : 1 ::$ the length of the longer string :
the length of the shorter.

Let $\therefore 5y$ and y = the lengths of the ropes (in yards);

$$\text{then } 4y = 2x + 280,$$

$$\text{and } \frac{5y}{5x} = \frac{y}{3x} + 12;$$

$$\text{or by transposition, } \frac{2y}{3x} = 12;$$

$$\therefore 4y = 72x,$$

$$\text{and } \therefore 72x = 2x + 280;$$

$$\text{by transposition, } 70x = 280,$$

$$\text{and } x = 4;$$

$$\therefore y = 72;$$

$\therefore$ the circumferences of the wheels are 20 and 12 yards, and the length of the strings 360 and 72 yards.

25. Three guineas were to be raised on two estates, to be charged proportionably to their values. Of this sum, *A*'s estate, which was 4 acres more than *B*'s, but worse by 2 shillings an acre, paid £1. 15s. But had *A* possessed 6 acres more, and *B*'s land been worth 3 shillings an acre less, it would have paid £2. 5s. Required the values of the estates.

Let x = the number of acres *B* had;

then $x + 4$ = the number *A* had.

Let y = the value of an acre of *A*'s land;

$\therefore y + 2$ = the value of an acre of *B*'s,

and $(x + 4) \cdot y : x \cdot (y + 2) :: 35 : 28 :: 5 : 4;$

$$\therefore 4xy + 16y = 5xy + 10x,$$

$$\text{and } xy = 16y - 10x.$$

Again, $(x + 10) \cdot y : x \cdot (y - 1) :: 45 : 18 :: 5 : 2;$

$$\therefore 2xy + 20y = 5xy - 5x,$$

and by transposition, $3xy = 20y + 5x,$

$$\text{whence } 48y - 30x = 20y + 5x;$$

by transposition, $28y = 35x,$

$$\text{and } 4y = 5x,$$

which value substituted in the first equation, gives

$$xy = 20x - 10x = 10x;$$

$$\text{and } \therefore y = 10,$$

$$\text{whence } x = \frac{4y}{5} = 8;$$

$\therefore$ the value of A 's estate $= (x + 4) \cdot y = 120 = \text{£}6$; and the value of B 's $= x \cdot (y + 2) = 96 = \text{£}4. 16s.$

26. A coach set out from Cambridge to London with a certain number of passengers, 4 more being on the outside than within. Seven outside passengers could travel at 2 shillings less expense than 4 inside. The fare of the whole amounted to $\text{£}9$. But at the end of half the journey, it took up 3 more outside and one more inside passengers; in consequence of which the fare of the whole became increased in the proportion of 17 to 15. Required the number of passengers, and the fare of the inside and outside.

$$\begin{aligned} &\text{Let } x = \text{the number of inside } \left. \begin{array}{l} \therefore x + 4 = \text{the number of outside} \\ \text{and } y = \text{the fare of an outside passenger;} \end{array} \right\} \text{passengers,} \\ &\therefore \frac{7y + 2}{4} = \text{the fare of an inside passenger,} \\ &\text{and } \frac{7xy + 2x}{4} + y \cdot (x + 4) = 180. \end{aligned}$$

$$\begin{aligned} \text{Also } \frac{3y}{2} + \frac{7y + 2}{8} &= \text{the fare of the passengers taken up half} \\ \text{way} &= \frac{19y + 2}{8}; \end{aligned}$$

$$\therefore \frac{19y + 2}{8} : 180 :: 2 : 15,$$

$$\text{or } \frac{19y + 2}{8} : 12 :: 2 : 1;$$

$$\therefore \frac{19y + 2}{8} = 24,$$

$$\text{and } 19y + 2 = 192;$$

∴ by transposition, $19y = 190$,

and $y = 10$;

∴ from the first equation, $\frac{70x + 2x}{4} + 10 \cdot (x + 4) = 180$;

by transposition, $28x = 140$,

and $x = 5$;

∴ there were 5 inside, and 9 outside passengers,
and the fares were 18 and 10 shillings, respectively.

27. In one of the corners of a rectangular garden there is a fish-pond, whose area is one-ninth part of the whole garden; the periphery of the garden exceeding that of the fish-pond by 200 yards. Also if the greater side be increased by 3 yards, and the other by 5 yards, the garden will be enlarged by 645 square yards. The fish-pond is a rectangle about the same diameter with the garden. Required the periphery of the garden, and the length of each side.

Let x = the length of the lesser side,

and y = the length of the greater;

∴ $\frac{x}{3}$ and $\frac{y}{3}$ = the lengths of the lesser and greater sides of

the fish-pond, (*Eucl. B. vi. Prop. 24.*)

Also $2 \cdot (x + y)$ = the periphery of the garden;

and $\frac{2 \cdot (x + y)}{3}$ = the periphery of the fish-pond;

$$\therefore 2 \cdot (x + y) - \frac{2}{3} \cdot (x + y) = 200,$$

$$\text{or } \frac{2}{3} \cdot (x + y) = 100;$$

$$\therefore x + y = 150.$$

Also $(y + 3) \cdot (x + 5) = xy + 645$;

∴ by transposition, $3x + 5y = 630$,

but from the former equation, $3x + 3y = 450$;

$\therefore$ by subtraction, $2y = 150$,

and $y = 90$;

$\therefore x = 150 - y = 60$;

and $\therefore$ the periphery = 300 yards, and the sides are 60 and 90 yards.

28. *A* and *B* each bought £300 into the stocks, *A* into the *three per cents.*, and *B* into the *fours*. These stocks were at such a price that *B* received one pound interest more than *A*. When afterwards each of the stocks rose 10 *per cent.*, they sold out their money, and *A* found himself £10 richer than *B*. Required the prices of the stocks.

Let x = the price of the three per cents.,
and y = the price of the fours;

$\therefore x : 300 :: 3 : A\text{'s interest} = \frac{900}{x}$,

and $y : 300 :: 4 : B\text{'s interest} = \frac{1200}{y}$;

$\therefore \frac{900}{x} + 1 = \frac{1200}{y}$.

Again, $x : x + 10 :: 300 : \text{what } A \text{ received when selling}$
 $= \frac{300 \cdot (x + 10)}{x}$;

and in the same way *B* received $\frac{300 \cdot (y + 10)}{y}$;

$\therefore \frac{300 \cdot (x + 10)}{x} = \frac{300 \cdot (y + 10)}{y} + 10$,

and $30 + \frac{300}{x} = 30 + \frac{300}{y} + 1$;

$\therefore \frac{300}{x} = \frac{300}{y} + 1$,

$$\text{and } \frac{900}{y} + 4 = \left(\frac{900}{x} + 1 \right) \frac{1200}{y} \text{ from the first equation ;}$$

$$\therefore \text{ by transposition, } 4 = \frac{300}{y},$$

$$\text{and } y = 75 ;$$

$$\therefore \frac{300}{x} = 5,$$

$$\text{and consequently, } x = 60 ;$$

$\therefore$ the prices of the stocks were 60 and 75 *per cent*.

29. £500 was to be lent out at simple interest in two separate sums, the smaller at 2 *per cent*. more than the other. The interest of the greater sum was afterwards increased, and that of the smaller sum diminished by 1 *per cent*. By this, the interest of the whole was augmented by one-fourth of the former value. But if the interest of the greater sum had been so increased, without any diminution of the less, the interest of the whole would have been increased one-third. What were the sums and the rate *per cent*. of each?

Let x = the less sum ;

$\therefore 500 - x$ = the greater ;

let $y + 1$ = the interest of the less ;

$\therefore y - 1$ = the interest of the greater,

$$\text{and } \frac{x \cdot (y + 1)}{100} + \frac{(500 - x) \cdot (y - 1)}{100} = \frac{x}{50} + 5y - 5 = \text{the}$$

former interest,

$$\text{and } \frac{xy}{100} + \frac{(500 - x) \cdot y}{100} = 5y = \text{the second interest ;}$$

$$\therefore 5y = \frac{5}{4} \cdot \left(\frac{x}{50} + 5y - 5 \right)$$

$$\text{and } \therefore 4y = \frac{x}{50} + 5y - 5;$$

$$\text{by transposition, } y = 5 - \frac{x}{50}.$$

$$\text{Again, the third interest} = \frac{x \cdot (y + 1)}{100} + \frac{(500 - x) \cdot y}{100} =$$

$$\frac{x}{100} + 5y;$$

$$\therefore \left(\frac{x}{100} + 5y \right) \cdot 3 = 4 \cdot \left(\frac{x}{50} + 5y - 5 \right),$$

$$\text{or } \frac{3x}{100} + 15y = \frac{8x}{100} + 20y - 20;$$

$$\text{by transposition, } \left(20 - \frac{5x}{100} \right) 20 - \frac{x}{20} = 5y = 25 - \frac{x}{10}$$

from the former equation;

$$\therefore \frac{x}{20} = 5,$$

$$\text{and } x = 100;$$

$$\text{also } y + 1 = 6 - \frac{x}{50} = 4;$$

$\therefore$ the sums were 100 and 400 pounds,
and the rates of interest £4 and £2, respectively.

SECTION VIII.

Examples of the Solution of Problems producing pure Equations.

1. WHAT two numbers are those, whose sum is to the greater as 10 to 7; and whose sum multiplied by the less produces 270?

Let $10x =$ their sum;
 $\therefore 7x =$ the greater number,
and $3x =$ the less;
whence $30x^2 = 270$,
and $x^2 = 9$;
 $\therefore x = \pm 3$,
and the numbers are ± 21 and ± 9 .

2. There are two numbers in the proportion of 4 to 5, the difference of whose squares is 81. What are those numbers?

Let $4x$ and $5x =$ the numbers;
then $(25x^2 - 16x^2) = 9x^2 = 81$;
 $\therefore x^2 = 9$,
and $x = \pm 3$,
and the numbers are ± 12 and ± 15 .

3. What two numbers are those, whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

Let $2x =$ their difference ;

$\therefore 9x =$ the greater,

and $7x =$ the less ;

$$\therefore (81x^2 - 49x^2) = 32x^2 = 128,$$

$$\text{and } x^2 = 4 ;$$

$$\therefore x = \pm 2,$$

and the numbers are ± 18 and ± 14 .

4. A Mercer bought a piece of silk for £16. 4s. ; and the number of shillings which he paid for a yard was to the number of yards as 4 : 9. How many yards did he buy, and what was the price of a yard ?

Let $4x =$ the number of shillings he paid for a yard ;

$\therefore 9x =$ the number of yards,

and $36x^2 =$ (the price of the whole $=$) 324 ;

$$\therefore x^2 = 9,$$

$$\text{and } \therefore x = \pm 3 ;$$

consequently there were 27 yards, at 12s. per yard.

5. It is required to divide the number 18 into two such parts, that the squares of those parts may be in the proportion of 25 to 16.

Let $x =$ the greater part ;

then $18 - x =$ the less ;

$$\therefore x^2 : (18 - x)^2 :: 25 : 16,$$

$$\text{and (Alg. 188.) } x : 18 - x :: 5 : 4 ;$$

$$\therefore x : 18 :: 5 : 9,$$

$$\text{and } x : 2 :: 5 : 1 ;$$

$$\therefore x = 10,$$

and the parts are 10 and 8.

6. Find three numbers in the proportion of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$;
the sum of whose squares is 724.

Reducing the fractions to a common denominator, the required numbers will evidently be in the proportion of 6, 8, and 9;

let $\therefore 6x, 8x,$ and $9x,$ represent the numbers;

$$\text{then } (36x^2 + 64x^2 + 81x^2) = 181x^2 = 724;$$

$$\therefore x^2 = 4,$$

$$\text{and } x = \pm 2,$$

and consequently, the numbers are $\pm 12, \pm 16,$ and ± 18 .

7. It is required to divide the number 14 into two such parts that the quotient of the greater part divided by the less, may be to the quotient of the less divided by the greater as 16 : 9.

Let $x =$ the greater part;

$\therefore 14 - x =$ the less,

$$\text{and } \frac{x}{14 - x} : \frac{14 - x}{x} :: 16 : 9;$$

$$\text{or } x^2 : (14 - x)^2 :: 16 : 9;$$

$$\therefore (\text{Alg. 188.}) x : 14 - x :: 4 : 3,$$

$$\text{and } x : 14 :: 4 : 7;$$

$$\therefore x : 2 :: 4 : 1,$$

$$\text{and } x = 8;$$

$\therefore$ the parts are 8 and 6.

8. What two numbers are those whose difference is to the less as 4 to 3; and their product multiplied by the less is equal to 504?

Let $4x =$ the difference ;
 then $3x =$ the less,
 and $7x =$ the greater ;
 whence $63x^2 = 504$,
 or $x^2 = 8$;
 $\therefore x = 2$;
 and the numbers are 14 and 6.

9. What two numbers are as 5 to 4, the sum of whose cubes is 5103 ?

Let $5x$ and $4x =$ the numbers ;
 $\therefore (125x^3 + 64x^3 =) 189x^3 = 5103$,
 and $x^3 = 27$;
 $\therefore x = 3$,
 and the numbers are 15 and 12.

10. A number of boys set out to rob an orchard, each carrying as many bags as there were boys in all, and each bag capable of containing 4 times as many apples as there were boys. They filled their bags, and found the number of apples was 2916. How many boys were there ?

Let $x =$ the number of boys ;
 then $x^2 =$ the number of bags,
 and $4x^3 =$ the number of apples ;
 $\therefore 4x^3 = 2916$,
 and $x^3 = 729$;
 $\therefore x = 9$;
 $\therefore$ there were 9 boys.

11. A person bought for one crown as many pounds of sugar as were equal to half the number of crowns he laid out. In selling the sugar he received for every 25 lbs. as many crowns as the whole had cost him; and he received on the whole 20 crowns. How many crowns did he lay out, and what did he give for a pound?

Let x = the number of crowns he laid out;

$\therefore \frac{x}{2}$ = the number of lbs. for one crown,

and $\frac{x^2}{2}$ = the number of lbs. in all,

and $\frac{x}{25}$ = the selling price of one lb.;

$$\therefore \frac{x^2}{2} \times \frac{x}{25} = 20,$$

$$\text{and } x^3 = 1000;$$

$$\text{whence } x = 10;$$

$\therefore$ he laid out 10 crowns, and gave one shilling for a lb.

12. A detachment from an army was marching in regular column with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in five lines. Required the number of men.

Let x = the number in front;

$\therefore x + 5$ = the number in depth,

$$\text{and } x^2 + 5x = 5x + 4225,$$

$$\therefore x^2 = 4225,$$

$$\text{and } x = \pm 65;$$

$\therefore$ the number of men $= 5x + 4225 = 4550$, the negative value not answering the conditions of the problem.

13. A number of shillings were placed at equal distances on a table so as to form the sides of an equilateral triangle; then from the middle of each side a number of shillings, equal to the square root of the number in the side were taken, and placed upon the corner shilling opposite to that side; it then appeared that the number on each side was to the number previously upon it, as 5 to 4. Required the number of shillings on one side at first.

Let x^2 = the number;

then $x^2 + x : x^2 :: 5 : 4$,

and $x : x^2 :: 1 : 4$;

$\therefore x^2 = 4x$,

and $x = 4$;

whence $x^2 = 16$ = the number required.

14. A certain sum of money is divided every week among the resident members of a corporation. It happened one week that the number resident was the square root of the number of pounds to be divided. Two men, however, coming into residence the week after, diminished the dividend of each of the former individuals £1. 6s. 8d. What was the sum to be divided?

Let x^2 = the number of pounds;

then x = the number of men resident, and also = the sum each received.

Hence, $x - \frac{4}{3} = \frac{x^2}{x + 2}$;

or $x^2 + \frac{2}{3}x - \frac{8}{3} = x^2$;

by transposition, $\frac{2}{3}x = \frac{8}{3}$,

and $x = 4$;

$\therefore x^2 = 16$ = the sum required.

15. Two partners *A* and *B* dividing their gain (£60.), *B* took £20; *A*'s money continued in trade 4 months, and if the number 50 be divided by *A*'s money, the quotient will give the number of months that *B*'s money, which was £100, continued in trade. What was *A*'s money, and how long did *B*'s money continue in trade?

Suppose *A*'s money was x pounds;

$\therefore \frac{50}{x}$ = the number of months *B*'s money was in trade,

and since *B* gained £20, *A* gained £40;

$$\therefore 4x : \frac{5000}{x} :: 2 : 1;$$

$$\text{or } x : \frac{2500}{x} :: 1 : 1;$$

$$\therefore x^2 = 2500,$$

$$\text{and } x = \pm 50;$$

$\therefore$ *A*'s money was £50, and *B*'s money was one month in trade.

16. Two workmen *A* and *B* were engaged to work for a certain number of days at different rates. At the end of the time, *A* who had played 4 of those days, had 75 shillings to receive; but *B* who had played 7 of those days, received only 48 shillings. Now had *B* only played 4 days, and *A* played 7 days, they would have received exactly alike. For how many days were they engaged; how many did each work, and what had each *per* day?

Let x = the number of days for which they were engaged;

$\therefore x - 4$ = the number *A* worked,

and $x - 7$ = the number *B* worked,

and $\frac{75}{x-4}$ = the number of shillings *A* received *per* day,

and $\frac{48}{x-7}$ = the number of shillings *B* received *per* day ;

$$\therefore \frac{75 \cdot (x-7)}{x-4} = \frac{48 \cdot (x-4)}{x-7},$$

$$\text{and } 25 \cdot (x-7)^2 = 16 \cdot (x-4)^2;$$

$$\therefore 5 \cdot (x-7) = \pm 4 \cdot (x-4);$$

$$\therefore x = 19, \text{ or } \frac{17}{3};$$

and $\therefore$ they were engaged to work 19 days,

and *A* worked 15, and *B* 12 days,

and *A* received 5 shillings, and *B* 4 shillings *per* day.

17. Two Travellers *A* and *B* set out to meet each other, *A* leaving the town *C* at the same time that *B* left *D*. They travelled the direct road *CD*, and on meeting it appeared that *A* had travelled 18 miles more than *B*; and that *A* could have gone *B*'s journey in $15\frac{1}{4}$ days, but *B* would have been 28 days in performing *A*'s journey. What was the distance between *C* and *D*?

Let x = the number of miles *A* has travelled ;

$\therefore x - 18$ = the number *B* has travelled,

and $x - 18 : x :: 15\frac{1}{4} : \text{the number of days } A \text{ travelled}$

$$= \frac{63x}{4 \cdot (x-18)};$$

also $x : x - 18 :: 28 : \text{the number of days } B \text{ travelled}$

$$= \frac{28 \cdot (x-18)}{x};$$

$$\therefore \frac{28 \cdot (x-18)}{x} = \frac{63x}{4 \cdot (x-18)};$$

$$\text{or } 16 \cdot (x - 18)^2 = 9x^2;$$

$$\therefore 4 \cdot (x - 18) = \pm 3x,$$

$$\text{and } x = 72, \text{ or } 10\frac{1}{2};$$

whence A travelled 72, and B 54 miles;

and $\therefore$ the whole distance CD 126 miles.

18. A and B lay out some money on speculation. A disposes of his bargain for £11, and gains as much *per cent.* as B lays out; B 's gain is £36, and it appears that A gains four times as much *per cent.* as B . Required the capital of each.

Let $4x = B$'s capital, and $\therefore A$'s gain *per cent.*;

then $x = B$'s gain *per cent.*,

$$\text{and } 100 : 4x :: x : 36;$$

$$\therefore 4x^2 = 36 \times 100,$$

$$\text{and } x^2 = 9 \times 100;$$

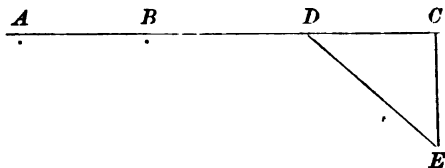
$$\therefore x = \pm 30,$$

$$\text{and } \therefore B\text{'s capital} = 120,$$

$$\text{and } 220 : 100 :: 11 : A\text{'s capital} = \frac{11 \times 10}{22} = 5.$$

19. The Captain of a privateer descrying a trading vessel 7 miles ahead, sailed 20 miles in direct pursuit of her, and then observing the trader steering in a direction perpendicular to her former course, changed his own course so as to overtake her without making another tack. On comparing their reckonings it was found, that the privateer had run at the rate of 10 knots in an hour, and the trading vessel at the rate of 8 knots in the same time. Required the distance sailed by the privateer.

Let A, B , be the original places of the privateer and trader, E the point of concourse, D the place where the



captain changed his course, CE being perpendicular to AC .
 $AB = 7, AD = 20$.

Now $(10 : 8 ::) 5 : 4 ::$ the velocity of the privateer
 : the velocity of the trader

$$:: AD : BC :: 20 : BC;$$

$$\therefore BC = \frac{20 \times 4}{5} = 16;$$

$$\therefore DC = 16 - BD = 16 - 13 = 3,$$

and $DE : CE :: 5 : 4$. Let $CE = x$;

$$\therefore \sqrt{9 + x^2} : x :: 5 : 4,$$

$$\text{and } 9 + x^2 : x^2 :: 25 : 16;$$

$$\therefore 9 : x^2 :: 9 : 16;$$

$$\therefore x^2 = 16, \text{ and } x = \pm 4,$$

$$\text{and } \therefore DE = 5, \text{ and } AD + DE = 25.$$

20. A Vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws off the same quantity of liquor as before, and so on, for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

Let x = the number of gallons drawn the first time;

$\therefore 256 - x$ = the quantity of wine left,

and $256 : 256 - x :: x : \text{the quantity of wine drawn the}$

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$$\text{second time} = \frac{x \cdot (256 - x)}{256};$$

$$\therefore 256 - x - \frac{x \cdot (256 - x)}{256} = \frac{(256 - x)^2}{256} = \text{the quantity left after the second draught.}$$

In the same way, $\left(\frac{256 - x}{256}\right)^2 \cdot x =$ the quantity drawn the third time,

$$\text{and } \frac{(256 - x)^3}{256^3} = \text{the quantity left,}$$

$$\text{and } \left(\frac{256 - x}{256}\right)^3 \cdot x \text{ and } \frac{(256 - x)^4}{256^4} = \text{the quantities drawn}$$

and left the fourth time;

$$\text{whence } \frac{(256 - x)^4}{256^4} = 81,$$

$$\text{and } 256 - x = 256^{\frac{1}{4}} \times 3 = 64 \times 3 = 192;$$

$$\therefore \text{by transposition, } 64 = x,$$

and the quantities drawn off each time were 64, 48, 36, and 27 gallons.

21. What two numbers are those, whose difference multiplied by the greater produces 40, and by the less 15?

Let $x =$ the greater,

and $y =$ the less;

$$\therefore x^2 - xy = 40,$$

$$\text{and } xy - y^2 = 15;$$

$$\therefore \text{by subtraction, } x^2 - 2xy + y^2 = 25,$$

$$\text{and } x - y = \pm 5;$$

$$\therefore \text{from the first equation, } \pm 5x = 40,$$

$$\text{and } x = \pm 8;$$

and from the second equation, $\pm 5y = 15$,
 and $y = \pm 3$;
 $\therefore$ the numbers are ± 8 , and ± 3 .

22. What two numbers are those, whose difference multiplied by the less produces 42, and by their sum 133?

Let x = the greater,
 and y = the less;
 $\therefore (x - y) \cdot y = 42$,
 and $(x - y) \cdot (x + y) = 133$;

$\therefore$ by subtraction, $(x - y) \cdot x = 91$, subtracting the first equation from this, $(x - y) \cdot (x - y) = 49$;
 or $(x - y)^2 = 49$;
 $\therefore x - y = \pm 7$;
 whence $\pm 7y = 42$,
 and $y = \pm 6$,
 and $x = \pm 7 + y = \pm 13$,
 $\therefore$ the numbers are ± 13 , and ± 6 .

23. In a mixture of rum and brandy, the difference between the quantities of each is to the quantity of brandy as 100 is to the number of gallons of rum; and the same difference is to the quantity of rum as 4 to the number of gallons of brandy. How many gallons are there of each?

Let x = the number of gallons of rum,
 and y = the number of gallons of brandy;

$\therefore x - y : y :: 100 : x$,
 and $x : x - y :: y : 4$;

$\therefore$ *ex æquali*, $x : y :: 25y : x$,
 and $x^2 = 25y^2$;

$\therefore x = \pm 5y$, the negative value not answering the conditions of the problem.

Now from the second proportion $5y : 4y :: y : 4$;

or $5 : 1 :: y : 1$;

$\therefore y = 5$,

and $x = 25$;

$\therefore$ there are 25 gallons of rum, and 5 of brandy.

24. What two numbers are those, whose difference being multiplied by the greater, and the product divided by the less, quotes 24; but if their difference be multiplied by the less, and the product divided by the greater, the quotient is 6?

Let $x =$ the greater,

and $y =$ the less;

then $(x - y) \cdot \frac{x}{y} = 24$,

and $(x - y) \cdot \frac{y}{x} = 6$;

dividing the first equation by the second, $\frac{x^2}{y^2} = 4$;

$\therefore \frac{x}{y} = \pm 2$;

or $x = \pm 2y$,

and $\therefore$ in the first case, $(x - y) \cdot y = 12$, and $x = 24$;

but if $x = -2y$; $-3y \times -2 = 24$;

$\therefore y = 4$, and $x = -8$.

25. It is required to find two numbers such, that the product of the greater and square root of the less may be equal to 48, and the product of the less and square root of the greater may be 36.

Let x and y be the two numbers;

$$\therefore xy = 48, \text{ and } xy^2 = 36;$$

$$\therefore \frac{48}{x} = (xy) \frac{36}{y},$$

$$\text{or } \frac{4}{x} = \frac{3}{y};$$

$$\therefore y = \frac{3x}{4};$$

$$\text{whence } \frac{3x^2}{4} = 48,$$

$$\text{and } x^2 = 64;$$

$$\therefore x = 8,$$

and consequently, $y = 3$;

$\therefore$ the numbers are 16, and 9.

26. Find two numbers such, that the square of the greater multiplied by the less may be equal to 448, and the square of the less multiplied by the greater may be 392.

Let x = the greater, and y = the less;

then $x^2y = 448$, and $xy^2 = 392$;

$$\therefore \frac{448}{x} = (xy) \frac{392}{y},$$

$$\text{or } \frac{8}{x} = \frac{7}{y};$$

$$\therefore x = \frac{8y}{7},$$

$$\text{and consequently, } \frac{8y^3}{7} = 392,$$

$$\text{or } \frac{y^3}{7} = 49;$$

$$\therefore y^3 = 343, \text{ and } y = 7;$$

$$\therefore x = 8;$$

$\therefore$ the numbers are 8, and 7.

27. *A* and *B* carried 100 eggs between them to market, and each received the same sum. If *A* had carried as many as *B*, he would have received 18 pence for them, and if *B* had taken only as many as *A*, he would have received only 8 pence. How many had each?

Let x = the number *A* had,

and y = the number *B* had;

then $\frac{18}{y}$ = the price of one egg of *A*'s (in pence),

and $\frac{8}{x}$ = the price of one of *B*'s;

$$\therefore \frac{18x}{y} = \frac{8y}{x},$$

$$\text{and } 9x^2 = 4y^2;$$

$\therefore 3x = \pm 2y$, the negative value of which will not answer the conditions of the problem.

$$\text{Now } (x + y =) x + \frac{3x}{2} = 100;$$

$$\therefore (2x + 3x =) 5x = 200,$$

$$\text{and } x = 40;$$

$$\text{and } \therefore y = 60.$$

28. What two numbers are those, which being both multiplied by 27 the first product is a square, and the second the root of that square: but being both multiplied by 3, the first product is a cube, and the second the root of that cube?

Let x and y be the numbers;

$$\text{then } \sqrt{(27x)} = 27y,$$

$$\text{and } \therefore x = 27y^2;$$

$$\text{also } \sqrt[3]{(3x)} = 3y,$$

$$\text{and } \therefore x = 9y^3;$$

$$\text{whence } 9y^2 = 27y^2,$$

$$\text{and } y = 3;$$

$$\therefore x = 27y^2 = 243;$$

$\therefore$ the numbers are 243, and 3.

29. It is required to find the three sides of a right-angled triangle from the following data. The number of square feet in the area is equal to the number of feet in the hypotenuse + the sum in the other two sides; and the square described upon the hypotenuse is less than the square described upon a line equal in length to the two sides, by half the product of the numbers representing the base and area.

Let x = the number of feet in the altitude,

and y = the number in the base;

$\therefore \sqrt{(x^2 + y^2)}$ = the number in the hypotenuse, (*Eucl. B. 1. p. 47.*)

and $\frac{xy}{2}$ = the area;

$$\therefore \frac{1}{2}xy = \sqrt{(x^2 + y^2)} + x + y;$$

$$\text{also } x^2 + y^2 = \{(x + y)^2 - \frac{1}{2}xy^2\} = x^2 + 2xy + y^2 - \frac{1}{2}xy^2;$$

$$\therefore \text{by transposition, } \frac{1}{2}xy^2 = 2xy,$$

$$\text{and } y = 8;$$

$$\text{hence from the first equation, } 4x = \sqrt{(x^2 + 64)} + x + 8,$$

$$\text{and by transposition, } 3x - 8 = \sqrt{(x^2 + 64)};$$

$$\therefore 9x^2 - 48x + 64 = x^2 + 64,$$

$$\text{and } 8x^2 = 48x;$$

$$\therefore x = 6;$$

$$\text{whence the hypotenuse} = \sqrt{(64 + 36)} = 10;$$

$\therefore$ the sides are 6, 8, and 10 feet, respectively.

30. A Farmer has 2 cubical stacks of hay. The side of one is 3 yards longer than the side of the other; and the dif-

ference of their contents is 117 solid yards. Required the side of each.

Let x = the side of the greater,

and y = the side of the less;

$$\therefore x^3 - y^3 = 117,$$

$$\text{and } x - y = 3;$$

cubing the latter equation, $x^3 - 3x^2y + 3xy^2 - y^3 = 27$;

$$\text{but } x^3 - y^3 = 117;$$

$$\therefore \text{ by subtraction, } 3x^2y - 3xy^2 = 90,$$

$$\text{and } xy \cdot (x - y) = 30,$$

$$\text{or } 3xy = 30;$$

$$\therefore xy = 10.$$

$$\text{Now } x^3 - 2xy + y^3 = 9,$$

$$\text{and } 4xy = 40;$$

$$\therefore \text{ by addition, } x^3 + 2xy + y^3 = 49,$$

$$\text{and } x + y = \pm 7;$$

$$\text{but } x - y = 3;$$

$$\therefore \text{ by addition, } 2x = 10, \text{ or } -4;$$

$$\therefore x = 5, \text{ or } -2,$$

$$\text{and by subtraction, } 2y = 4, \text{ or } -10;$$

$$\therefore y = 2, \text{ or } -5,$$

and the sides of the stacks are 5, and 2 yards, respectively.

31. When a parish was enclosed, the allotment of one of the proprietors consisted of two pieces of ground; one of which was in the form of a right-angled triangle; the other was a rectangle, one of the sides of which was equal to the hypotenuse of the triangle, the other, to half the greater side; but wishing to have his land in one

piece, he exchanged his allotments for a square piece of ground of equal area, one side of which equalled the greater of the sides of the triangle which contained the right angle. By this exchange he found that he had saved ten poles of railing. What are the respective areas of the triangle and rectangle; and what is the length of each of their sides?

Let $2x$ = the greater side of the triangle,

and y = the less;

$\therefore \sqrt{(4x^2 + y^2)}$ = the hypotenuse; and also the greater side of the rectangle,

and x = the less side of the rectangle;

$\therefore xy$ = the area of the triangle,

and $x\sqrt{(4x^2 + y^2)}$ = the area of the rectangle;

$$\therefore 4x^2 = xy + x\sqrt{(4x^2 + y^2)},$$

$$\text{or } 4x - y = \sqrt{(4x^2 + y^2)};$$

also $8x + 10 = 2x + y + \sqrt{(4x^2 + y^2)} + 2x + 2\sqrt{(4x^2 + y^2)}$,
or $4x + 10 = y + 3\sqrt{(4x^2 + y^2)}$; in which equation substituting the value of $\sqrt{(4x^2 + y^2)}$ found above;

$$\therefore 4x + 10 = y + 3(4x - y) = 12x - 2y;$$

$$\therefore \text{by transposition, } 2y = 8x - 10,$$

$$\text{and } y = 4x - 5;$$

$$\therefore \text{from the first equation, } 5 = \sqrt{\{4x^2 + (4x - 5)^2\}},$$

$$\text{and } 25 = 4x^2 + 16x^2 - 40x + 25;$$

$$\text{by transposition, } 40x = 20x^2;$$

$$\therefore 2 = x,$$

$$\text{and } y = 4x - 5 = 3;$$

$\therefore$ the sides of the triangle are 3, 4, and 5; the sides of the rectangle are 2, and 5; and the areas of the triangle and rectangle are 6, and 10, respectively.

SECTION IX.

Examples of the Solution of Problems producing Adfected Quadratic Equations.

1. A MERCHANT sold a quantity of brandy for £39, and gained as much *per cent.* as the brandy cost him. What was the price of the brandy?

Let x = the price of the brandy;

then $100 : x :: x : \text{the gain} = \frac{x^2}{100}$,

$$\text{and } \therefore \frac{x^2}{100} = 39 - x,$$

$$\text{or } x^2 = 3900 - 100x;$$

by transposition, $x^2 + 100x = 3900$,

$$\begin{aligned} \text{completing the square, } x^2 + 100x + 50^2 &= 3900 + 2500 \\ &= 6400; \end{aligned}$$

extracting the root, $x + 50 = \pm 80$;

$$\therefore x = 30, \text{ or } -130;$$

$\therefore$ the price was £30.

2. There are two numbers whose difference is 9, and their sum multiplied by the greater produces 266. What are those numbers?

Let x = the greater;

$\therefore x - 9$ = the less,

$$\text{and } x \cdot (2x - 9) = 266;$$

$$\therefore x^2 - \frac{9}{2} \cdot x = \frac{266}{2};$$

completing the square, $x^2 - \frac{9}{2}x + \frac{81}{16} = \frac{266}{2} + \frac{81}{16} = \frac{2209}{16}$;

extracting the root, $x - \frac{9}{4} = \pm \frac{47}{4}$;

$$\therefore x = 14, \text{ or } -\frac{19}{2};$$

$$\therefore x - 9 = 5, \text{ or } -\frac{37}{2},$$

and both values answer the conditions of the problem.

3. It is required to find two numbers, the first of which may be to the second as the second is to 16; and the sum of the squares of the numbers may be equal to 225.

Let x = the first number;

$\therefore \sqrt{16x}$ = the second,

and $x^2 + 16x = 225$;

completing the square, $x^2 + 16x + 64 = 225 + 64 = 289$;

extracting the root, $x + 8 = \pm 17$;

$$\therefore x = 9, \text{ or } -25;$$

but as this latter value of x makes the second number an impossible quantity, 9 is the only value of x which answers the conditions, and therefore the numbers are 9 and 12.

4. Bought two sorts of linen for 6 crowns. An ell of the finer cost as many shillings as there were ells of the finer. Also 28 ells of the coarser (which was the whole quantity) were at such a price that 8 ells cost as many shillings as 1 ell of the finer. How many ells were there of the finer, and what was the value of each piece?

Let x = the number of ells of the finer;

$\therefore x^2$ = the price of the finer (in shillings),

and $8 : 28 :: x : \text{the price of the coarser} = \frac{7x}{2}$;

$$\therefore x^2 + \frac{7x}{2} = 30;$$

completing the square, $x^2 + \frac{7x}{2} + \frac{49}{16} = 30 + \frac{49}{16} = \frac{529}{16}$;

extracting the root, $x + \frac{7}{4} = \pm \frac{23}{4}$,

$$\text{and } x = 4, \text{ or } -\frac{15}{2},$$

and $\therefore$ the price of the finer = 16 shillings, and of the coarser = 14 shillings.

5. Two partners *A* and *B* gained £18 by trade. *A*'s money was in trade 12 months, and he received for his principal and gain £26. Also *B*'s money, which was £30, was in trade 16 months. What money did *A* put into trade?

Let x = the number of pounds he put in;

$\therefore 26 - x$ = the number he gained,

$$\text{and } 12x + 16 \times 30 : 12x :: 18 : 26 - x;$$

$$\therefore x + 40 : x :: 18 : 26 - x,$$

$$\text{and } 18x = 1040 - 14x - x^2;$$

$$\text{by transposition, } x^2 + 32x = 1040;$$

$$\text{completing the square, } x^2 + 32x + 16^2 = 1040 + 256 \\ = 1296;$$

$$\text{extracting the root, } x + 16 = \pm 36;$$

$$\therefore x = 20, \text{ or } -52,$$

and consequently *A* put £20 into trade.

6. A person bought some sheep for £72; and found that if he had bought 6 more for the same money, he would

have paid £1 less for each. How many did he buy, and what was the price of each?

Let x = the number of sheep bought;

then $\frac{72}{x}$ = the price of one (in pounds),

and $\frac{72}{x+6}$ = the price of one, if he had bought 6 more;

$$\therefore \frac{72}{x+6} + 1 = \frac{72}{x};$$

$$\therefore 72x + x^2 + 6x = 72x + 432;$$

$$\therefore x^2 + 6x = 432;$$

completing the square, $x^2 + 6x + 9 = 441$;

extracting the root, $x + 3 = \pm 21$,

and $x = 18$, or -24 ,

and $\therefore$ he bought 18, and the price of one = $\frac{72}{18}$

= 4 pounds.

7. The plate of a looking-glass is 18 inches by 12, and is to be framed with a frame of equal width, whose area is to be equal to that of the glass. Required the width of the frame.

The area of the glass = $12 \times 18 = 216$.

Let x = the width of the frame (in inches);

then the area of the frame = $(18 + 2x) \cdot (12 + 2x) - 216$,

and $\therefore (18 + 2x) \cdot (12 + 2x) - 216 = 216$,

or $4x^2 + 60x = 216$,

and $x^2 + 15x = 54$;

completing the square, $x^2 + 15x + \left(\frac{15}{2}\right)^2 = 54 + \frac{225}{4} = \frac{441}{4}$;

extracting the root, $x + \frac{15}{2} = \pm \frac{21}{2}$;

$\therefore x = 3$, or -18 ,

and $\therefore$ the width must be 3 inches.

8. There are two square buildings, that are paved with stones, a foot square each. The side of one building exceeds that of the other by 12 feet, and both their pavements taken together contain 2120 stones. What are the lengths of them separately?

Let x and $x + 12$ = the number of feet in the sides of each;

$\therefore x^2$ and $(x + 12)^2$ = the number of stones in the squares,

and $x^2 + x^2 + 24x + 144 = 2120$;

by transposition, $2x^2 + 24x = 1976$,

or $x^2 + 12x = 988$;

completing the square, $x^2 + 12x + 36 = 988 + 36 = 1024$;

extracting the root, $x + 6 = \pm 32$;

$\therefore x = 26$, or -38 ,

whence the lengths are 26, and 38 feet, respectively.

9. A labourer dug two trenches, one of which was 6 yards longer than the other, for £17. 16s. and the digging of each of them cost as many shillings *per* yard as there were yards in its length. What was the length of each?

Let x and $x + 6$ = the number of yards in each;

$\therefore x^2 + (x + 6)^2 = 356$ shillings,

or $2x^2 + 12x + 36 = 356$;

by transposition, $2x^2 + 12x = 320$;

or $x^2 + 6x = 160$;

completing the square, $x^2 + 6x + 9 = 169$;

extracting the root, $x + 3 = \pm 13$,

and $x = 10$, or -16 ;

$\therefore$ the lengths were 10, and 16 yards.

10. A company at a tavern had £8. 15s. to pay; but before the bill was paid, two of them sneaked off, when those who remained had each 10 shillings more to pay. How many were in the company at first?

Let x = the number;

then $\frac{175}{x}$ = the number of shillings each had to pay at first,

and $\frac{175}{x-2}$ = the number each had to pay, after two had sneaked off;

$$\therefore 10 = \frac{175}{x-2} - \frac{175}{x} = 175 \times \left(\frac{1}{x-2} - \frac{1}{x} \right);$$

$$\text{and } \therefore 10 \cdot x \cdot (x-2) = 175 \times 2,$$

$$\text{or } x^2 - 2x = 35;$$

completing the square, $x^2 - 2x + 1 = 36$;

extracting the root, $x - 1 = \pm 6$,

and $\therefore x = 7$, or -5 ;

consequently there were 7 at first.

11. A grazier bought as many sheep as cost him £60; out of which he reserved 15, and sold the remainder for £54, gaining 2 shillings a head by them. How many sheep did he buy, and what was the price of each?

Let x = the number;

$\therefore \frac{60}{x}$ = the price of each in pounds,

$$\text{and } (x - 15) \cdot \left(\frac{60}{x} + \frac{1}{10} \right) = 54;$$

$$\text{or } (x - 15) \cdot (600 + x) = 540x,$$

$$\text{and } x^2 + 585x - 9000 = 540x;$$

$$\text{by transposition, } x^2 + 45x = 9000;$$

$$\text{completing the square, } x^2 + 45x + \left(\frac{45}{2}\right)^2 = 9000 + \frac{2025}{4}$$

$$= \frac{38025}{4};$$

$$\text{extracting the root, } x + \frac{45}{2} = \pm \frac{195}{2},$$

$$\text{and } x = 75, \text{ or } -120;$$

$$\text{and } \therefore \text{ the number bought was } 75,$$

$$\text{and the price} = \frac{4}{5}\mathcal{L} = 16 \text{ shillings.}$$

12. *A* and *B* set out from two towns which were at the distance of 247 miles, and travelled the direct road till they met. *A* went 9 miles a day; and the number of days, at the end of which they met, was greater by 3 than the number of miles which *B* went in a day. How many miles did each go?

Let x = the number of days they travelled;

$\therefore 9x$ = the number of miles *A* went,

and $247 - 9x$ = the number *B* went,

and $\frac{247 - 9x}{x}$ = the number *B* went *per* day;

$$\therefore x - 3 = \frac{247 - 9x}{x},$$

$$\text{and } x^2 - 3x = 247 - 9x;$$

$$\text{by transposition, } x^2 + 6x = 247;$$

$$\text{completing the square, } x^2 + 6x + 9 = 256,$$

$$\text{extracting the root, } x + 3 = \pm 16,$$

$$\text{and } x = 13, \text{ or } -19,$$

$$\text{and } \therefore \text{ } A \text{ went } 117, \text{ and } B \text{ } 130 \text{ miles.}$$

13. A person bought two pieces of cloth of different sorts; whereof the finer cost 4 shillings a yard more than the other; for the finer he paid £18; but the coarser, which exceeded the finer in length by 2 yards, cost only £16. How many yards were there in each piece, and what was the price of a yard of each?

Let x = the number of yards of the finer;

$\therefore x + 2$ = the number of yards of the coarser,

and $\frac{18}{x}$ = the price of a yard of the finer (in pounds);

also $\frac{16}{x+2}$ = the price of a yard of the coarser;

$$\therefore \frac{18}{x} = \frac{16}{x+2} + \frac{1}{5},$$

$$\text{and } 90x + 180 = 80x + x^2 + 2x;$$

$$\text{by transposition, } x^2 - 8x = 180;$$

$$\text{completing the square, } x^2 - 8x + 16 = 180 + 16 = 196;$$

$$\text{extracting the root, } x - 4 = \pm 14;$$

$$\therefore x = 18, \text{ or } -10,$$

consequently, there were 18 yards of the finer, and 20 of the coarser; and the prices were £1, and 16 shillings, respectively.

14. A set out from C towards D, and travelled 7 miles a day. After he had gone 32 miles, B set out from D towards C, and went every day $\frac{1}{9}$ th of the whole journey; and after he had travelled as many days as he went miles in one day, he met A. Required the distance of the places C and D.

Suppose the distance was x miles;

$\therefore \frac{x}{19}$ = the number of miles B travelled *per* day; and also
= the number of days he travelled before he met A.

$$\therefore \frac{x^2}{361} + 32 + \frac{7x}{19} = x;$$

$$\text{by transposition, } \frac{x^2}{361} - \frac{12x}{19} = -32;$$

$$\text{completing the square, } \frac{x^2}{361} - \frac{12x}{19} + 36 = 36 - 32 = 4;$$

$$\text{extracting the root, } \frac{x}{19} - 6 = \pm 2;$$

$$\therefore \frac{x}{19} = 8, \text{ or } 4,$$

and $x = 152$, or 76 , both which values answer the conditions of the problem. The distance therefore of C from D was 152 , or 76 miles.

15. A and B sold 130 ells of silk, (of which 40 ells were A 's, and 90 B 's,) for 42 crowns. Now A sold for a crown one-third of an ell more than B did. How many ells did each sell for a crown?

$$\left. \begin{array}{l} \text{Let } x = \text{the number } B \text{ sold} \\ \therefore x + \frac{1}{3} = \text{the number } A \text{ sold} \end{array} \right\} \text{for a crown,}$$

$$\text{and } x : 90 :: 1 : \text{the price of 90 ells} = \frac{90}{x},$$

$$\text{and } x + \frac{1}{3} : 40 :: 1 : \text{the price of 40 ells} = \frac{120}{3x + 1};$$

$$\therefore 42 = \frac{90}{x} + \frac{120}{3x + 1},$$

$$\text{or } 7 = \frac{15}{x} + \frac{20}{3x + 1};$$

$$\text{whence, } 21x^2 + 7x = 45x + 15 + 20x;$$

$$\text{by transposition, } 21x^2 - 58x = 15,$$

$$\text{and } x^2 - \frac{58}{21}x = \frac{15}{21};$$

$$\text{completing the square, } x^2 - \frac{58}{21}x + \left(\frac{29}{21}\right)^2 = \frac{15}{21} + \frac{841}{21^2} = \frac{1156}{21^2};$$

$$\text{extracting the root, } x - \frac{29}{21} = \pm \frac{34}{21};$$

$$\therefore x = 3, \text{ or } -\frac{5}{21};$$

whence, B sold 3 ells, and A $3\frac{1}{3}$, for a crown.

16. Three Merchants, A , B , and C , made a joint stock, by which they gained a sum less than that stock by £80. A 's share of the gain was £60; and his contribution to the stock was £17 more than B 's. Also B and C together contributed £325. How much did each contribute?

Let x = the number of pounds that A contributed;

$\therefore x - 17$ = the number that B contributed,

and $325 - (x - 17) = 342 - x$ = the number that C contributed;

$\therefore 325 + x$ = the whole stock,

and $325 + x - 80 = 245 + x$ = the whole gain;

$\therefore 325 + x : x :: 245 + x : 60$,

and $x^2 + 245x = 60x + 19500$;

by transposition, $x^2 + 185x = 19500$;

completing the square, $x^2 + 185x + \left(\frac{185}{2}\right)^2 = 19500$

$$+ \frac{34225}{4} = \frac{112225}{4};$$

extracting the root, $x + \frac{185}{2} = \pm \frac{335}{2}$,

and $x = 75$, or -260 ;

$\therefore$ the stocks of A , B , and C were 75, 58, and 267 pounds, respectively.

17. The joint stock of 2 partners A and B was £416. A 's money was in trade 9 months, and B 's 6 months: when they shared stock and gain, A received £228, and B £252. What was each man's stock?

Let $x = A$'s stock;

$\therefore 228 - x =$ his gain;

also $416 - x = B$'s stock;

and $x - 164 =$ his gain;

and $\therefore 64 =$ the whole gain,

and $9x + 6 \cdot (416 - x) : 9x :: 64 : 228 - x$;

or $3x + 2 \cdot (416 - x) : 3x :: 64 : 228 - x$;

$\therefore 192x = (x + 832) \cdot (228 - x) = 189696 - 604x - x^2$;

by transposition, $x^2 + 796x = 189696$;

completing the square, $x^2 + 796x + 398^2 = 189696 + 158404 = 348100$;

extracting the root, $x + 398 = \pm 590$;

and $x = 192$, or -988 ;

$\therefore$ the stocks were £192, and £224.

18. A body of men were formed into a hollow square, three deep, when it was observed, that with the addition of 25 to their number, a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square. Required the number of men in the hollow square.

Let $x =$ the number of men in a side of the hollow square;

$$\begin{aligned}
 \therefore x^2 - (x - 6)^2 &= \text{the whole number of men,} \\
 \text{and } x^2 - (x - 6)^2 + 25 &= (x^{\frac{1}{2}} + 2)^2, \\
 \text{or } 12x - 36 + 25 &= x + 44x^{\frac{1}{2}} + 484; \\
 \therefore \text{by transposition, } 11x - 44x^{\frac{1}{2}} &= 495, \\
 \text{or } x - 4x^{\frac{1}{2}} &= 45; \\
 \text{completing the square, } x - 4x^{\frac{1}{2}} + 4 &= 49; \\
 \text{extracting the root, } x^{\frac{1}{2}} - 2 &= \pm 7; \\
 \therefore x^{\frac{1}{2}} &= 9, \text{ or } -5, \\
 \text{and } x &= 81, \text{ or } 25, \\
 \text{and } \therefore \text{the whole number} &= 936.
 \end{aligned}$$

19. A Mercer bought a number of pieces of two different kinds of silk for £92. 3s. There were as many pieces bought of each kind, and as many shillings paid *per* yard for them, as a piece of that kind contained yards. Now 2 pieces, one of each kind, together measured 19 yards. How many yards were there in each?

Let x = the number of yards in one piece; and $\therefore$ = the number of pieces, and also the number of shillings *per* yard;

$$\begin{aligned}
 \therefore 19 - x &= \text{the number in the other,} \\
 \text{and } x^2, \text{ and } (19 - x)^2 &= \text{the whole prices of each kind;} \\
 \therefore x^2 + (19 - x)^2 &= 1843, \\
 \text{or } 57x^2 - 1083x + 6589 &= 1843; \\
 \text{by transposition, } 57x^2 - 1083x &= -5016; \\
 \text{or } x^2 - 19x &= -88; \\
 \text{completing the square, } x^2 - 19x + \left(\frac{19}{2}\right)^2 &= \frac{361}{4} - 88 = \frac{9}{4}; \\
 \text{extracting the root, } x - \frac{19}{2} &= \pm \frac{3}{2}; \\
 \therefore x &= 11, \text{ or } 8; \\
 \therefore 19 - x &= 8, \text{ or } 11,
 \end{aligned}$$

both which values answer the conditions of the problem ;

$\therefore$ there were 11 yards in one, and 8 in the other.

20. A square court-yard has a rectangular gravel-walk round it. The side of the court wants 2 yards of being 6 times the breadth of the gravel-walk ; and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 92. Required the area of the court.

Let x = the breadth of the walk (in yards),

$\therefore 6x - 2$ = the side of the court,

and $4x - 2$ = the side of the interior square ;

$\therefore (6x - 2)^2 - (4x - 2)^2$ = the area of the walk,

and $20x^2 - 8x - 92 = 4 \times (6x - 2)$;

by transposition, $20x^2 - 32x = 84$;

$$\therefore x^2 - \frac{8}{5}x = \frac{21}{5} ;$$

completing the square, $x^2 - \frac{8}{5}x + \frac{16}{25} = \frac{21}{5} + \frac{16}{25} = \frac{121}{25}$;

extracting the root, $x - \frac{4}{5} = \pm \frac{11}{5}$;

$$\therefore x = 3, \text{ or } -\frac{7}{5},$$

and $(6x - 2)^2 = 16^2 = 256$, the area required.

21. A Merchant bought 54 gallons of Cogniac brandy, and a certain quantity of British. For the former he gave half as many shillings *per* gallon as there were gallons of British, and for the latter 4 shillings *per* gallon less. He sold the mixture at 10 shillings *per* gallon, and lost £28. 16s. by his bargain. Required the price of the Cogniac, and the number of gallons of British.

Let $2x$ = the number of gallons of British;

$\therefore x$ = the number of shillings one gallon of Cogniac cost,

and $54x$ = the price of all the Cogniac;

also $x - 4$ = the number of shillings one gallon of British cost,

and $2x^2 - 8x$ = the price of all the British;

$$\therefore 2x^2 - 8x + 54x = 10 \cdot (54 + 2x) + 576;$$

$$\text{by transposition, } 2x^2 + 26x = 1116,$$

$$\text{or } x^2 + 13x = 558;$$

$$\text{completing the square, } x^2 + 13x + \left(\frac{13}{2}\right)^2 = 558 +$$

$$\frac{169}{4} = \frac{2041}{4};$$

$$\text{extracting the root, } x + \frac{13}{2} = \pm \frac{49}{2},$$

$$\text{and } x = 18, \text{ or } -31;$$

$\therefore$ he bought 36 gallons of British: the Cogniac cost 18 shillings *per* gallon,

$$\text{and } \therefore \text{ the whole price} = \text{£}48. 12s.$$

22. During the time that the shadow on a sun-dial, which shows true time, moves from one o'clock to five, a clock, which is too fast a certain number of hours and minutes, strikes a number of strokes = that number of hours and minutes, and it is observed that the number of minutes is less by 41 than the square of the number which the clock strikes at the last time of striking. The clock does not strike twelve during the time. How much is it too fast?

Let x = the number of hours too fast;

then the clock strikes $(x + 2) + (x + 3) + (x + 4)$

$$+ (x + 5) \text{ times} = 4x + 14,$$

and the number of minutes $= x^2 + 10x + 25 - 41 =$

$$x^2 + 10x - 16;$$

$$\therefore x + x^2 + 10x - 16 = 4x + 14;$$

$$\text{by transposition, } x^2 + 7x = 30;$$

$$\text{completing the square, } x^2 + 7x + \frac{7^2}{2} = 30 + \frac{49}{4} = \frac{169}{4};$$

$$\text{extracting the root, } x + \frac{7}{2} = \pm \frac{13}{2};$$

$$\therefore x = 3, \text{ or } -10,$$

and the number of minutes $= 23$;

$\therefore$ the clock is too fast 3 hours and 23 minutes.

23. A Vintner sold 7 dozen of sherry and 12 dozen of claret for £50. He sold 3 dozen more of sherry for £10 than he did of claret for £6. Required the price of each.

Let x = the price of a dozen of sherry (in pounds);

$\therefore x : 10 :: 1$: the number of dozens of sherry for

$$£10, = \frac{10}{x},$$

and $\frac{10}{x} - 3 = \frac{10 - 3x}{x}$ = the number of dozens of claret for £6;

$\therefore \frac{10 - 3x}{x} : 1 :: 6$: the price of a dozen of claret =

$$\frac{6x}{10 - 3x};$$

$$\therefore 7x + \frac{72x}{10 - 3x} = 50,$$

$$\text{and } 70x - 21x^2 + 72x = 500 - 150x;$$

$$\text{by transposition, } 292x - 21x^2 = 500,$$

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$$\text{or } x^2 - \frac{292}{21} \cdot x = -\frac{500}{21};$$

completing the square,

$$x^2 - \frac{292}{21} \cdot x + \left| \frac{146}{21} \right|^2 = \frac{21316}{21^2} - \frac{500}{21} = \frac{10816}{21^2};$$

$$\text{extracting the root, } x - \frac{146}{21} = \pm \frac{104}{21},$$

$$\text{and } x = 2, \text{ or } \frac{250}{21};$$

$\therefore$ the price of a dozen of sherry was £2, and the price of a

$$\text{dozen of claret} = \frac{6x}{10 - 3x} = \text{£3}.$$

24. *A* and *B* hired a pasture into which *A* put 4 horses, and *B* as many as cost him 18 shillings a week. Afterwards *B* put in two additional horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired?

Let x = the number of *B*'s horses at first;

then $\frac{18}{x}$ = the pay of each *per* week (in shillings);

$$\therefore \frac{72}{x} = \text{what } A \text{ paid,}$$

and $\frac{72}{x} + 18$ = the price of the pasture;

also $x + 6$ = the whole number of horses in the second case;

$$\therefore x + 6 : x + 2 :: \frac{72}{x} + 18 : 20 :: 72 + 18x : 20x;$$

$$\therefore 20x^2 + 120x = 18x^2 + 108x + 144;$$

$$\text{by transposition, } 2x^2 + 12x = 144,$$

$$\text{or } x^2 + 6x = 72;$$

$$\text{completing the square, } x^2 + 6x + 9 = 72 + 9 = 81;$$

$$\text{extracting the root, } x + 3 = \pm 9,$$

$$\text{and } x = 6, \text{ or } -12;$$

$\therefore$ B had 6 horses in the pasture at first, and $\frac{72}{x} + 18 = 30$ shillings, *per week*, was the price of the pasture.

25. An Upholsterer has two square carpets divided into square yards by the lines of the pattern. Now he observes, that if he subtracts from the number of squares in the smaller carpet, the number of yards in the side of the other, the square of the remainder will exceed the difference of the number of squares in the smaller carpet, and the number of yards in its side, by 88. Also the difference of the lengths of the sides of the carpets is 6 feet. Required the size of each carpet.

Let x = the number of yards in a side of the less;

$\therefore x + 2$ = the number in a side of the greater,

$$\text{and } (x^2 - x - 2)^2 = x^2 - x + 88 = x^2 - x - 2 + 90;$$

$$\text{by transposition, } (x^2 - x - 2)^2 - (x^2 - x - 2) = 90;$$

$$\text{completing the square, } (x^2 - x - 2)^2 - (x^2 - x - 2) + \frac{1}{4} = 90 + \frac{1}{4} = \frac{361}{4};$$

$$\text{extracting the root, } x^2 - x - 2 - \frac{1}{2} = \pm \frac{19}{2},$$

and $x^2 - x = 12$, or -7 , the former of which only will give a possible value of x ; and $\therefore$

$$\text{completing the square, } x^2 - x + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4};$$

$$\text{extracting the root, } x - \frac{1}{2} = \pm \frac{7}{2},$$

and $x = 4$, or -3 ;

consequently the carpets contain 16 and 36 square yards, respectively.

26. A Man playing at hazard won at the first throw as much money as he had in his pocket; at the second throw he won 5 shillings more than the square root of what he then had; at the third throw he won the square of all he then had; and then he had £112. 16s. What had he at first?

Let x = the number of shillings he had at first;

$\therefore 2x$ = the number he had after the first throw,

$\sqrt{2x} + 5$ = the number won the second throw,

and $2x + \sqrt{2x} + 5$ = the number he had after the second throw;

also $(2x + \sqrt{2x} + 5)^2$ = the number won the third throw;

$\therefore (2x + \sqrt{2x} + 5)^2 + (2x + \sqrt{2x} + 5) = 2256$;

completing the square,

$$\begin{aligned} (2x + \sqrt{2x} + 5)^2 + (2x + \sqrt{2x} + 5) + \frac{1}{4} \\ = 2256 + \frac{1}{4} = \frac{9025}{4}; \end{aligned}$$

extracting the root, $2x + \sqrt{2x} + 5 + \frac{1}{2} = \pm \frac{95}{2}$;

$\therefore 2x + \sqrt{2x} = 42$, or -53 , the latter of which gives impossible values of x , and $\therefore 2x + \sqrt{2x} = 42$, to answer the conditions of the problem;

completing the square, $2x + \sqrt{2x} + \frac{1}{4} = 42 + \frac{1}{4} = \frac{169}{4}$;

extracting the root, $\sqrt{2x} + \frac{1}{2} = \pm \frac{13}{2}$;

$$\therefore \sqrt{2x} = 6, \text{ or } -7,$$

$$\text{and } 2x = 36, \text{ or } 49;$$

$$\therefore x = 18, \text{ or } \frac{49}{2};$$

and consequently he began with 18 shillings.

27. What number is that, which being divided by the product of its two digits, the quotient is two, and if 27 be added to it, the digits will be inverted?

Let x and y be the digits;

$$\therefore 10x + y = \text{the number},$$

$$\text{and } \frac{10x + y}{xy} = 2;$$

$$\therefore 10x + y = 2xy;$$

$$\text{also } 10x + y + 27 = 10y + x;$$

$$\text{by transposition, } 9x + 27 = 9y,$$

$$\text{or } x + 3 = y;$$

which value of y being substituted in the first equation,

$$10x + x + 3 = 2x \cdot (x + 3),$$

$$\text{or } 11x + 3 = 2x^2 + 6x;$$

$$\text{by transposition, } 2x^2 - 5x = 3,$$

$$\text{or } x^2 - \frac{5}{2}x = \frac{3}{2};$$

$$\text{completing the square, } x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{3}{2} + \frac{25}{16} = \frac{49}{16};$$

$$\text{extracting the root, } x - \frac{5}{4} = \pm \frac{7}{4};$$

and $x = 3$, or $-\frac{1}{2}$, the first of which only answers the conditions; $\therefore y = 6$, and the number is 36.

28. There are three numbers, the difference of whose differences is 8; their sum is 41; and the sum of their squares 699. What are the numbers?

Let x = the second number,

and y = the difference of the second and the least;

$\therefore x - y, x$, and $x + y + 8$ are the numbers,

and their sum $= 3x + 8 = 41$;

by transposition, $3x = 33$,

and $x = 11$;

$\therefore (11 - y)^2 + 121 + (19 + y)^2 = 699$;

or $603 + 16y + 2y^2 = 699$,

by transposition, $2y^2 + 16y = 96$,

and $y^2 + 8y = 48$;

completing the square, $y^2 + 8y + 16 = 64$;

extracting the root, $y + 4 = \pm 8$,

and $y = 4$, or -12 , both which values answer the conditions;
and the numbers are 7, 11, and 23.

29. There are three numbers, the difference of whose differences is 5; their sum is 44; and continual product is 1950. What are the numbers?

Let x = the second number,

and y = the difference of the second and the least;

$\therefore$ the numbers are $x - y, x, x + y + 5$,

and $(x - y + x + x + y + 5) = 3x + 5 = 44$;

by transposition, $3x = 39$,

and $x = 13$;

$\therefore (13 - y) \cdot 13 \cdot (18 + y) = 1950$,

and $(13 - y) \cdot (18 + y) = 150$,

or $234 - 5y - y^2 = 150$;

$\therefore y^2 + 5y = 84$;

completing the square, $y^2 + 5y + \frac{25}{4} = 84 + \frac{25}{4} = \frac{361}{4}$;

extracting the root, $y + \frac{5}{2} = \pm \frac{19}{2}$,

and $y = 7$, or -12 , both which values answer the conditions;
 $\therefore$ the numbers are 6, 13, and 25.

30. There were two rows of counters, of which the upper row exceeded the lower by one. A certain number having been taken from the upper row, and as many as then remained from the lower row, it was found that the square of the number remaining in the lower row, added to the square root of that number, is equal to 72 divided by the excess of the number taken from the upper row above unity. Required the number of counters taken from the upper row.

Let $x + 1 =$ the number in the upper row;

$\therefore x =$ the number in the lower;

$y + 1 =$ the number taken from the upper row;

$\therefore x - y =$ the remainder in the upper row,

and $y =$ the remainder in the lower row;

$$\therefore y^2 + \sqrt{y} = \frac{72}{y},$$

$$\text{and } y^3 + y^{\frac{1}{2}} = 72;$$

completing the square, $y^3 + y^{\frac{1}{2}} + \frac{1}{4} = 72 + \frac{1}{4} = \frac{289}{4}$;

extracting the root, $y^{\frac{3}{2}} + \frac{1}{2} = \pm \frac{17}{2}$;

$$\therefore y^{\frac{3}{2}} = 8, \text{ or } -9,$$

$$\text{and } y^3 = 64, \text{ or } 81;$$

$\therefore y = 4$, or $\sqrt[3]{81}$, the latter of which is excluded by the nature of the question.

31. A Grocer sold 80 pounds of mace, and 100 pounds of cloves for £65; but he sold 60 pounds more of cloves for £20, than he did of mace for £10. What was the price of a pound of each?

Let x = the price of a pound of mace (in pounds),

and y = the price of a lb. of cloves;

then $x : 10 :: 1 : \text{the number of lbs. of mace for } £10 = \frac{10}{x}$.

In the same way $\frac{20}{y}$ = the number of lbs. of cloves for £20;

$$\therefore \frac{20}{y} = 60 + \frac{10}{x},$$

$$\text{or } \frac{2}{y} = 6 + \frac{1}{x} = \frac{6x + 1}{x},$$

$$\text{and } \therefore y = \frac{2x}{6x + 1}.$$

$$\text{Again, } 80x + 100y = 65,$$

$$\text{or } 16x + 20y = 13;$$

$$\therefore 16x + \frac{40x}{6x + 1} = 13,$$

$$\text{and } 96x^2 + 56x = 78x + 13;$$

$$\text{by transposition, } 96x^2 - 22x = 13,$$

$$\text{and } x^2 - \frac{22}{96} \cdot x = \frac{13}{96};$$

$$\text{completing the square, } x^2 - \frac{22}{96} \cdot x + \left| \frac{11}{96} \right|^2 = \frac{121}{96^2} +$$

$$\frac{13}{96} = \frac{1369}{96^2};$$

$$\text{extracting the root, } x - \frac{11}{96} = \pm \frac{37}{96};$$

$$\text{and } \therefore x = \frac{1}{2}, \text{ or } -\frac{13}{48}, \text{ which last does not}$$

answer the conditions; and $y = \frac{1}{4}$.

∴ the price of a pound of mace is 10 shillings, and of a pound of cloves is 5 shillings.

32. *A* and *B* engage to reap a field for £4. 10s.; and as *A* alone could reap it in 9 days, they promise to complete it in 5 days. They found however that they were obliged to call in *C*, an inferior workman, to assist them for the two last days, in consequence of which *B* received 3s. 9d. less than he otherwise would have done. In what time could *B* or *C* alone reap the field?

Let x = the number of days in which *B* could reap the field,
and y = the number in which *C* could reap it;

then $\frac{1}{9} + \frac{1}{x} : \frac{1}{x} :: 90 : \text{the number of shillings } B \text{ would have}$
received $= \frac{810}{9+x}$,

and $\frac{5}{x} \times 90 = \frac{450}{x}$ = the number he did receive;

$$\therefore \frac{810}{9+x} - \frac{450}{x} = 3\frac{3}{4} = \frac{15}{4},$$

$$\text{or } \frac{54}{9+x} - \frac{30}{x} = \frac{1}{4};$$

$$\therefore 216x - 1080 - 120x = 9x + x^2;$$

$$\text{by transposition, } x^2 - 87x = -1080;$$

$$\text{completing the square, } x^2 - 87x + \frac{7569}{4} = \frac{7569}{4} -$$

$$1080 = \frac{3249}{4};$$

$$\text{extracting the root, } x - \frac{87}{2} = \pm \frac{57}{2},$$

$$\text{and } x = 72, \text{ or } 15.$$

$$\text{Let } x = 15, \text{ then } \frac{5}{9} + \frac{5}{15} + \frac{2}{y} = 1;$$

$$\text{by transposition, } \frac{2}{y} = 1 - \frac{5}{9} - \frac{1}{3} = \frac{1}{9};$$

$\therefore y = 18$ the number of days in which C could reap the field. The other value of x is excluded by the nature of the question.

33. Throwing out the three court cards from a suit of spades, and placing the remainder in two heaps, I find the sum of the pips in the smaller heap is to the sum in the greater, as the number of cards in the greater heap is to the number of cards in the smaller. But if I add the seventh card to the smaller heap, the difference of the number of pips in the two heaps is equal the square of the number of cards in the smaller. Required the number of pips and cards in each.

The whole number of cards = 10,

and the whole number of pips = 55.

If $\therefore x$ = the number of cards in the larger heap ;

$10 - x$ = the number in the smaller,

and if y = the number of pips in the smaller ;

$55 - y$ = the number in the larger ;

$$\therefore y : 55 - y :: x : 10 - x,$$

$$\text{and (Alg. 179.) } y : 55 :: x : 10;$$

$$\text{or } y : 11 :: x : 2;$$

$$\therefore 2y = 11x.$$

Again, after the change, $y + 7$ = the number of pips in the smaller heap,

and $55 - y - 7 = 48 - y$ = the number of pips in the larger heap,

$$\text{and } \therefore \text{their difference} = 2y - 41 = (11 - x)^2,$$

and by substituting for $2y$ its value, $11x - 41 = (11 - x)^2$
 $= 121 - 22x + x^2$,

and $\therefore$ by transposition, $x^2 - 33x = -162$;

completing the square, $x^2 - 33x + \left(\frac{33}{2}\right)^2 = \frac{1089}{4} - 162 = \frac{441}{4}$;

extracting the root, $x - \frac{33}{2} = \pm \frac{21}{2}$,

and $x = 6$, or 27 ; but 27 being inconsistent with the nature of the problem, the number of cards in the larger heap $= 6$, and $\therefore$ the number in the smaller heap $= 4$; and the number of pips in the smaller $= \frac{11x}{2} = 33$, and $\therefore$ the number in the greater heap $= 22$.

34. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the periphery of each wheel be increased one yard, it will make only 4 revolutions more than the hind-wheel in the same space. Required the circumference of each.

Let x = the number of yards in the circumference of the larger,

and y = the number in the circumference of the less;

$$\text{then } \frac{120}{x} = \frac{120}{y} - 6,$$

$$\text{or } 20y = 20x - xy;$$

$$\therefore \text{ by transposition, } xy = 20x - 20y.$$

$$\text{Again, } \frac{120}{x+1} = \frac{120}{y+1} - 4,$$

$$\text{or } 30 \cdot (y+1) = (x+1) \cdot (29-y),$$

$$\text{or } 30y + 30 = 29x + 29 - xy - y;$$

$$\text{by transposition, } xy = 29x - 1 - 31y,$$

$$\text{and } \therefore 29x - 1 - 31y = 20x - 20y;$$

$$\text{by transposition, } 9x = 11y + 1,$$

$$\text{or } x = \frac{11y + 1}{9};$$

$$\therefore \text{ by substitution, } \frac{11y^2 + y}{9} = \frac{220y + 20}{9} - 20y,$$

$$\text{or } 11y^2 + y = 220y + 20 - 180y = 40y + 20;$$

$$\therefore y^2 - \frac{39}{11}y = \frac{20}{11};$$

$$\text{completing the square, } y^2 - \frac{39}{11}y + \left(\frac{39}{22}\right)^2 = \frac{1521}{(22)^2} +$$

$$\frac{20}{11} = \frac{2401}{(22)^2};$$

$$\text{extracting the root, } y - \frac{39}{22} = \pm \frac{49}{22};$$

$$\therefore y = 4, \text{ or } -\frac{5}{11};$$

$\therefore$ the number of yards in the circumference of the less = 4
and the number in the circumference of the greater

$$= \frac{11y + 1}{9} = 5.$$

35. On the late jubilee, a gentleman treated his tenantry at the following rate. He allowed for each poor child a certain number of sixpences, for each poor woman sixpence more, and for each poor man sixpence still in addition. The number of women was one-fourth greater than the number of men; the number of children was equal to twice the square of the difference between the numbers of men and women; and the whole expence was £s. 2s. But had

each child been allowed as much as each woman, the expence on their account added to nine times the difference of what the men and women cost, would have been £4. 18s. Required the number of men, women, and children, and the allotment to each.

Let $4x$ = the number of men ;

$\therefore 5x$ = the number of women,

and $2x^2$ = the number of children.

Let y = the number of sixpences each child had ;

$\therefore y + 1$ = the number each woman had ;

and $y + 2$ = the number each man had ;

$$\therefore 2x^2y + 9xy + 13x = 324 ;$$

$$\text{also } 2x^2y + 2x^2 + 9xy - 27x = 196 ;$$

$$\therefore \text{by subtraction, } 2x^2 - 40x = -128,$$

$$\text{and } x^2 - 20x = -64 ;$$

completing the square, $x^2 - 20x + 100 = 100 - 64 = 36 ;$

extracting the root, $x - 10 = \pm 6,$

and $x = 16$, or 4 , the former of which will not answer the conditions of the problem ; $\therefore$ the number of men was 16, of women 20, and of children 32.

$$\text{Also } 32y + 36y + 52 = 324,$$

$$\text{or } 68y = 272 ;$$

$$\therefore y = 4 ;$$

$\therefore$ each child had 2 shillings, each woman 2s. 6d., and each man 3s.

36. *A* and *B* were going to market, the first with cucumbers, and the second with three times as many eggs ; and they find that if *B* gave all his eggs for the cucumbers, *A* would lose 10 pence, according to the rate at which they were then selling. *A* therefore reserves two-fifths of his

cucumbers; by which B would lose sixpence, according to the same rate. But B , selling the cucumbers at sixpence apiece, gains upon the whole the price of six eggs. Required the number of eggs and cucumbers, and their price.

Let x = the number of cucumbers,

and y = the price of one;

$\therefore 3x$ = the number of eggs,

and $\frac{xy - 10}{3x}$ = the price of one egg;

$$\text{also } \frac{3}{5}xy = xy - 16,$$

$$\text{or } 3xy = 5xy - 80;$$

$$\therefore 2xy = 80,$$

$$\text{and } xy = 40;$$

$$\text{also } \frac{3}{5} \cdot 6x - (xy - 10) = \frac{2 \cdot (xy - 10)}{x},$$

$$\text{or } \frac{18}{5}x^2 - 30x = 60,$$

$$\text{and } 9x^2 - 75x = 150;$$

$$\begin{aligned} \text{completing the square, } 9x^2 - 75x + \frac{25}{4} &= \frac{8625}{4} + 150 \\ &= \frac{1225}{4}; \end{aligned}$$

$$\text{extracting the root, } 3x - \frac{25}{2} = \pm \frac{35}{2},$$

and $3x = 30$, or -5 , the latter of which is excluded by the nature of the problem; $\therefore x = 10$,

$$\text{and } y = \frac{40}{x} = 4.$$

Hence the number of eggs was 30, and of cucumbers 10;

$\therefore$ the price of a cucumber was 4 pence, and of an egg
 $= \frac{xy - 10}{3x} = 1$ penny.

37. A person bought a certain number of larks and sparrows for 6 shillings. He gave as many pence *per* dozen for larks as there were sparrows, and as many pence *per* score for sparrows as there were larks. If he had bought 10 more of each, (the price of larks remaining the same,) and had given as much *per* dozen for sparrows as he gave *per* score for larks, they would have cost £1. 5s. 5d. Required the number of each.

Let x = the number of larks, and $\therefore$ = the number of pence *per* score for sparrows,

y = the number of sparrows, and $\therefore$ = the number of pence *per* dozen for larks;

$$\therefore \left(\frac{xy}{12} + \frac{xy}{20} \right) \frac{2xy}{15} = 72,$$

$$\text{and } xy = 15 \times 36 = 540.$$

Again, if $x + 10$ = the number of larks,

and $y + 10$ = the number of sparrows;

$$\text{then the price of the larks} = y \times \frac{x + 10}{12} = \frac{xy + 10y}{12}$$

$$= 10 \cdot \frac{54 + y}{12},$$

$$\text{and the price per dozen of sparrows} = \frac{y}{12} \times 20 = \frac{5y}{3};$$

$$\therefore \text{ the price of the sparrows} = \frac{5y^2 + 50y}{3 \times 12},$$

$$\text{and } \frac{54 + y}{12} \times 10 + \frac{5 \cdot (y^2 + 10y)}{3 \times 12} = 305,$$

$$\text{and } (54 + y) 30 + 5 \cdot (y^2 + 10y) = 36 \times 305,$$

$$\text{or } y^2 + 16y + 324 = 36 \times 61 = 2196;$$

$$\text{by transposition, } y^2 + 16y = 1872;$$

$$\text{completing the square, } y^2 + 16y + 64 = 1872 + 64 = 1936;$$

$$\text{extracting the root, } y + 8 = \pm 44;$$

$\therefore y = 36$, or -52 , the latter of which will not answer the conditions of the problem,

$$\text{and } x = \frac{15 \times 36}{y} = 15;$$

$\therefore$ he bought 15 larks, and 36 sparrows.

38. A Poulterer bought a certain number of ducks and 18 turkeys for £5. 10s.; each turkey costing within one shilling as much as three ducks. He afterwards bought as many ducks and 5 over, and 20 turkeys, giving one shilling a piece more for each duck and turkey than before; and finds that the value of his former purchase is to the value of the present one $\therefore 2 : 3$. Required the number of ducks, and the prices of the ducks and turkeys at the first purchase.

Let x = the number of ducks required,

and y = the price of a duck;

$$\therefore 3y - 1 = \text{the price of a turkey,}$$

$$\text{and } xy + 54y - 18 = 110,$$

$$\text{or } xy + 54y = 128.$$

Now at the second purchase, $x + 5$ = the number of ducks, $y + 1$, and $3y$ = the price of a duck and turkey, respectively;

$$\therefore 110 : xy + x + 65y + 5 :: 2 : 3,$$

$$\text{or } 55 : xy + x + 65y + 5 :: 1 : 3;$$

$$\therefore xy + x + 65y + 5 = 165,$$

$$\text{and } xy + x + 65y = 160;$$

$$\text{but } xy + 54y = 128;$$

$$\therefore \text{ by subtraction, } x + 11y = 32,$$

or $x = 32 - 11y$, which being substituted in the first equation,

$$32y - 11y^2 + 54y = 128,$$

$$\text{or } 11y^2 - 86y = -128,$$

$$\text{and } y^2 - \frac{86}{11}y = -\frac{128}{11};$$

$$\text{completing the square, } y^2 - \frac{86}{11}y + \left(\frac{43}{11}\right)^2 = \frac{1849}{121} -$$

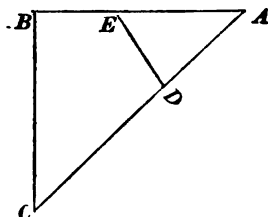
$$\frac{128}{11} = \frac{441}{821};$$

$$\text{extracting the root, } y - \frac{43}{11} = \pm \frac{21}{11},$$

and $y = \frac{64}{11}$, or 2, the former of which makes x negative, and $\therefore$ the price of a duck is 2 shillings, and the price of a turkey = 5 shillings. Also the number of ducks = $32 - 11y = 10$.

39. There are three towns A , B , and C ; the road from B to A forming a right angle with that from B to C . Now a person has to go from B to A , but after travelling a certain distance towards A , he crosses over by the nearest way to the road which leads from C to A , and when on this road he is 3 miles from A and 7 from C . He then proceeds to A , and when arrived there he finds that he has gone a distance, equal to one-fourth of the distance from B to C , more than he would have done, had he gone the direct road from B to A . Required the distance of B from A and C .

Let $BC = x$, $BA = y$, $AC = 10$,



and since the shortest path from a given point to a given straight line is a perpendicular drawn from that point, draw ED perpendicular to AC ; E being the point where he leaves the road BA ; $\therefore D$ is the point where he enters the road CA ; $\therefore CD = 7$, and $DA = 3$.

By similar Δ 's $BA : CA :: DA : AE$,

$$\text{or } y : 10 :: 3 : AE = \frac{30}{y},$$

and $BA : BC :: DA : DE$,

$$\text{or } y : x :: 3 : DE = \frac{3x}{y};$$

$$\therefore \frac{3x}{y} + 3 = \frac{30}{y} + \frac{x}{4},$$

$$\text{or } 3 \cdot (x + y) = 30 + \frac{1}{4}xy,$$

$$\text{and } 2xy - 24 \cdot (x + y) = -240;$$

$$\text{but } x^2 + y^2 = 100;$$

$\therefore$ by addition, $x^2 + 2xy + y^2 - 24 \cdot (x + y) = -140$;
 completing the square, $(x + y)^2 - 24 \cdot (x + y) + 144 =$
 $144 - 140 = 4$;

extracting the root, $x + y - 12 = \pm 2$;

$\therefore x + y = 14$, or 10 , the former of which only

answers the conditions, $\therefore x^2 + 2xy + y^2 = 196$,
but $2x^2 \quad \quad + 2y^2 = 200$;

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 4$,
whence $x - y = \pm 2$,
but $x + y = 14$;

$\therefore$ by addition, $2x = 16$, or 12 ,
and by subtraction, $2y = 12$, or 16 ;
 $\therefore x = 8$, or 6 ,
and $y = 6$, or 8 ;

$\therefore$ the distance of B from C is 8 , or 6 miles, and from A is 6 , or 8 miles.

-
40. In a garden is a square bowling-green, a side of which is 30 yards, and near to it is a rectangular grass-plot. The number of square yards in the area of the grass-plot is a mean proportional between $\frac{192}{79}$, and the number of square yards contained in the grass-plot and bowling-green together. Also the number of square yards contained in the square described on the diameter of the grass-plot is a mean proportional between 10, and the number of square yards contained in the aforesaid square increased by the number contained in the bowling-green. Required the area and sides of the grass-plot.

Let x and y = the number of yards in the sides ;

$\therefore xy$ = the area,

and $\frac{192}{79} : xy :: xy : xy + 900$;

$$\therefore x^2 y^2 = \frac{192}{79} \cdot xy + \frac{172800}{79};$$

s 2

by transposition, $x^2y^3 - \frac{192}{79}xy = \frac{172800}{79}$;

completing the square, $x^2y^3 - \frac{192}{79}xy + \left[\frac{96}{79}\right]^2 = \frac{172800}{79} + \left[\frac{96}{79}\right]^2$
 $= \frac{13660416}{79^3}$;

extracting the root, $xy - \frac{96}{79} = \pm \frac{3696}{79}$;

$\therefore xy = 48$, or $-\frac{3600}{79}$, the former of which only answers the conditions of the problem.

Again, $10 : x^3 + y^3 :: x^3 + y^3 : x^3 + y^3 + 900$;

$\therefore (x^3 + y^3)^2 = 10 \cdot (x^3 + y^3) + 9000$;

by transposition, $(x^3 + y^3)^2 - 10 \cdot (x^3 + y^3) = 9000$;

completing the square, $(x^3 + y^3)^2 - 10 \cdot (x^3 + y^3) + 25 = 9025$;

extracting the root, $x^3 + y^3 - 5 = 95$;

$\therefore x^3 + y^3 = 100$, or -90 , the latter of which will not answer the conditions;

but, $2xy = 96$;

$\therefore$ by addition, $x^3 + 2xy + y^3 = 196$, and $x + y = \pm 14$;

by subtraction, $x^3 - 2xy + y^3 = 4$, and $x - y = \pm 2$;

$\therefore$ by addition, $2x = \pm 16$, or ± 12 ;

$\therefore x = \pm 8$, or ± 6 ;

by subtraction, $2y = \pm 12$, or ± 16 ;

$\therefore y = \pm 6$, or ± 8 ;

but the positive values will only answer the conditions, and the area is 48 square yards.

41. A rectangular vat, 3 feet deep, when filled to the depth of 2 feet, holds less than when completely filled by a

number of cubic feet equal to 24, together with half the number of feet in the perimeter of the base. It is also observed, that the length of a pole, which reaches from one of the corners of the top to the opposite corner of the bottom of the vat, is equal to one-eighth of the number of feet in the square inscribed on the diagonal of the bottom. Required the dimensions of the vat.

Let x and y be the number of feet in the sides of the base ;

$$\text{then } (3xy - 2xy =) xy = 24 + x + y,$$

$$\text{and } \sqrt{9 + x^2 + y^2} = \frac{1}{8} \cdot (x^2 + y^2) ;$$

$$\therefore x^2 + y^2 - 8\sqrt{9 + x^2 + y^2} = 0,$$

$$\text{and } (9 + x^2 + y^2) - 8\sqrt{9 + x^2 + y^2} = 9 ;$$

$$\text{completing the square, } (9 + x^2 + y^2) - 8\sqrt{9 + x^2 + y^2} + 16 = 25 ;$$

$$\text{extracting the root, } \sqrt{9 + x^2 + y^2} - 4 = \pm 5,$$

$$\text{and } \sqrt{9 + x^2 + y^2} = 9, \text{ or } -1 ;$$

$$\therefore 9 + x^2 + y^2 = 81, \text{ or } 1 ;$$

by transposition, $x^2 + y^2 = 72$, or -8 , the latter of which is impossible ;

$$\text{but } 2xy = 48 + 2 \cdot (x + y) ;$$

$$\therefore \text{ by addition, } x^2 + 2xy + y^2 = 120 + 2 \cdot (x + y) ;$$

$$\text{by transposition, } (x + y)^2 - 2 \cdot (x + y) = 120 ;$$

$$\text{completing the square, } (x + y)^2 - 2 \cdot (x + y) + 1 = 121 ;$$

$$\text{extracting the root, } x + y - 1 = \pm 11,$$

and $x + y = 12$, or -10 , the latter of which is impossible ;

$$\therefore x^2 + 2xy + y^2 = 144 ;$$

$$\text{but } 4xy = 144;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 0,$$

$$\text{and } x - y = 0;$$

$$\text{but } x + y = 12;$$

$$\therefore \text{ by addition, } 2x = 12,$$

$$\text{and } x = 6;$$

$$\therefore y = x = 6, \text{ and the base is a square whose side is 6 feet.}$$

42. A person bought two cubical stacks of hay for £41, each of which cost as many shillings *per* solid yard as there were yards in a side of the other, and the greater stood on more ground than the less by 9 square yards. What was the price of each?

Let x = the number of yards in a side of the larger,

and y = the number in a side of the less;

then x^3 and y^3 = the number of solid yards in the stacks,

and x^2 , and y^2 = the number of square yards in their bases;

$$\therefore x^2 - y^2 = 9,$$

$$\text{and } x^3y + y^3x = 820;$$

$$\therefore x^2 + y^2 = \frac{820}{xy},$$

$$\text{and } x^4 + 2x^2y^2 + y^4 = \frac{820^2}{x^2y^2};$$

$$\text{but } x^4 - 2x^2y^2 + y^4 = 81;$$

$$\therefore \text{ by subtraction, } 4x^2y^2 = \frac{820^2}{x^2y^2} - 81,$$

$$\text{or } x^2y^2 = \frac{820^2}{4} - \frac{81x^2y^2}{4};$$

by transposition, $x^2y^2 + \frac{81}{4} \cdot x^2y^2 = \left(\frac{820}{2}\right)^2 = 410^2 = 168100$;

completing the square, $x^2y^2 + \frac{81}{4} \cdot x^2y^2 + \frac{81}{8} = 168100$
 $+ \frac{6561}{64} = \frac{10764961}{64}$;

extracting the root, $x^2y^2 + \frac{81}{8} = \pm \frac{3281}{8}$;

$\therefore x^2y^2 = 400$, or $-\frac{1681}{4}$, the latter of which is impossible;

$\therefore xy = \pm 20$, the positive value only answering the conditions of the problem;

$$\text{and } x^2 + y^2 = \frac{820}{xy} = \frac{820}{20} = 41;$$

$$\text{but } 2xy = 40;$$

$\therefore$ by addition, $x^2 + 2xy + y^2 = 81$, and $x + y = \pm 9$,
 and by subtraction, $x^2 - 2xy + y^2 = 1$; $\therefore x - y = \pm 1$;

$\therefore$ by addition, $2x = \pm 10$, or ± 8 ,

and $x = \pm 5$, or ± 4 ;

by subtraction, $2y = \pm 8$, or ± 10 ,

and $y = \pm 4$, or ± 5 ;

$\therefore$ the prices were £25, and £16.

43. *A* and *B* put out different sums to interest, amounting together to £200. *B*'s rate of interest was £1 *per cent.* more than *A*'s. At the end of 5 years, *B*'s accumulated simple interest wanted but £4 to be double of *A*'s. At the end of 10 years, *A*'s principal and interest was to *B*'s as 5 : 8. Required the separate sums put out by each, and the rate *per cent.*

Examples of the Solution of Problems

Let $4x = A$'s money (in pounds);

$\therefore 4 \cdot (50 - x) = B$'s money;

$y = A$'s rate of interest;

$\therefore y + 1 = B$'s rate;

$\therefore \frac{xy}{5} = A$'s interest after 5 years,

and $\frac{(50 - x) \cdot (y + 1)}{5} = B$'s interest after 5 years;

$$\therefore \frac{(50 - x) \cdot (y + 1)}{5} + 4 = 2 \cdot \frac{xy}{5},$$

$$\text{or } 50y - xy + 50 - x + 20 = 2xy;$$

$$\therefore 3xy = 50y + 70 - x \quad . \quad . \quad . \quad (1).$$

Again, after 10 years, A 's capital and interest =

$$4x + \frac{2xy}{5} = 2 \cdot \frac{10x + xy}{5},$$

$$\text{and } B's = 2 \cdot \frac{(50 - x) \cdot (y + 11)}{5};$$

$$\therefore 2 \cdot \frac{10x + xy}{5} : 2 \cdot \frac{(50 - x) \cdot (y + 11)}{5} :: 5 : 8;$$

$$\therefore 80x + 8xy = 250y - 5xy + 2750 - 55x;$$

$$\text{by transposition, } 13xy = 250y + 2750 - 135x \quad (2)$$

$$\text{but from (1) } 15xy = 250y + 350 - 5x;$$

$$\therefore \text{by subtraction, } 2xy = 130x - 2400 \quad . \quad . \quad . \quad (3)$$

multiplying (1) by 13, and (2) by 3;

$$\therefore 650y + 910 - 13x = (39xy =) 750y + 8250 - 405x;$$

$$\text{by transposition, } 100y = 392x - 7340,$$

or $50y = 196x - 3670$, which being multiplied by x ,

$$50xy = 196x^2 - 3670x; \text{ but (3) being multiplied by 25,}$$

$$50xy = 3250x - 60000;$$

$$\therefore 196x^2 - 3670x = 3250x - 60000;$$

$$\text{by transposition, } 196x^2 - 6920x = -60000;$$

$$\begin{aligned} &\text{completing the square, } 196x^2 - 6920x + \frac{1730}{7} \Big| ^2 \\ &= \frac{2992900}{49} - 60000 = \frac{52900}{49}; \end{aligned}$$

$$\text{extracting the root, } 14x - \frac{1730}{7} = \pm \frac{230}{7};$$

$$\therefore 14x = 280, \text{ or } \frac{1500}{7};$$

$$\therefore x = 20;$$

$$\therefore A\text{'s money} = 4x = 80 \text{ pounds, and } B\text{'s} = 120 \text{ pounds,}$$

$$\text{and } y = \frac{196 \times 20 - 3670}{50} = 5;$$

$$\therefore A\text{'s rate of interest was } 5 \text{ per cent. and } B\text{'s } 6 \text{ per cent.}$$

44. When the price of brandy was three times the price of British spirit, a merchant made two mixtures of brandy and British spirit, and the prices *per* gallon were in the ratio of 9 to 10. He afterwards mixed twice as much brandy with the same quantity of British spirit in each case, and the relative price was the same as before. Required the ratio of the quantities mixed.

Suppose at first x gallons of British spirit were mixed with one gallon of brandy, in one case; and y gallons of British spirit with one gallon of brandy in the other case; then

$$\begin{aligned} &x + 1 : 1 :: x + 3 : \frac{x+3}{x+1} \Bigg\} \begin{array}{l} \text{the relative prices of the first} \\ \text{mixtures per gallon;} \end{array} \\ &\text{and } y + 1 : 1 :: y + 3 : \frac{y+3}{y+1} \end{aligned}$$

$$\text{hence, } \frac{x+3}{x+1} : \frac{y+3}{y+1} :: 9 : 10.$$

$$\text{In the same manner } \frac{x+6}{x+2} : \frac{y+6}{y+2} :: 9 : 10,$$

$$\text{and } \therefore xy + x + 3y + 3 : xy + 3x + y + 3 :: 9 : 10,$$

$$\text{div}^{\text{do}}. xy + x + 3y + 3 : 2x - 2y :: 9 : 1;$$

$$\therefore xy + x + 3y + 3 = 18x - 18y;$$

$$\text{by transposition, } xy - 17x + 21y + 3 = 0;$$

$$\text{also } xy + 2x + 6y + 12 : xy + 6x + 2y + 12 :: 9 : 10;$$

$$\text{div}^{\text{do}}. xy + 2x + 6y + 12 : 4x - 4y :: 9 : 1;$$

$$\therefore xy + 2x + 6y + 12 = 36x - 36y;$$

$$\text{by transposition, } xy - 34x + 42y + 12 = 0;$$

$$\text{but } xy - 17x + 21y + 3 = 0;$$

$$\therefore \text{by subtraction, } 17x - 21y - 9 = 0;$$

$$\text{also } 2xy - 34x + 42y + 6 = 0,$$

$$\text{and } xy - 34x + 42y + 12 = 0;$$

$$\therefore \text{by subtraction, } xy - 6 = 0,$$

$$\text{or } xy = 6.$$

$$\text{Now } x = \frac{21y + 9}{17};$$

$$\therefore \frac{21y^2 + 9y}{17} = (xy =) 6,$$

$$\text{and } 21y^2 + 9y = 102,$$

$$\text{or } y^2 + \frac{3}{7}y = \frac{34}{7};$$

$$\text{completing the square, } y^2 + \frac{3}{7}y + \frac{9}{196} = \frac{34}{7} + \frac{9}{196} = \frac{961}{196};$$

$$\text{extracting the root, } y + \frac{3}{14} = \pm \frac{31}{14};$$

$$\therefore y = 2, \text{ or } -\frac{17}{7};$$

$$\therefore x = 3;$$

$\therefore$ the first mixtures were in the ratio of 3 to 1, and 2 to 1;
and the second in the ratio of 3 to 2, and of equality.

SECTION X.

Examples of the Solution of Problems in Arithmetical and Geometrical Progressions.

1. A PERSON bought 7 books, the particular prices of which (in shillings) were in arithmetical progression. The price of the next above the cheapest was 8 shillings, and the price of the dearest 23 shillings. What was the price of each book?

Let x = the price of the cheapest,
and y = the common difference;
then $x + y$ = the price of the second = 8,
and $x + 6y$ = the price of the dearest = 23;

$$\therefore \text{by subtraction, } 5y = 15,$$

$$\text{and } y = 3;$$

$$\therefore x = 8 - y = 8 - 3 = 5,$$

and $\therefore$ the prices are 5, 8, 11, 14, 17, 20, 23 shillings, respectively.

2. A number consists of 3 digits, which are in arithmetical progression; and this number divided by the sum of its digits is equal to 26; but if 198 be added to it, the digits will be inverted. Required the number.

$$\left. \begin{array}{l} \text{Let } x - y \\ x \\ x + y \end{array} \right\} \text{ represent the digits;}$$

then the number will be $100 \cdot (x - y) + 10x + x + y = 111x - 99y$;

$$\therefore \frac{111x - 99y}{3x} = 26,$$

$$\text{or } 37x - 33y = 26x,$$

and $\therefore$ by transposition, $11x = 33y$,

$$\text{and } x = 3y.$$

Again, $111x - 99y + 198 = 100 \cdot (x + y) + 10x + (x - y) = 111x + 99y$;

$\therefore$ by transposition, $198y = 198$,

$$\text{and } y = 1;$$

$$\therefore x = 3y = 3;$$

$\therefore$ the digits are 2, 3, and 4, and the number = 234.

3. The sum of £1. 7s. was to be raised by subscription by three persons *A*, *B*, and *C*; the sums to be subscribed by them respectively forming an arithmetical progression. But *C* dying before the money was paid, the whole fell to *A* and *B*; and *C*'s share was raised between them in the proportion of 3 : 2, when it appeared that the whole sum subscribed by *A* was to the whole sum subscribed by *B* :: 4 : 5. Required the original subscriptions of *A*, *B*, and *C*.

Let $x - y$, x , $x + y$, be the respective subscriptions of *A*, *B*, and *C*;

then $3x = 27$; and $\therefore x = 9$.

Now $5 : 2 :: (C\text{'s share}) 9 + y : \text{the part paid by } B$
 $= \frac{2}{5} \cdot (9 + y),$

and $5 : 3 :: 9 + y : \text{the part paid by } A = \frac{3}{5} \cdot (9 + y),$

and consequently, A paid upon the whole $9 - y + \frac{3}{5} \cdot (9 + y)$
 $= \frac{72 - 2y}{5};$

also B paid upon the whole $9 + \frac{2}{5}(9 + y) = \frac{63 + 2y}{5};$

hence, $72 - 2y : 63 + 2y :: 4 : 5,$

and (*Alg.* 179.) $135 : 63 + 2y :: 9 : 5,$

and (*Alg.* 186.) $15 : 63 + 2y :: 1 : 5;$

$\therefore (21) 63 + 2y = 75;$

by transposition, $2y = 12,$

and $y = 6;$

$\therefore$ the sums to be subscribed originally were 3, 9, and 15 shillings.

4. Four numbers are in arithmetical progression. The sum of their squares is equal to 276, and the sum of the numbers themselves is equal to 32. What are the numbers?

Let $2y =$ the common difference,

and $\left. \begin{array}{l} x + 3y \\ x + y \\ x - y \\ x - 3y \end{array} \right\}$ be the numbers;

then their sum $= 4x = 32,$

and $\therefore x = 8;$

also the sum of their squares $= 4x^2 + 20y^2 = 276,$ in which substituting the value of x found above,

$256 + 20y^2 = 276;$

by transposition, $20y^2 = 20;$

$\therefore y^2 = 1;$

$$\text{and } y = \pm 1;$$

hence the numbers are 11, 9, 7, 5.

5. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

Supposing as before, $x + 3y$, $x + y$, $x - y$, and $x - 3y$, to be the numbers;

$$\text{then } 2x^2 + 18y^2 = 200,$$

$$\text{and } 2x^2 + 2y^2 = 136;$$

$$\therefore \text{ by subtraction, } 16y^2 = 64,$$

$$\text{and } 4y = \pm 8;$$

$$\therefore y = \pm 2;$$

$$\text{whence } x^2 = 68 - y^2 = 68 - 4 = 64,$$

$$\text{and } x = \pm 8,$$

and $\therefore$ the numbers are $\pm 14, \pm 10, \pm 6, \pm 2$.

6. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. Required the numbers.

Let x, xy, xy^2, xy^3 , be the numbers;

$$\therefore x + xy = 15,$$

$$\text{and } xy^2 + xy^3 = 60,$$

$$\text{or } y^2 \cdot (x + xy) = 60,$$

$$\text{or } 15y^2 = 60;$$

$$\therefore y^2 = 4,$$

$$\text{and } y = \pm 2,$$

$$\text{and } (x + 2x) 3x = 15;$$

$$\therefore x = 5,$$

and the numbers are 5, 10, 20, 40.

7. The sum of four numbers in geometrical progression is equal to the common ratio + 1 ; and the first term = $\frac{1}{17}$.

Required the numbers.

Let x = the common ratio ;

$$\therefore \text{the numbers are } \frac{1}{17}, \frac{x}{17}, \frac{x^2}{17}, \frac{x^3}{17},$$

$$\text{and } 1 + x = \frac{1 + x + x^2 + x^3}{17} = \frac{(1 + x^3) \cdot (1 + x)}{17},$$

$$\text{and } 1 = \frac{1 + x^3}{17} ;$$

$$\therefore 17 = 1 + x^3,$$

$$\text{and } 16 = x^3 ;$$

$$\therefore \pm 4 = x,$$

$$\text{and the numbers are } \frac{1}{17}, \frac{4}{17}, \frac{16}{17}, \frac{64}{17}.$$

8. A regiment of militia was just sufficient to form an equilateral wedge. It was afterwards doubled by the supplementary, but was still found to want 385 men to complete a square containing 5 more men in a side, than in a side of the wedge. How many did the regiment at first contain ?

Let x = the number of men in a side of the wedge ;

$$\therefore (\text{Alg. 212.}) (x + 1) \cdot \frac{x}{2} = \text{the number of men in the wedge ;}$$

$$\therefore (x + 1) \cdot x + 385 = (x + 5)^2,$$

$$\text{or } x^2 + x + 385 = x^2 + 10x + 25 ;$$

$$\therefore \text{by transposition, } 360 = 9x,$$

$$\text{and } 40 = x ;$$

$$\therefore \text{the number of men} = 820.$$

9. After *A*, who travelled at the rate of 4 miles an hour, had been set out two hours and three-quarters, *B* set out to overtake him, and in order thereto went four miles and a half the first hour, four and three-quarters the second, five the third; and so on, gaining a quarter of a mile every hour. In how many hours would he overtake *A*?

Let x = the number of hours;

$\therefore$ (*Alg.* 212.) $\left(9 + (x - 1) \frac{1}{4}\right) \times \frac{x}{2}$ = the whole number of miles he travelled;

but $11 + 4x$ = the whole number *A* travelled;

$$\therefore \left(9 + \frac{1}{4}(x - 1)\right) \cdot \frac{x}{2} = 11 + 4x,$$

$$\text{or } 9x + \frac{x^2}{4} - \frac{x}{4} = 22 + 8x,$$

$$\text{and by transposition, } \frac{x^2}{4} + \frac{3x}{4} = 22;$$

$$\text{completing the square, } \frac{x^2}{4} + \frac{3x}{4} + \frac{9}{16} = 22 + \frac{9}{16} = \frac{361}{16};$$

$$\text{extracting the root, } \frac{x}{2} + \frac{3}{4} = \pm \frac{19}{4};$$

$$\therefore \frac{x}{2} = 4, \text{ or } -\frac{11}{2};$$

$$\therefore x = 8, \text{ or } -11;$$

hence in 8 hours he would overtake him. — 11 not answering the conditions of the problem.

10. The base of a right-angled triangle is 6, and the sides are in arithmetical progression; it is required to find the other two sides.

Let $6 - x$, 6, and $6 + x$ be the sides;

then $36 - 12x + x^2 + 36 = 36 + 12x + x^2$
(Eucl. B. I. p. 48.)

by transposition, $24x = 36$,

$$\text{and } x = \frac{3}{2};$$

$\therefore$ the sides are $\frac{9}{2}$, $\frac{12}{2}$, and $\frac{15}{2}$.

But if 6 be the first term of the progression; let,

$6, 6 + x, 6 + 2x$ be the sides;

then $36 + 24x + 4x^2 = 36 + 36 + 12x + x^2$;

by transposition, $3x^2 + 12x = 36$,

$$\text{or } x^2 + 4x = 12;$$

completing the square, $x^2 + 4x + 4 = 16$;

extracting the root, $x + 2 = \pm 4$,

and $x = 2$, or -6 ,

and the sides are 6, 8, and 10; or 0, 6, and -6 .

The problem is not properly restricted; the algebraical expression, in this instance, is more precise than the language in which the problem is stated.

11. *A* and *B* set out from London at the same time, to go round the world (23661 miles), one going East, the other West. *A* goes one mile the first day, two the second, and so on. *B* goes 20 miles a day. In how many days will they meet; and how many miles will be travelled by each?

Let x = the number of days;

then (*Alg.* 212.) $(x + 1) \cdot \frac{x}{2}$ = the number of miles *A* goes,

and $20x$ = the number *B* goes;

$$\therefore \frac{x^2 + x}{2} + 20x = 23661,$$

$$\text{and } x^2 + 41x = 47322;$$

completing the square, $x^2 + 41x + \left(\frac{41}{2}\right)^2 = 47322 +$

$$\frac{1681}{4} = \frac{190969}{4};$$

$$\therefore \text{extracting the root, } x + \frac{41}{2} = \pm \frac{437}{2};$$

$$\therefore x = 198, \text{ or } -239;$$

$\therefore$ they travel 198 days; *A* goes 19701, and *B* 3960 miles.

12. A traveller sets out for a certain place, and travels one mile the first day, two the second, and so on. In 5 days afterwards another sets out, and travels 12 miles a day. How long and how far must he travel to overtake the first?

Let x = the number of days;

then $x + 5$ = the number the first travels,

and $\therefore$ (*Alg.* 212.) $(x + 6) \cdot \frac{x + 5}{2}$ = the distance he travels,

and $12x$ = the distance the second travels;

$$\therefore (x + 6) \cdot \frac{x + 5}{2} = 12x,$$

$$\text{and } x^2 + 11x + 30 = 24x;$$

$$\therefore \text{by transposition, } x^2 - 13x = -30;$$

$$\text{completing the square, } x^2 - 13x + \frac{169}{4} = \frac{169}{4} - 30 = \frac{49}{4};$$

$$\text{extracting the root, } x - \frac{13}{2} = \pm \frac{7}{2};$$

$$\text{and } x = 3, \text{ or } 10.$$

$\therefore$ they are together at the end of 3, and 10 days after the second sets out; and 36 and 120 miles is the distance travelled.

13. *A* and *B*, 165 miles distant from each other, set out with a design to meet; *A* travels one mile the first day, two the second, three the third, and so on; *B* travels 20 miles the first day, 18 the second, 16 the third, and so on. How soon will they meet?

Let x = the number of days required;

then $1 + 2 + 3 + \dots + x = (1 + x) \cdot \frac{x}{2}$ = the number of miles *A* travelled.

and $20 + 18 + \dots + 20 - 2x + 2 = (42 - 2x) \cdot \frac{x}{2}$ = the number *B* travelled;

$$\therefore (43 - x) \cdot \frac{x}{2} = 165,$$

$$\text{or } x^2 - 43x = -330;$$

$$\text{completing the square, } x^2 - 43x + \frac{1849}{4} = \frac{1849}{4} - 330 = \frac{529}{4};$$

$$\text{extracting the root, } x - \frac{43}{2} = \pm \frac{23}{2};$$

$$\therefore x = 10, \text{ or } 33.$$

Hence it appears that they meet in 10 days. On the 10th day *B* travels 2 miles, and the next day he rests; the following day he returns 2 miles; the succeeding day 4, and so on, increasing two miles every day; and on the 33d day he again comes up with *A*, who has been travelling forward, every day's journey being one mile longer than that of the preceding day.

14. There are four numbers in arithmetical progression whose continual product is 1680, and common difference is 4. Required the numbers.

Let $x + 6$, $x + 2$, $x - 2$, and $x - 6$, be the numbers ;

$$\text{then } (x^2 - 36) \cdot (x^2 - 4) = 1680,$$

$$\text{or } x^4 - 40x^2 + 144 = 1680 ;$$

$$\therefore \text{ by transposition, } x^4 - 40x^2 = 1536 ;$$

$$\text{completing the square, } x^4 - 40x^2 + 400 = 1936,$$

$$\text{extracting the root, } x^2 - 20 = \pm 44 ;$$

$$\therefore x^2 = 64, \text{ or } -24,$$

$$\text{and } x = \pm 8, \text{ or } \pm 2\sqrt{-6},$$

and $\therefore$ the numbers are $\pm 14, \pm 10, \pm 6, \pm 2$; the two other values of x being impossible.

15. The product of five numbers in arithmetical progression is 945, and their sum is 25. Required the numbers.

Let $x + 2y$, $x + y$, x , $x - y$, $x - 2y$, be the numbers ;

$$\text{then } 5x = 25, \text{ and } \therefore x = 5 ;$$

$$\text{also, } x \cdot (x^2 - y^2) \cdot (x^2 - 4y^2) = 945, \text{ or dividing by } x = 5,$$

$$(x^2 - y^2) \cdot (x^2 - 4y^2) = 189,$$

$$\text{or } x^4 - 5x^2y^2 + 4y^4 = 189,$$

$$\text{and } 4y^4 - 125y^2 + 625 = 189 ;$$

$$\text{by transposition, } 4y^4 - 125y^2 = -436 ;$$

$$\text{completing the square, } 4y^4 - 125y^2 + \frac{125^2}{4} = \frac{15625}{16} - 436 = \frac{8649}{16} ;$$

$$\text{extracting the root, } 2y^2 - \frac{125}{4} = \pm \frac{93}{4} ;$$

$$\therefore 2y^2 = \frac{109}{2}, \text{ or } 8 ;$$

$$\therefore y^2 = \frac{109}{4}, \text{ or } 4 ;$$

$$\therefore y = \pm \frac{\sqrt{109}}{2}, \text{ or } \pm 2,$$

and the numbers are 9, 7, 5, 3, 1.

16. A Gentleman divided £210 among three servants, in geometrical progression; the first had £90 more than the last. How much had each?

Let xy^2 , xy , x = the number of pounds each had;

$$\text{then } xy^2 = x + 90,$$

$$\text{and } x + xy + xy^2 = 210;$$

$$\text{or } 2x + xy + 90 = 210;$$

$$\text{by transposition, } 2x + xy = 120.$$

$$\text{Now from the first equation, } x = \frac{90}{y^2 - 1},$$

$$\text{and from the last } x = \frac{120}{y + 2};$$

$$\therefore \frac{120}{y + 2} = \frac{90}{y^2 - 1};$$

$$\text{or } \frac{4}{y + 2} = \frac{3}{y^2 - 1};$$

$$\therefore 4y^2 - 4 = 3y + 6;$$

$$\text{by transposition, } 4y^2 - 3y = 10;$$

$$\text{completing the square, } 4y^2 - 3y + \frac{9}{16} = 10 + \frac{9}{16} = \frac{169}{16};$$

$$\text{extracting the root, } 2y - \frac{3}{4} = \pm \frac{13}{4},$$

$$\text{and } 2y = 4, \text{ or } -\frac{5}{2};$$

$$\text{and } \therefore y = 2, \text{ or } -\frac{5}{4};$$

$$\text{whence } x = \frac{120}{y+2} = 30, \text{ or } 160,$$

and the sums are 120, 60, and 30 pounds.

17. The sum of three numbers in geometrical progression is 35; and the mean term is to the difference of the extremes as 2 to 3. Required the numbers.

Let $\frac{x}{y}$, x , and xy , be the three numbers;

$$\therefore \frac{x}{y} + x + xy = 35,$$

$$\text{and } x : xy - \frac{x}{y} :: 2 : 3,$$

$$\text{or } 1 : y - \frac{1}{y} :: 2 : 3;$$

$$\therefore 2y - \frac{2}{y} = 3,$$

$$\text{and } y^2 - 1 = \frac{3}{2} \cdot y;$$

$$\text{by transposition, } y^2 - \frac{3}{2} \cdot y = 1;$$

$$\text{completing the square, } y^2 - \frac{3}{2} y + \frac{9}{16} = 1 + \frac{9}{16} = \frac{25}{16};$$

$$\text{extracting the root, } y - \frac{3}{4} = \pm \frac{5}{4};$$

$$\therefore y = 2, \text{ or } -\frac{1}{2}, \text{ which last does not answer the conditions;}$$

$$\therefore \left(\frac{x}{2} + x + 2x \right) \frac{7x}{2} = 35;$$

$$\therefore x = 10,$$

and the numbers are 5, 10, and 20.

18. There are three numbers in geometrical progression, the greatest of which exceeds the least by 15. Also the difference of the squares of the greatest and least is to the sum of the squares of all the three numbers as 5 : 7. Required the numbers.

Let x , xy , xy^2 , be the numbers ;

$$\text{then } xy^2 - x = 15,$$

$$\text{and } x^2y^4 - x^2 : x^2y^4 + x^2y^2 + x^2 :: 5 : 7,$$

$$\text{or } y^4 - 1 : y^4 + y^2 + 1 :: 5 : 7;$$

$$\therefore y^4 - 1 : y^2 + 2 :: 5 : 2,$$

$$\text{and } y^4 - 1 = \frac{5y^2}{2} + 5;$$

$$\text{by transposition, } y^4 - \frac{5}{2}y^2 = 6;$$

$$\text{completing the square, } y^4 - \frac{5}{2}y^2 + \frac{25}{16} = 6 + \frac{25}{16} = \frac{121}{16};$$

$$\text{extracting the root, } y^2 - \frac{5}{4} = \pm \frac{11}{4};$$

$$\therefore y^2 = 4, \text{ or } -\frac{3}{2}, \text{ which last is impossible,}$$

$$\text{and } y = \pm 2;$$

$$\therefore \text{ from the first equation, } (4x - x) 3x = 15,$$

$$\text{and } x = 5;$$

$$\therefore \text{ the numbers are 5, 10, and 20.}$$

19. The sum of three numbers in geometrical progression is 33, and the product of the mean and the sum of the extremes is 30. Required the numbers.

Let the numbers be $\frac{x}{y}$, x , and xy ;

$$\text{then } \frac{x}{y} + x + xy = 13,$$

$$\text{and } \left(\frac{x}{y} + xy \right) \cdot x = 30;$$

$$\therefore \text{ by transposition, } 13 - x = \frac{x}{y} + xy = \frac{30}{x};$$

$$\text{and } 13x - x^2 = 30,$$

$$\text{or } x^2 - 13x = -30,$$

$$\text{completing the square, } x^2 - 13x + \frac{169}{4} = \frac{169}{4} - 30 = \frac{49}{4};$$

$$\text{extracting the root, } x - \frac{13}{2} = \pm \frac{7}{2},$$

$$\text{and } \therefore x = 10, \text{ or } 3.$$

$$\text{If } x = 3; \text{ then } \frac{3}{y} + 3y = 13 - 3 = 10,$$

$$\text{or } 3 + 3y^2 = 10y;$$

$$\text{by transposition, } 3y^2 - 10y = -3,$$

$$\text{or } y^2 - \frac{10}{3} \cdot y = -1;$$

$$\text{completing the square, } y^2 - \frac{10}{3} \cdot y + \frac{25}{9} = \frac{25}{9} - 1 = \frac{16}{9};$$

$$\text{extracting the root, } y - \frac{5}{3} = \pm \frac{4}{3};$$

$$\therefore y = 3, \text{ or } \frac{1}{3},$$

$$\text{and the numbers are } 1, 3, 9.$$

If the other value of x be taken, the corresponding values of y are impossible.

20. There are three numbers in arithmetical progression, and the square of the first added to the product of the other

two is 16; the square of the second added to the product of the other two is 14. What are the numbers?

Let $x - y$, x , $x + y$, be the numbers;

$$\text{then } 2x^2 - xy + y^2 = 16,$$

$$\text{and } 2x^2 - y^2 = 14;$$

$$\therefore \text{ by subtraction, } 2y^2 - xy = 2,$$

$$\text{and by addition, } 4x^2 - xy = 30,$$

$$\text{or } 2y^2 = 2 + xy,$$

$$\text{and } 4x^2 = 30 + xy;$$

$$\therefore \text{ by multiplication, } 8x^2y^2 = 60 + 32xy + x^2y^2;$$

$$\text{by transposition, } 7x^2y^2 - 32xy = 60,$$

$$\text{or } x^2y^2 - \frac{32}{7} \cdot xy = \frac{60}{7};$$

$$\text{completing the square, } x^2y^2 - \frac{32}{7} \cdot xy + \frac{256}{49} = \frac{60}{7}$$

$$+ \frac{256}{49} = \frac{676}{49};$$

$$\text{extracting the root, } xy - \frac{16}{7} = \pm \frac{26}{7};$$

$$\therefore xy = 6, \text{ or } -\frac{10}{7};$$

$$\therefore 2y^2 = 2 + xy = 8,$$

$$\text{and } y^2 = 4;$$

$$\therefore y = \pm 2,$$

$$\text{and } 4x^2 = 30 + xy = 36;$$

$$\therefore 2x = \pm 6,$$

$$\text{and } x = \pm 3;$$

$\therefore$ the numbers are 1, 3, 5; or -5 , -3 , -1 . The other value of xy was introduced in the operation, and does not answer the conditions of the question.

21. The sum of four whole numbers in arithmetical progression is 20, and the sum of their reciprocals is $\frac{25}{24}$. Required the numbers.

Let $x - 3y, x - y, x + y, x + 3y$, be the numbers ;

$$\text{then } 4x = 20,$$

$$\text{or } x = 5.$$

$$\text{Again, } \frac{1}{x-3y} + \frac{1}{x-y} + \frac{1}{x+y} + \frac{1}{x+3y} = \frac{25}{24},$$

$$\text{or } \frac{4x^3 - 20xy^2}{x^4 - 10x^2y^2 + 9y^4} = \frac{25}{24};$$

$$\therefore 25 \times (9y^4 - 250y^2 + 625) = 24 \times (500 - 100y^2),$$

$$\text{or } 9y^4 - 250y^2 + 625 = 24 \times (20 - 4y^2);$$

$$\text{by transposition, } 9y^4 - 154y^2 = -145,$$

$$\text{completing the square, } 9y^4 - 154y^2 + \frac{5929}{9} = \frac{5929}{9} - 145$$

$$= \frac{4624}{9};$$

$$\text{extracting the root, } 3y^2 - \frac{77}{3} = \pm \frac{68}{3},$$

$$\text{and } 3y^2 = 3, \text{ or } \frac{145}{3};$$

$$\therefore y^2 = 1, \text{ or } \frac{145}{9},$$

$$\text{and } y = \pm 1, \text{ or } \frac{\pm \sqrt{(145)}}{3},$$

and $\therefore$ the numbers are 2, 4, 6, 8.

22. There is a number consisting of 3 digits, the first of which is to the second as the second to the third; the number itself is to the sum of its digits as 124 to 7; and

if 594 be added to it, the digits will be inverted. Required the number.

Let the digits be represented by x, xy, xy^2 ;

then $100x + 10xy + xy^2 : x + xy + xy^2 :: 124 : 7$,

or $100 + 10y + y^2 : 1 + y + y^2 :: 124 : 7$;

div^{do}. $99 + 9y : 1 + y + y^2 :: 117 : 7$,

or $11 + y : 1 + y + y^2 :: 13 : 7$;

$\therefore 13y^2 + 13y + 13 = 7y + 77$;

by transposition, $13y^2 + 6y = 64$,

$$\text{or } y^2 + \frac{6}{13} \cdot y = \frac{64}{13};$$

completing the square, $y^2 + \frac{6}{13} \cdot y + \frac{3}{13} \Big| = \frac{64}{13} + \frac{9}{13^2} = \frac{841}{169}$;

extracting the root, $y + \frac{3}{13} = \pm \frac{29}{13}$;

$$\therefore y = 2, \text{ or } -\frac{32}{13};$$

also $100x + 10xy + xy^2 + 594 = 100xy^2 + 10xy + x$;

by transposition, $99x + 594 = 99xy^2$,

or $x + 6 = xy^2 = 4x$;

$\therefore$ by transposition, $6 = 3x$,

and $2 = x$;

$\therefore$ the digits are 2, 4, 8, and the number is 248.

23. There are five whole numbers, the three first of which are in geometric progression; the three last in arithmetic progression, the second number being the second difference. The sum of the four last = 40, and the product of the second and last = 64. Required the numbers.

Let x = the first,

and y = the common ratio of the three first;

$\therefore$ the numbers are $x, xy, xy^2, xy^2 + xy, xy^2 + 2xy$;

$$\therefore 3xy^2 + 4xy = 40,$$

$$\text{and } x^2y^2 + 2x^2y^2 = 64.$$

Multiplying the first equation by xy , and the second by 3,

$$3x^2y^2 + 4x^2y^2 = 40xy,$$

$$\text{and } 3x^2y^2 + 6x^2y^2 = 192;$$

$\therefore$ by subtraction, $2x^2y^2 = 192 - 40xy$;

by transposition, $2x^2y^2 + 40xy = 192$,

$$\text{or } x^2y^2 + 20xy = 96;$$

completing the square, $x^2y^2 + 20xy + 100 = 196$;

extracting the root, $xy + 10 = \pm 14$;

$$\therefore xy = 4, \text{ or } -24.$$

Now from the first equation, $xy \cdot (3y + 4) = 40$,

$$\text{or } 4 \cdot (3y + 4) = 40;$$

$$\therefore 3y + 4 = 10;$$

by transposition, $3y = 6$,

$$\text{and } y = 2;$$

$$\therefore \text{also } x = 2,$$

and the numbers are 2, 4, 8, 12, 16.

24. There are two casks A and B ; of which, A the greater holds 312 gallons. Into A a certain quantity of wine is put, and B is filled with water; then water is conveyed out of B into A in the following manner. First, a number of gallons is taken, which is less by two than the square root of the number of gallons in A , then a quantity less than the former by two gallons, and so on. Now when B is in this manner exactly emptied, A is exactly full: and it is known that 8 gallons were taken out of B at one time,

after which the quantity left in B was 12 gallons. Required the number of gallons of wine in A .

Since the quantities taken out of B are in a decreasing progression, whose common difference is 2, and one term of this progression is 8, therefore the next terms are 6, 4, 2, the sum of which is = 12; and therefore the quantity last drawn out is 2 gallons. Let x^3 = the number of gallons of wine in A , then $x - 2$, $x - 4$, &c. are the numbers of gallons drawn each successive time; and the number of terms is evidently $\frac{x}{2} - 1$; and therefore the whole quantity drawn from B is

$$\frac{x}{2} \cdot \left(\frac{x}{2} - 1 \right) = \frac{x^3}{4} - \frac{x}{2}.$$

$$\therefore x^3 + \frac{x^3}{4} - \frac{x}{2} = 312,$$

$$\text{or } \frac{5x^3}{4} - \frac{x}{2} = 312;$$

$$\therefore x^3 - \frac{2}{5}x = \frac{1248}{5};$$

$$\text{completing the square, } x^3 - \frac{2}{5}x + \frac{1}{25} = \frac{1248}{5} + \frac{1}{25} = \frac{6241}{25};$$

$$\text{extracting the root, } x - \frac{1}{5} = \pm \frac{79}{5},$$

$$\text{and } x = 16, \text{ or } -\frac{78}{5}, \text{ which last will not}$$

answer the conditions; therefore $x^3 = 256$ = the number required.

25. The diagonals of 4 squares are in an increasing geometrical progression, and the product of the squares of the diagonals of the extremes is to the product of the diagonals of the means as a side of the third is to the square root of the common ratio divided by $4\sqrt{2}$. Required

the diagonal of the third square, and the common ratio, supposing their difference equal to 45.

Let $\frac{x}{y}$, x , xy , and xy^2 = the diagonals ;

then since the diagonal : a side :: $\sqrt{2} : 1$;

the side of the third = $\frac{xy}{\sqrt{2}}$,

and $\frac{x^3}{y^3} \times x^2y^4 : x \times xy :: \frac{xy}{\sqrt{2}} : \frac{\sqrt{y}}{4\sqrt{2}}$;

$\therefore x^3y : 1 :: 4x\sqrt{y} : 1$;

$\therefore x^3y = 4x\sqrt{y}$,

and $xy^4 = 4$.

Now $y - xy = 45$,

or $y - 4y^{\frac{1}{4}} = 45$;

completing the square, $y - 4y^{\frac{1}{4}} + 4 = 49$;

extracting the root, $y^{\frac{1}{4}} - 2 = \pm 7$;

$\therefore y^{\frac{1}{4}} = 9$, or -5 , which last does not agree with the conditions ; and $\therefore y = 81$;

whence $x = \frac{4}{y^{\frac{1}{4}}} = \frac{4}{9}$;

and $\therefore$ the diagonal of the third square = 36.

26. Two persons, *A* and *B*, traded together. *A* gained every year £3 more than the preceding year, and the last year he gained £17. His whole gain was £57. *B* in the four first years gained £52, and if what *A* put into the common stock be added to what *B* gained the second year, the sum will be £13. How many years did they remain in trade, and what were their original stocks ?

Since A 's annual gains are in an increasing arithmetical progression, whose common difference is 3, the last term 17, and sum 57, if n = the number of terms, then (*Alg.* 212),

$$\{ 34 (n - 1) \cdot 3 \} \cdot \frac{n}{2} = 57,$$

$$\text{or } 37n - 3n^2 = 114;$$

$$\therefore n^2 - \frac{37}{3}n = -\frac{114}{3};$$

$$\text{completing the square, } n^2 - \frac{37}{3}n + \left[\frac{37}{6} \right]^2 = \frac{1369}{36} - \frac{114}{3} = \frac{1}{36};$$

$$\text{extracting the root, } n - \frac{37}{6} = \pm \frac{1}{6};$$

$$\therefore n = 6, \text{ or } \frac{19}{3};$$

hence they remained 6 years in trade, and consequently A 's gain the first year was £2, and his gain in four years was £26.

Let $\therefore x$ = A 's stock,

and $26 : 52 : x : B$'s stock = $2x$,

and A 's gain the second year being £5,

$x : 2x :: 5 : B$'s gain = 10;

$$\therefore x + 10 = 13,$$

and $x = 3$;

$\therefore A$'s stock was £3, and B 's £6.

27. A pyramidal pile of cannon-balls, the base of which was an equilateral triangle, was all used in an engagement, except the three lowest layers, and 4 balls of the next layer; these were afterwards formed into a pile with a rectangular base, having as many balls in one side of the lowest layer, as there were in the side of the lowest layer of the pyramidal pile, and 4 in the adjacent side.

What was the number of balls; and what the number of layers in each pile when complete?

Let x = the number of balls in a side of the lowest layer;

$$\therefore x \cdot \frac{x+1}{2} = \text{the number of balls in that layer,}$$

$$\text{and } (x-1) \cdot \frac{x}{2} = \text{the number in the next,}$$

$$\text{and } (x-2) \cdot \frac{x-1}{2} = \text{the number in the third;}$$

$$\therefore \frac{3x^2 - 3x + 2}{2} + 4 = \text{the whole number left.}$$

Now since there were only 4 balls in one side of the second pile, there can only be four layers, which will contain $4x$, $3 \cdot (x-1)$, $2 \cdot (x-2)$, and $x-3$ balls respectively;

$$\therefore \frac{3x^2 - 3x + 2}{2} + 4 = 10x - 10,$$

$$\text{or } 3x^2 - 3x + 2 + 8 = 20x - 20;$$

$$\text{by transposition, } 3x^2 - 23x = -30,$$

$$\text{or } x^2 - \frac{23}{3}x = -10;$$

$$\text{completing the square, } x^2 - \frac{23}{3}x + \left(\frac{23}{6}\right)^2 = \frac{529}{36} - 10 = \frac{169}{36};$$

$$\text{extracting the root, } x - \frac{23}{6} = \pm \frac{13}{6},$$

and $x = 6$, or $\frac{5}{3}$, which last cannot answer the conditions of the problem. Hence there were 6 layers in the first pile, and they contained 1, 3, 6, 10, 15, 21 balls, respectively; $\therefore$ the whole number of balls in the first pile was 56, and in the second 50.

(31.) In the preceding solutions it may be observed, that, in many instances, values of the unknown quantities are deduced, which do not agree with the conditions of the problems. This is always the case when the roots of the equations are negative; and the circumstance arises from that peculiar quality of an algebraic expression, by which it is denominated either positive or negative. The product of two or any even number of such quantities, whether all of them are positive or all negative, will only be affected with a positive sign: thus the quantity xy will represent the product of $+x \times +y$, or of $-x \times -y$; and a^2 , of $+a \times +a$, or of $-a \times -a$; consequently, in the reduction of such quantities to their constituent factors by the rules of division or evolution, these factors may be considered either as all positive or all negative. But in common language, in which the conditions of a problem are expressed, quantity or number is from its very nature what in Algebra is meant by the term positive, *i. e.* it increases any homogeneous quantity to which it is added, and diminishes any one from which it is subtracted. Hence it may be understood, why, when quadratic equations are formed to express the conditions of a problem, the resulting roots may exceed in number what appear to be required as answers to the problem, and why such as are negative cannot be applied to its conditions.

These roots or values, however, though inapplicable in their present shape, will, if assumed as positive, become correct answers to the problem under a different modification of the conditions. In the equations thence deduced, these former negative values will appear as positive roots, and the former positive values as negative roots. Thus, if Prob. 12, page 213, be transformed into the following, "A detachment from an army was marching in regular column with 5 fewer in depth than in front; but upon the enemy coming in sight the front was increased till it became = 845 — the original front; and by this movement the detachment was drawn up in 5 lines. Required the number of men;" from the solution of this problem the number is found to be 3900, answering to the number which would be found from using

the negative value of x in the original problem; and the equation for determining this ($x^2 - 5x = 4225 - 5x$) differs from the other only in the sign of x .

In Prob. 19, page 217, x is found to be equal to ± 4 , where the negative value shows, that if the trading vessel had turned out of its first course in a direction contrary to CE , or on the opposite side of the line AC , it would have been taken after sailing 4 miles in that direction.

In Prob. 1, page 227, the negative value of x is found to be -130 . But if the problem be modified so as to become "A merchant sold a quantity of brandy, by which he lost £39 *more* than the prime cost, and found that his loss was as much *per cent.* as the brandy cost him. What was that price?" the equation for determining the price, is $\frac{x^2}{100} = 39 + x$, which is deduced from the equation to the original problem by changing the sign of x , the positive value of which is in this case 130.

Also, in Prob. 4, page 228, the negative value of x is $-\frac{15}{2}$. Now if the problem were, "Bought two sorts of linen, for the finer of which I gave 6 crowns *more than for the other*. An ell of the finer cost as many shillings as there were ells of the finer. Also 28 ells of the coarser (which was the whole quantity) sold at such a price, that 8 ells cost as many shillings as one ell of the finer. How many ells were there of the finer; and what was the value of each piece?" an equation arises differing from the equation to the original problem only in the sign of x , and whose positive root is $\frac{15}{2}$; whence there were $7\frac{1}{2}$ ells of the finer at 7s. 6d. *per* ell, the whole price of which was therefore £2. 16s. 3d., and the whole price of the coarser was £1. 6s. 3d. And in the very same manner, all the other problems may be transformed.

(32.) The same reasoning will apply to the case in which all the roots of the resulting equation are negative. None of its values can in this case be applied to satisfy the conditions of the problem; but if the conditions are properly modified, equations may be deduced, of which these values rendered positive will become roots, and will satisfy such conditions.

The same observation holds, if the resulting values be the square roots of negative quantities, with this exception, that such roots can never be applied to satisfy the conditions of the problem under any modification whatever.

SECTION XI.

PRAXIS.

I. *Simple Equations involving only one unknown Quantity.*

1. GIVEN $19x + 13 = 59 - 4x$, to find the value of x .

ANSWER $x = 2$.

2. Given $3x + 4 - \frac{x}{3} = 46 - 2x$, to find the value of x .

ANS. $x = 9$.

3. Given $x^2 + 15x = 35x - 3x^2$, to find the value of x .

ANS. $x = 5$.

4. Given $\frac{x}{6} - \frac{x}{4} + 10 = \frac{x}{3} - \frac{x}{2} + 11$, to find the value of x .

ANS. $x = 12$.

5. Given $\frac{x+1}{5} + 3 = \frac{2x-3}{3}$, to find the value of x .

ANS. $x = 9$.

6. Given $\frac{7x+2}{3} + 5x = 28 + \frac{5x-6}{7}$, to find the value of x .

ANS. $x = 4$.

7. Given $\frac{3x+4}{5} + 2x = \frac{22-x}{5} + 16$, to find the value of x .

Ans. $x = 7$.

8. Given $\frac{7-x}{2} + 4 = \frac{3x-11}{4} + \frac{8x+15}{6}$, to find the value of x .

Ans. $x = 3$.

9. Given $\frac{2x-5}{18} + \frac{19-x}{3} = \frac{10x-7}{9} - \frac{5}{2}$, to find the value of x .

Ans. $x = 7$.

10. Given $x - \frac{2x+1}{3} = \frac{x+3}{4}$, to find the value of x .

Ans. $x = 13$.

11. Given $\frac{3x+5}{8} - \frac{21+x}{3} = 39 - 5x$, to find the value of x .

Ans. $x = 9$.

12. Given $4x - \frac{19+2x}{5} = 15 - \frac{7x+11}{4}$, to find the value of x .

Ans. $x = 3$.

13. Given $\frac{21-3x}{3} - \frac{4x+6}{9} = 6 - \frac{5x+1}{4}$, to find the value of x .

Ans. $x = 3$.

14. Given $7\frac{1}{2} + \frac{3x-1}{4} - \frac{7x+3}{16} = \frac{8x+19}{8}$, to find the value of x .

Ans. $x = 7$.

15. Given $\frac{6x+8}{11} - \frac{5x+3}{2} = \frac{27-4x}{3} - \frac{3x+9}{2}$, to find the value of x .

Ans. $x = 6$.

16. Given $x + \frac{27-9x}{4} - \frac{5x+2}{6} = \frac{61}{12} - \frac{2x+5}{3} - \frac{29+4x}{12}$, to find the value of x .

Ans. $x = 5$.

17. Given $\frac{7x-8}{11} + \frac{15x+8}{13} = 3x - \frac{31-x}{2}$, to find the value of x .

Ans. $x = 9$.

18. Given $\frac{5x-1}{2} - \frac{7x-2}{10} = 6\frac{1}{2} - \frac{x}{2}$, to find the value of x .

Ans. $x = 3$.

19. Given $\frac{3x-3}{4} - \frac{3x-4}{3} = 5\frac{1}{2} - \frac{27+4x}{9}$, to find the value of x .

Ans. $x = 9$.

20. Given $\frac{4x-34}{17} - \frac{258-5x}{3} = \frac{69-x}{2}$, to find the value of x .

Ans. $x = 51$.

21. Given $2x - \frac{4x-2}{13} = \frac{2x+11}{5} - \frac{7-8x}{7}$, to find the value of x .

Ans. $x = 7$.

22. Given $\frac{2x+1}{29} - \frac{402-3x}{12} = 9 - \frac{471-6x}{2}$, to find the value of x .

Ans. $x = 72$.

23. Given $\frac{7x+9}{8} - \frac{3x+1}{7} = \frac{9x-13}{4} - \frac{249-9x}{14}$, to find the value of x .

$$\text{Ans. } x = 9.$$

24. Given $5 - \frac{3x+25}{4} - \frac{17-6x}{9} = 2\frac{1}{6} + x - \frac{9x+40}{8}$, to find the value of x .

$$\text{Ans. } x = 4.$$

25. Given $\frac{7x-43}{12} + 13\frac{1}{4} - \frac{5+4x}{6} = 256 - \frac{3x-12}{9} - \frac{5x+29}{8} - 12x$, to find the value of x .

$$\text{Ans. } x = 19.$$

26. Given $4x + \frac{1}{10} - \frac{3x-13}{16} - \frac{12+7x}{9} = 7x - 33 - \frac{9+5x}{10} - \frac{11x-17}{8}$, to find the value of x .

$$\text{Ans. } x = 15.$$

27. Given $\frac{31+4x}{3} - \frac{3x+47}{8} - \frac{3x-19}{16} = 47\frac{3}{4} + \frac{16-10x}{11} - \frac{5x+20}{7}$, to find the value of x .

$$\text{Ans. } x = 17.$$

28. Given $\frac{3a+x}{x} - 5 = \frac{6}{x}$, to find the value of x .

$$\text{Ans. } x = \frac{3a-6}{4}.$$

29. Given $\frac{5}{6} \cdot ab + \frac{4}{5}ac - \frac{2}{3} \cdot cx = \frac{3}{4} \cdot ac + 2ab - 6cx$, to find the value of x .

$$\text{Ans. } x = \frac{70ab-3ac}{320c}.$$

30. Given $\frac{11x-13}{25} + \frac{19x+3}{7} - \frac{5x-25\frac{1}{2}}{4} = 28\frac{1}{7} - \frac{17x+4}{21}$, to find the value of x .

$$\text{Ans. } x = 8.$$

31. Given $\frac{\frac{3x}{2} - 4}{6} - \frac{4x-7}{9} + x = \frac{8 - \frac{x+4}{2}}{3}$, to find the value of x .

$$\text{Ans. } x = 4.$$

32. Given $\frac{x - 14\frac{1}{2}}{2} - \frac{2-6x}{13} = x - \frac{5x - \frac{10-3x}{4}}{39}$, to find the value of x .

$$\text{Ans. } x = 11.$$

33. Given $\frac{a^2x}{bc} - \frac{d^2}{a} + bx = \frac{ex}{f} - b + (d+b) \cdot x$, to find the value of x .

$$\text{Ans. } x = \frac{(d^2 - ab) \cdot bcf}{a^3f - abce - abcd}.$$

34. Given $\frac{ax}{b} + \frac{cx}{d} + \frac{ex}{f} - g = h$, to find the value of x .

$$\text{Ans. } x = \frac{(g+h) \cdot bdf}{adf + bcf + bde}.$$

35. Given $\frac{a^2x}{b-c} - dc = bx - ac$, to find the value of x .

$$\text{Ans. } x = \frac{c \cdot (b-c) \cdot (d-a)}{a^2 - b^2 + bc}.$$

36. Given $\frac{2x}{3} - \frac{1 - \frac{x}{2}}{4x} = \frac{x-1}{2} + \frac{x}{6} + \frac{7}{12}$, to find the value of x .

Ans. $x = 6$.

37. Given $\frac{9x+20}{36} = \frac{4x-12}{5x-4} + \frac{x}{4}$, to find the value of x .

Ans. $x = 8$.

38. Given $\frac{20x+36}{25} + \frac{5x+20}{9x-16} = \frac{4x}{5} + \frac{86}{25}$, to find the value of x .

Ans. $x = 4$.

39. Given $\frac{10x+17}{18} - \frac{12x+2}{13x-16} = \frac{5x-4}{9}$, to find the value of x .

Ans. $x = 4$.

40. Given $\frac{18x-19}{28} + \frac{11x+21}{6x+14} = \frac{9x+15}{14}$, to find the value of x .

Ans. $x = 7$.

41. Given $\frac{2x+8\frac{1}{2}}{9} - \frac{13x-2}{17x-32} + \frac{x}{3} = \frac{7x}{12} - \frac{x+16}{36}$, to find the value of x .

Ans. $x = 4$.

42. Given $\frac{7x+6}{28} - \frac{2x+4\frac{3}{4}}{23x-6} + \frac{x}{4} = \frac{11x}{21} - \frac{x-3}{42}$, to find the value of x .

Ans. $x = 4$.

43. Given $\frac{6-5x}{15} - \frac{7-2x^2}{14 \cdot (x-1)} = \frac{1+3x}{21} - \frac{2x-\frac{11}{5}}{6} + \frac{1}{105}$,
to find the value of x .

Ans. $x = 4$.

44. Given $\frac{a \cdot (b^2 + x^2)}{bx} = ac + \frac{ax}{b}$, to find the value of x .

Ans. $x = \frac{b}{c}$.

45. Given $\frac{cx^m}{a+bx} = \frac{dx^m}{e+fx}$, to find the value of x .

Ans. $x = \frac{ad-ce}{cf-bd}$.

46. Given $\frac{a}{bx} + \frac{c}{dx} + \frac{e}{fx} + \frac{g}{hx} = k$, to find the value of x .

Ans. $x = \frac{adfh + bcfh + bdeh + bdfg}{bdfhk}$.

47. Given $(a+x) \cdot (b+x) - a \cdot (b+c) = \frac{a^2c}{b} + x^2$, to find
the value of x .

Ans. $x = \frac{ac}{b}$.

48. Given $\frac{10+x}{5} : \frac{4x-9}{7} :: 14 : 5$, to find the value of x .

Ans. $x = 4$.

49. Given $\frac{17-4x}{4} : \frac{15+2x}{3} - 2x :: 5 : 4$, to find the value
of x .

Ans. $x = 3$.

50. Given $16x + 5 : \frac{4x + 14}{9x + 31} :: 36x + 10 : 1$, to find the value of x .

$$\text{Ans. } x = 5.$$

51. Given $\frac{4x + 3}{6x - 43} : 1 :: 2x + 19 : 3x - 19$, to find the value of x .

$$\text{Ans. } x = 8.$$

52. Given $5x + \frac{7x + 9}{4x + 3} = 9 + \frac{10x^2 - 18}{2x + 3}$, to find the value of x .

$$\text{Ans. } x = 3.$$

53. Given $\sqrt[3]{(10x + 35)} - 1 = 4$, to find the value of x .

$$\text{Ans. } x = 9.$$

54. Given $\sqrt[3]{(9x - 4)} + 6 = 8$, to find the value of x .

$$\text{Ans. } x = 4.$$

55. Given $\sqrt{(x + 16)} = 2 + \sqrt{x}$, to find the value of x .

$$\text{Ans. } x = 9.$$

56. Given $\sqrt{(x - 32)} = 16 - \sqrt{x}$, to find the value of x .

$$\text{Ans. } x = 81.$$

57. Given $\sqrt{(4x + 21)} = 2\sqrt{x} + 1$, to find the value of x .

$$\text{Ans. } x = 25.$$

58. Given $a\sqrt[3]{(bx - c)} = d \cdot \sqrt[3]{(ex + fx - g)}$, to find the value of x .

$$\text{Ans. } x = \frac{a^3c - d^3g}{a^3b - d^3 \cdot (e + f)}.$$

59. Given $\sqrt[3]{(a^3 + c)} = \sqrt[3]{\left(\frac{a^3 + c}{d \cdot (x + b)}\right)}$, to find the value of x .

$$\text{Ans. } x = \frac{1}{d \sqrt[3]{(a^3 + c)}} - b.$$

60. Given $\sqrt[m]{(a + x)} = \sqrt[m]{x^2 + 5ax + b^2}$, to find the value of x .

$$\text{Ans. } x = \frac{a^2 - b^2}{3a}.$$

61. Given $a + b \cdot \sqrt[m]{(x + d)} = c$, to find the value of x .

$$\text{Ans. } x = \left(\frac{c - a}{b}\right)^m - d.$$

62. Given $\frac{\sqrt{9x - 4}}{\sqrt{x + 2}} = \frac{15 + \sqrt{9x}}{\sqrt{x + 40}}$, to find the value of x .

$$\text{Ans. } x = 4.$$

63. Given $\frac{\sqrt{x} + \sqrt{b}}{\sqrt{x} - \sqrt{b}} = \frac{a}{b}$, to find the value of x .

$$\text{Ans. } x = b \cdot \left(\frac{a + b}{a - b}\right)^2.$$

64. Given $\frac{\sqrt{6x - 2}}{\sqrt{6x + 2}} = \frac{4\sqrt{6x - 9}}{4\sqrt{6x + 6}}$, to find the value of x .

$$\text{Ans. } x = 6.$$

65. Given $\frac{5x - 9}{\sqrt{5x + 3}} - 1 = \frac{\sqrt{5x - 3}}{2}$, to find the value of x .

$$\text{Ans. } x = 5.$$

66. Given $\sqrt{1 + x\sqrt{(x^2 + 12)}} = 1 + x$, to find the value of x .

$$\text{Ans. } x = 2.$$

67. Given $\frac{ax}{b} \cdot \sqrt{(c^2x^2 + d^2)} + \frac{acx^2}{b} = ex$, to find the value of x .

$$\text{Ans. } x = \frac{b^2e^2 - a^2d^2}{2abce}.$$

68. Given $\sqrt{x} + \sqrt{(x-9)} = \frac{36}{\sqrt{(x-9)}}$, to find the value of x .

$$\text{Ans. } x = 25.$$

II. Simple Equations involving two unknown Quantities.

1. Given $x + 15y = 53,$
and $y + 3x = 27,$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 8, \\ y = 3. \end{cases}$$

2. Given $4x + 9y = 51,$
and $8x - 13y = 9,$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 6, \\ y = 3. \end{cases}$$

3. Given $\frac{x}{6} + \frac{y}{4} = 6,$
and $\frac{x}{4} + \frac{y}{6} = 5\frac{2}{3},$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 12, \\ y = 16. \end{cases}$$

4. Given $\frac{x}{8} + 8y = 194,$
and $\frac{y}{8} + 8x = 131,$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 16, \\ y = 24. \end{cases}$$

$$5. \quad \text{Given } \left. \begin{aligned} \frac{3x-1}{5} + 3y-4 &= 15, \\ \text{and } \frac{3y-5}{6} + 2x-8 &= 7\frac{1}{3}, \end{aligned} \right\} \begin{array}{l} \text{to find the values of } x \\ \text{and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 7, \\ y = 5. \end{cases}$$

$$6. \quad \text{Given } \left. \begin{aligned} 9x + \frac{8y}{5} &= 70, \\ \text{and } 7y - \frac{13x}{3} &= 44, \end{aligned} \right\} \begin{array}{l} \text{to find the values of } x \text{ and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 6, \\ y = 10. \end{cases}$$

$$7. \quad \text{Given } \left. \begin{aligned} \frac{7+x}{5} - \frac{2x-y}{4} &= 3y-5, \\ \text{and } \frac{5y-7}{2} + \frac{4x-3}{6} &= 18-5x, \end{aligned} \right\} \begin{array}{l} \text{to find the values of } x \\ \text{and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 3, \\ y = 2. \end{cases}$$

$$8. \quad \text{Given } \left. \begin{aligned} x + 1 - \frac{3y+4x}{7} &= 7 - \frac{9y+33}{14}, \\ \text{and } y - 3 - \frac{5x-4y}{2} &= x - \frac{11y-19}{4}, \end{aligned} \right\} \begin{array}{l} \text{to find the values} \\ \text{of } x \text{ and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 6, \\ y = 5. \end{cases}$$

$$9. \quad \text{Given } \left. \begin{aligned} 4x + \frac{15-x}{4} &= 2y + 5 + \frac{7x+11}{16}, \\ \text{and } 3y - \frac{2x+y}{5} &= 2x + \frac{2y+4}{3}, \end{aligned} \right\} \begin{array}{l} \text{to find the values} \\ \text{of } x \text{ and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 3, \\ y = 4. \end{cases}$$

$$10. \quad \left. \begin{array}{l} \text{Given } x - \frac{3x+5y}{17} + 17 = 5y + \frac{4x+7}{3}, \\ \text{and } \frac{22-6y}{3} - \frac{5x-7}{11} = \frac{x+1}{6} - \frac{8y+5}{18}, \end{array} \right\} \begin{array}{l} \text{to find the} \\ \text{values of} \\ x \text{ and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 8, \\ y = 2. \end{cases}$$

$$11. \quad \left. \begin{array}{l} \text{Given } \frac{7x-21}{6} + \frac{3y-x}{3} = 4 + \frac{3x-19}{2}, \\ \text{and } \frac{2x+y}{2} - \frac{9x-7}{8} = \frac{3y+9}{4} - \frac{4x+5y}{16}, \end{array} \right\} \begin{array}{l} \text{to find the} \\ \text{values of} \\ x \text{ and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 9, \\ y = 4. \end{cases}$$

$$12. \quad \left. \begin{array}{l} \text{Given } \frac{a}{b+y} = \frac{b}{3a+x}, \\ \text{and } ax + 2by = c, \end{array} \right\} \begin{array}{l} \text{to find the values of } x \text{ and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = \frac{2b^2 - 6a^2 + c}{3a}, \\ y = \frac{3a^2 - b^2 + c}{3b}. \end{cases}$$

$$13. \quad \left. \begin{array}{l} \text{Given } \frac{7x+6}{11} + \frac{4y-9}{3} = 3x - \frac{13-x}{2} - \frac{3y-x}{5}, \\ \text{and } 3x+4 : 2y-3 :: 5 : 3, \end{array} \right\} \begin{array}{l} \text{to find the values of } x \text{ and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 7, \\ y = 9. \end{cases}$$

$$14. \quad \left. \begin{array}{l} \text{Given } \frac{5x+13}{2} - \frac{8y-3x-5}{6} = 9 + \frac{7x-3y+1}{3}, \\ \text{and } \frac{x+7}{3} : \frac{3y-8}{4} + 4x :: 4 : 21, \end{array} \right\} \begin{array}{l} \text{to find the values of } x \text{ and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 5, \\ y = 4. \end{cases}$$

$$15. \quad \text{Given } x + y : 4x + y :: 4 : 7, \\ \text{and } \frac{\frac{11y}{6} - 2x}{5} - \frac{21 - 3y}{4} = \frac{2 + \frac{x}{10}}{3} - \frac{1}{12}, \quad \left. \vphantom{\frac{11y}{6} - 2x} \right\} \text{ to find the values of } x \text{ and } y.$$

$$\text{Ans. } \begin{cases} x = 2, \\ y = 6. \end{cases}$$

$$16. \quad \text{Given } \frac{3x + 4y + 3}{10} - \frac{2x + 7 - y}{15} = 5 + \frac{y - 8}{5}, \\ \text{and } \frac{9y + 5x - 8}{12} - \frac{x + y}{4} = \frac{7x + 6}{11}, \quad \left. \vphantom{\frac{3x + 4y + 3}{10}} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 7, \\ y = 9. \end{cases}$$

$$17. \quad \text{Given } 13x + \frac{4y - 17 + x}{12} - \frac{15 - 3x}{4} = \frac{12y + 11}{3} \\ - \frac{12x + 7y + 28}{6}, \\ \text{and } \frac{9x + 18}{4} - \frac{12 + 5y - 6x}{5} = \frac{15x - 3y - 5}{8} - \frac{7x + y - 10}{15} \quad \left. \vphantom{\frac{4y - 17 + x}{12}} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 2, \\ y = 11. \end{cases}$$

$$18. \quad \text{Given } 3x + 5y = \frac{(8a - 2b) \cdot ab}{a^2 - b^2}, \\ \text{and } a^2x - \frac{acb^3}{a + b} + (a + b + c) \cdot by = b^2x + (a + 2b)ab, \quad \left. \vphantom{\frac{(8a - 2b) \cdot ab}{a^2 - b^2}} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = \frac{ab}{a - b}, \\ y = \frac{ab}{a + b}. \end{cases}$$

$$19. \quad \left. \begin{aligned} \text{Given } \frac{2x+y}{9} + \frac{7y+6x+11}{18} &= 9\frac{1}{2} - \frac{5x-17}{6}, \\ \text{and } \frac{5x+3y+2}{7} : \frac{9y+6}{2} &:: 1 : 3, \end{aligned} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 7, \\ y = 4. \end{cases}$$

$$20. \quad \left. \begin{aligned} \text{Given } \frac{3x-5y}{3} - \frac{2x-8y-9}{12} &= \frac{y}{2} + \frac{1}{3} + \frac{1}{4}, \\ \text{and } \frac{x}{7} + \frac{y}{4} + 1\frac{1}{3} : 4x - \frac{y}{8} - 24 &:: 3\frac{1}{2} : 3\frac{1}{2}, \end{aligned} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 7, \\ y = 4. \end{cases}$$

$$21. \quad \left. \begin{aligned} \text{Given } (x+5) \cdot (y+7) &= (x+1) \cdot (y-9) + 112, \\ \text{and } 2x+10 &= 3y+1, \end{aligned} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 3, \\ y = 5. \end{cases}$$

$$22. \quad \left. \begin{aligned} \text{Given } \frac{6x+9}{4} + \frac{3x+5y}{4x-6} &= 3\frac{1}{2} + \frac{3x+4}{2}, \\ \text{and } \frac{8y+7}{10} + \frac{6x-3y}{2y-8} &= 4 + \frac{4y-9}{5}, \end{aligned} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 7, \\ y = 9. \end{cases}$$

$$23. \quad \left. \begin{aligned} \text{Given } 4x - 34\frac{1}{2} - \frac{4y + 13x}{27 - 6y} &= \frac{12x + 8}{3}, \\ \text{and } 3x + \frac{21 - 4y}{4x - 10} &= \frac{18x + 13}{6} - 2\frac{1}{3}, \end{aligned} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 7, \\ y = 5. \end{cases}$$

$$24. \quad \left. \begin{aligned} \text{Given } 16x + 6y - 1 &= \frac{128x^2 - 18y^2 + 217}{8x - 3y + 2}, \\ \text{and } \frac{10x + 10y - 35}{2x + 2y + 3} &= 5 - \frac{54}{3x + 2y - 1}, \end{aligned} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 6, \\ y = 5. \end{cases}$$

$$25. \quad \left. \begin{aligned} \text{Given } 4x + 3y + \frac{24 + \frac{11y}{2}}{2x + 1} &= \frac{16x^2 + 12xy - 8x + 5y + 28}{4x - 2}, \\ \text{and } 2x + 4 &= 3y + \frac{8x^2 - 18y^2 + 108}{4x + 6y + 3}, \end{aligned} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 3, \\ y = 2. \end{cases}$$

$$26. \quad \left. \begin{aligned} \text{Given } 3x + 6y + 1 &= \frac{6x^2 + 130 - 24y^2}{2x - 4y + 3}, \\ \text{and } 3x - \frac{151 - 16x}{4y - 1} &= \frac{9xy - 110}{3y - 4}, \end{aligned} \right\} \quad \begin{array}{l} \text{to find the} \\ \text{values of} \\ x \text{ and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 9, \\ y = 2. \end{cases}$$

$$\begin{aligned}
 27. \quad & \text{Given } \frac{x-6}{7y} + \frac{4x+7}{24} - \frac{x-\frac{y}{7}}{6} = \frac{19+y}{42} - \frac{\frac{11x}{3}+6}{56y}, \\
 & \text{and } 12x - 15y + \frac{13}{4} : 10y - 8x + \frac{86}{3} :: 93 - 9x : 6x - \frac{14}{5},
 \end{aligned}
 \left. \vphantom{\begin{aligned} 27. \quad & \text{Given } \frac{x-6}{7y} + \frac{4x+7}{24} - \frac{x-\frac{y}{7}}{6} = \frac{19+y}{42} - \frac{\frac{11x}{3}+6}{56y}, \right.} \\
 & \text{to find the values of } x \text{ and } y.
 \end{aligned}}$$

$$\text{Ans. } \begin{cases} x = 9, \\ y = 7. \end{cases}$$

$$\begin{aligned}
 28. \quad & \text{Given } \frac{\frac{7x}{4} + 6y}{5} - \frac{\frac{3y+6}{5} - \frac{3x-2}{10}}{8} = 5 - \frac{x}{16}, \\
 & \text{and } \frac{3x}{2} + \frac{2y}{3} + 2\frac{1}{2} : \frac{x}{2} - \frac{y}{3} + \frac{1}{6} :: 10\frac{1}{2} : 1\frac{1}{6},
 \end{aligned}
 \left. \vphantom{\begin{aligned} 28. \quad & \text{Given } \frac{\frac{7x}{4} + 6y}{5} - \frac{\frac{3y+6}{5} - \frac{3x-2}{10}}{8} = 5 - \frac{x}{16}, \right.} \\
 & \text{to find the values of } x \text{ and } y.
 \end{aligned}}$$

$$\text{Ans. } \begin{cases} x = 4, \\ y = 3. \end{cases}$$

$$\begin{aligned}
 29. \quad & \text{Given } \frac{4x-2y+3}{3} - \frac{18-x+5y}{7} = \frac{x}{4} - \frac{y}{5} - \frac{1}{7} - 7\frac{1}{10}, \\
 & \text{and } 2x - y + 15 : y - 2x + 15 :: \frac{x}{3} - \frac{y}{4} + \frac{3}{4} : \frac{y}{4} - \frac{x}{3} + \frac{1}{12},
 \end{aligned}
 \left. \vphantom{\begin{aligned} 29. \quad & \text{Given } \frac{4x-2y+3}{3} - \frac{18-x+5y}{7} = \frac{x}{4} - \frac{y}{5} - \frac{1}{7} - 7\frac{1}{10}, \right.} \\
 & \text{to find the values of } x \text{ and } y.
 \end{aligned}}$$

$$\text{Ans. } \begin{cases} x = 18, \\ y = 24. \end{cases}$$

$$\begin{aligned}
 30. \quad & \text{Given } 4 + \frac{12y - \frac{6y+2}{x}}{11} = y + \frac{\frac{3xy-31}{11} + 10x + 13}{3x}, \\
 & \text{and } \frac{2x}{3} - \frac{3x-5}{y+7} = \frac{4xy + \frac{170}{3}}{6y+27},
 \end{aligned}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 7, \\ y = 2. \end{cases}$$

31. Given $\sqrt{y} - \sqrt{(y-x)} = \sqrt{(20-x)}$, } to find the values
and $\sqrt{(y-x)} : \sqrt{(20-x)} :: 3 : 2$, } of x and y .

$$\text{Ans. } \begin{cases} x = 16, \\ y = 25. \end{cases}$$

III. *Pure Quadratics and others which may be solved without completing the Square.*

1. GIVEN $3x^2 - 4 = 28 + x^2$, to find the values of x .

$$\text{Ans. } x = \pm 4.$$

2. Given $x + y : y :: 3 : 1$, } to find the values of x and y .
and $xy = 18$, }

$$\text{Ans. } \begin{cases} x = \pm 6, \\ y = \pm 3. \end{cases}$$

3. Given $x - y : y :: 4 : 5$, } to find the values of x and y .
and $x^2 + 4y^2 = 181$, }

$$\text{Ans. } \begin{cases} x = \pm 9, \\ y = \pm 5. \end{cases}$$

4. Given $x + y : x - y :: a : b$, } to find the values of x and y .
and $xy = c^2$, }

$$\text{Ans. } \begin{cases} x = \pm c \sqrt{\left(\frac{a+b}{a-b}\right)}, \\ y = \pm c \sqrt{\left(\frac{a-b}{a+b}\right)}. \end{cases}$$

5. Given $x^2 + y^2 : x^2 - y^2 :: 17 : 8$, } to find the values of x
and $xy^2 = 45$, } and y .

$$\text{Ans. } \begin{cases} x = 5, \\ y = 3. \end{cases}$$

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6. Given $x^2 - xy = 54,$
and $xy - y^2 = 18,$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = \pm 9, \\ y = \pm 3. \end{cases}$$

7. Given $x + y : x^2 - y^2 :: 1 : 4,$
and $xy = 21,$ } and y .

$$\text{Ans. } \begin{cases} x = 7, \text{ or } -3, \\ y = 3, \text{ or } -7. \end{cases}$$

8. Given $ax^2 + bxy = c^2,$
and $x - y : x :: m : n,$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = \pm c \cdot \sqrt{\left(\frac{nc}{na + nb - mb}\right)}, \\ y = \pm \frac{n-m}{n} \cdot c \cdot \sqrt{\left(\frac{nc}{na + nb - mb}\right)}. \end{cases}$$

9. Given $x^2 + y^2 : x^2 - y^2 :: 559 : 127,$
and $x^2y = 294,$ } and y .

$$\text{Ans. } \begin{cases} x = 7, \\ y = 6. \end{cases}$$

10. Given $x^2 - xy : xy - y^2 :: 3 : 7,$
and $xy^2 = 147,$ } and y .

$$\text{Ans. } \begin{cases} x = 3, \\ y = 7. \end{cases}$$

11. Given $\sqrt{x} + \sqrt{y} : \sqrt{x} - \sqrt{y} :: 4 : 1,$
and $x - y = 16,$ } of x and y .

$$\text{Ans. } \begin{cases} x = 25, \\ y = 9. \end{cases}$$

12. Given $\sqrt[4]{x} - \sqrt[4]{y} = 3,$
and $\sqrt[4]{x} + \sqrt[4]{y} = 7,$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 625, \\ y = 16. \end{cases}$$

13. Given $x - y : \sqrt{x} - \sqrt{y} :: 8 : 1,$ } to find the values of x
and $\sqrt{xy} = 15,$ } and y .

$$\text{Ans. } \begin{cases} x = 25, \text{ or } 9, \\ y = 9, \text{ or } 25. \end{cases}$$

14. Given $x^2 - y^2 : x^2y - xy^2 :: 7 : 2,$ } to find the values of x
and $x + y = 6,$ } and y .

$$\text{Ans. } \begin{cases} x = 4, \text{ or } 2, \\ y = 2, \text{ or } 4. \end{cases}$$

15. Given $\frac{1}{x} + \frac{1}{y} = \frac{1}{2},$ } to find the values of x and y .
and $\frac{2}{xy} = \frac{1}{9},$ }

$$\text{Ans. } \begin{cases} x = 6, \text{ or } 3, \\ y = 3, \text{ or } 6. \end{cases}$$

16. Given $x^4 - y^4 = 369,$ } to find the values of x and y .
and $x^2 - y^2 = 9,$ }

$$\text{Ans. } \begin{cases} x = \pm 5, \\ y = \pm 4. \end{cases}$$

17. Given $x^2 - y^2 = 56,$ } to find the values of x and y .
and $x - y = \frac{16}{xy},$ }

$$\text{Ans. } \begin{cases} x = 4, \text{ or } -2, \\ y = 2, \text{ or } -4. \end{cases}$$

18. Given $\frac{1}{1 - \sqrt{1 - x^2}} - \frac{1}{1 + \sqrt{1 - x^2}} = \frac{\sqrt{3}}{x^2},$
to find the values of x .

$$\text{Ans. } x = \pm \frac{1}{2}.$$

19. Given $x^2y + y^2 = 116,$ } to find the values of x and y .
and $xy^2 + y = 14,$ }

$$\text{Ans. } \begin{cases} x = 5, \text{ or } 2\sqrt{\frac{2}{5}}, \\ y = 4, \text{ or } 10. \end{cases}$$

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20. Given $\sqrt[3]{x} + \sqrt[3]{y} = 6,$
and $x + y = 72,$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 64, \text{ or } 8, \\ y = 8, \text{ or } 64. \end{cases}$$

21. Given $4x^2 + \frac{5}{2} = \frac{x^2}{y} + 10y,$ } to find the values of x and y .
and $x^2 + 3y = 55,$ }

$$\text{Ans. } \begin{cases} x = \pm 5, \\ y = 10. \end{cases}$$

22. Given $\frac{1}{x + \sqrt{(2-x^2)}} + \frac{1}{x - \sqrt{(2-x^2)}} = ax,$ to find the values of x .

$$\text{Ans. } x = \pm \sqrt{\left(\frac{a+1}{a}\right)}.$$

23. Given $\frac{x}{\sqrt{(a^2 + x^2)} - x} = b,$ to find the values of x .

$$\text{Ans. } x = \pm \frac{ab}{\sqrt{(2b+1)}}.$$

24. Given $\frac{1}{2}\sqrt{(4y-x)} + \frac{1}{2}\sqrt{(y-x)} = \sqrt{(2y-x)},$ }
and $\frac{3}{4}\sqrt{(x^2-6y)} + \sqrt{(y^2-9x)} : \sqrt{(x^2-6y)} :: 1\frac{3}{4} : 1,$ }
to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 7, \\ y = 8. \end{cases}$$

25. Given $\frac{9}{8} \cdot \frac{\sqrt[3]{(x+y)}}{y} + \frac{9}{8} \cdot \frac{\sqrt[3]{(x+y)}}{x} = 1\frac{1}{7},$ } to find the values of
and $\frac{7}{4} \cdot \frac{\sqrt[3]{(x-y)}}{y} - \frac{7}{4} \cdot \frac{\sqrt[3]{(x-y)}}{x} = \frac{1}{9},$ } x and y .

$$\text{Ans. } \begin{cases} x = \frac{7}{2}, \\ y = \frac{9}{2}. \end{cases}$$

26. Given $x^4 + y^4 = 20$,
and $x^2 + y^2 = 6$, } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = \pm 8, \text{ or } \pm \sqrt{8}, \\ y = 32, \text{ or } 1024. \end{cases}$$

27. Given $x^4 + 2x^2y^2 + y^4 = 1296 - 4xy \cdot (x^2 + xy + y^2)$,
and $x - y = 4$, }
to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 5, \text{ or } -1, \\ y = 1, \text{ or } -5. \end{cases}$$

28. Given $(a^{4b} + 1) \cdot (x^4 - 1)^2 = 2 \cdot (x + 1)$, to find the values of x .

$$\text{Ans. } x = \left(\frac{a^{2b} \pm 1}{a^{2b} \mp 1} \right)^2.$$

29. Given $\frac{\sqrt{a} - \sqrt{a-x}}{\sqrt{a} + \sqrt{a-x}} = a$, to find the value of x .

$$\text{Ans. } x = \frac{4a^2}{(a+1)^3}.$$

30. Given $\frac{\sqrt{x} + \sqrt{x-y}}{\sqrt{x} - \sqrt{x-y}} = 4$, } to find the values of x
and $\sqrt{x} : \sqrt{y} :: \sqrt{y} : 4$, } and y .

$$\text{Ans. } \begin{cases} x = \frac{625}{16}, \\ y = 25. \end{cases}$$

31. Given $\frac{1}{y} - \frac{1}{x} = \frac{1}{4}$, } to find the values of x and y .
and $x^2y - xy^2 = 16$, }

$$\text{Ans. } \begin{cases} x = 4, \text{ or } -2, \\ y = 2, \text{ or } -4. \end{cases}$$

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32. Given $\frac{\sqrt{(4x+1)} + \sqrt{4x}}{\sqrt{(4x+1)} - \sqrt{4x}} = 9$, } to find the value of x .

$$\text{Ans. } x = \frac{4}{9}.$$

33. Given $\frac{a+x+\sqrt{(2ax+x^2)}}{a+x-\sqrt{(2ax+x^2)}} = b$, } to find the values of x .

$$\text{Ans. } x = \pm a \cdot \frac{(\sqrt{b} \mp 1)^2}{2\sqrt{b}}.$$

34. Given $\sqrt{(x-y)} + \frac{1}{2}\sqrt{(x+y)} = \frac{x-1}{\sqrt{(x-y)}}$, } to find the
and $x^2 + y^2 : xy :: 34 : 15$, } values of
 x and y .

$$\text{Ans. } \begin{cases} x = 5, \\ y = 3. \end{cases}$$

35. Given $x^4y^2 - x^2y^4 = 216$, } to find the values of x and y .
and $x^2y - xy^2 = 6$, }

$$\text{Ans. } \begin{cases} x = 3, \text{ or } -2, \\ y = 2, \text{ or } -3. \end{cases}$$

36. Given $x^3 + x\sqrt[3]{(xy^3)} = 208$, } to find the values of x and y .
and $y^3 + y\sqrt[3]{(x^3y)} = 1053$, }

$$\text{Ans. } \begin{cases} x = \pm 8, \\ y = \pm 27. \end{cases}$$

37. Given $x^4 + x^2y^2 + y^4 = 1009$, } to find the values of x
and $x^3 + x^2y^2 + y^3 = 582193$, } and y .

$$\text{Ans. } \begin{cases} x = 81, \text{ or } 16, \\ y = 16, \text{ or } 81. \end{cases}$$

38. Given $x^3 + y^3 + xy \cdot (x+y) = 68$, } to find the values of
and $x^3 + y^3 - 3x^2 = 12 + 3y^2$, } x and y .

$$\text{Ans. } \begin{cases} x = 4, \text{ or } 2, \\ y = 2, \text{ or } 4. \end{cases}$$

39. Given $xy \cdot (x + y) = 84,$
and $x^2y^2 \cdot (x^2 + y^2) = 3600,$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 4, \text{ or } 3, \\ y = 3, \text{ or } 4. \end{cases}$$

40. Given $\frac{x^2 + xy + y^2}{x + y} = 7,$
and $\frac{x^2 - xy + y^2}{x - y} = 9,$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 6, \\ y = 3. \end{cases}$$

41. Given $\frac{x^{(m-n)^2} + x^{-4mn}}{x^{(m-n)^2} - x^{-4mn}} = a^{\frac{r}{s}},$ to find the value of x .

$$\text{Ans. } x = \left(\frac{a^{\frac{r}{s}} + 1}{a^{\frac{r}{s}} - 1} \right)^{\frac{1}{(m+n)^2}}.$$

IV. Affected Quadratics involving only one unknown Quantity.

1. Given $x^2 + 4x = 140,$ to find the values of x .
Ans. $x = 10,$ or $-14.$
2. Given $x^2 - 6x + 8 = 80,$ to find the values of x .
Ans. $x = 12,$ or $-6.$
3. Given $x^2 - 10x + 17 = 1,$ to find the values of x .
Ans. $x = 8,$ or $2.$
4. Given $x^2 - x - 40 = 170,$ to find the values of x .
Ans. $x = 15,$ or $-14.$

5. Given $3x^2 - 9x - 4 = 80$, to find the values of x .

Ans. $x = 7$, or -4 .

6. Given $7x^2 - 21x + 13 = 293$, to find the values of x .

Ans. $x = 8$, or -5 .

7. Given $\frac{x^2}{3} + \frac{4x}{5} - 19 = 15\frac{1}{5}$, to find the values of x .

Ans. $x = 9$, or $-\frac{57}{5}$.

8. Given $\frac{2x^2}{3} + 3\frac{1}{2} = \frac{x}{2} + 8$, to find the values of x .

Ans. $x = 3$, or $-\frac{9}{4}$.

9. Given $x + 4 + \frac{7x - 8}{x} = 13$, to find the values of x .

Ans. $x = 4$, or -2 .

10. Given $4x - \frac{36 - x}{x} = 46$, to find the values of x .

Ans. $x = 12$, or $-\frac{3}{4}$.

11. Given $16 - \frac{5 - x}{2} = \frac{9 + 3x}{x} + 3x$, to find the values of x .

Ans. $x = 3$, or $\frac{6}{5}$.

12. Given $\frac{x + 3}{2} + \frac{16 - 2x}{2x - 5} = 5\frac{1}{5}$, to find the values of x .

Ans. $x = 5$, or $\frac{69}{10}$.

13. Given $14 + 4x - \frac{x+7}{x-7} = 3x + \frac{9+4x}{3}$, to find the values of x .

Ans. $x = 9$, or 28 .

14. Given $\frac{x+4}{3} - \frac{7-x}{x-3} = \frac{4x+7}{9} - 1$, to find the values of x .

Ans. $x = 21$, or 5 .

15. Given $\frac{15-x}{4} - \frac{12-3x}{4x-5} = 7x - \frac{23x+60}{7}$, to find the values of x .

Ans. $x = 3$, or $\frac{229}{148}$.

16. Given $\frac{x+11}{x} + \frac{9+4x}{x^2} = 7$, to find the values of x .

Ans. $x = 3$, or $-\frac{1}{2}$.

17. Given $\frac{2x+9}{9} + \frac{4x-3}{4x+3} = 3 + \frac{3x-16}{18}$, to find the values of x .

Ans. $x = 6$, or $-\frac{19}{4}$.

18. Given $\frac{x}{x+60} = \frac{7}{3x-5}$, to find the values of x .

Ans. $x = 14$, or -10 .

19. Given $\frac{3x-7}{x} + \frac{4x-10}{x+5} = 3\frac{1}{4}$, to find the values of x .

Ans. $x = 7$, or $-\frac{10}{7}$.

20. Given $\frac{x+2}{x-1} - \frac{4-x}{2x} = 2\frac{1}{2}$, to find the values of x .

Ans. $x = 3$, or $-\frac{4}{5}$.

21. Given $\frac{8x}{x+2} - 6 = \frac{20}{3x}$, to find the values of x .

Ans. $x = 10$, or $-\frac{2}{3}$.

22. Given $\frac{40}{x-5} + \frac{27}{x} = 13$, to find the values of x .

Ans. $x = 9$, or $\frac{15}{13}$.

23. Given $\frac{5x-12}{9} + \frac{3x-24}{4x-12} = 9 - \frac{7x-34}{15}$, to find the values of x .

Ans. $x = 12$, or $\frac{477}{184}$.

24. Given $\frac{2x}{x-4} + \frac{2x-5}{x-3} = 8\frac{1}{2}$, to find the values of x .

Ans. $x = 6$, or $\frac{40}{13}$.

25. Given $\frac{2x+3}{10-x} = \frac{2x}{25-3x} - 6\frac{1}{2}$, to find the values of x .

Ans. $x = 8$, or $13\frac{3}{4}$.

26. Given $\frac{4x-5}{x} - \frac{3x-7}{3x+7} = \frac{9x+23}{13x}$, to find the values of x .

Ans. $x = 2$, or $-\frac{154}{45}$.

27. Given $2x + 18 - \frac{8x^2 + 16}{4x + 7} = 27 - \frac{12x - 11}{2x - 3}$, to find the values of x .

Ans. $x = 8$, or 5 .

28. Given $\frac{x}{x+9} + \frac{5}{2x+18} = \frac{x - \frac{x^2 + 20}{x+8}}{2}$, to find the values of x .

Ans. $x = 4$, or $-\frac{55}{6}$.

29. Given $\frac{x+4}{x+6} + \frac{5}{2x+4} = \frac{3x+7}{3x+4}$, to find the values of x .

Ans. $x = 8$, or $-\frac{2}{3}$.

30. Given $\frac{4}{2x+3} + \frac{3x+6}{5x+18} = \frac{3x+5}{5x}$, to find the values of x .

Ans. $x = 6$, or $-\frac{45}{2}$.

31. Given $\frac{8}{9+5x} + \frac{8x-17}{2+4x} = \frac{4x+3}{2x+12}$, to find the values of x .

Ans. $x = 3$, or $-\frac{283}{137}$.

32. Given $\frac{12}{5-x} + \frac{8}{4-x} = \frac{32}{x+2}$, to find the values of x .

Ans. $x = 2$, or $\frac{58}{13}$.

33. Given $\frac{2x-1}{3-x} = \frac{8-x^2}{2x-2} + \frac{x}{2}$, to find the values of x .

Ans. $x = 2$, or $-\frac{11}{3}$.

34. Given $\frac{3}{6x-x^2} + \frac{6}{x^2+2x} = \frac{11}{5x}$, to find the values of x .

Ans. $x = 3$, or $\frac{26}{11}$.

35. Given $\frac{4x^2+7x}{19} + \frac{5x-x^2}{3+x} = \frac{4x^2}{9}$, to find the values of x .

Ans. $x = 3$, or $-\frac{87}{10}$.

36. Given $\frac{x^4+2x^2+8}{x^2+x-6} = x^2+x+8$, to find the values of x .

Ans. $x = 4$, or $-\frac{14}{3}$.

37. Given $\frac{x+12}{x} + \frac{x}{x+12} = 5\frac{3}{15}$, to find the values of x .

Ans. $x = 3$, or -15 .

38. Given $\sqrt{4x+5} \times \sqrt{7x+1} = 30$, to find the values of x .

Ans. $x = 5$, or $-\frac{179}{28}$.

39. Given $\frac{\sqrt{x+9}}{\sqrt{x}} = \frac{\sqrt{9x}-3\frac{4}{5}}{9-\sqrt{x}}$, to find the values of x .

Ans. $x = 5$, or $\frac{6561}{400}$.

40. Given $\frac{x + \sqrt{x}}{x - \sqrt{x}} = \frac{x^2 - x}{4}$, to find the values of x .

Ans. $x = 4$, or 1 , or $\frac{-3 \pm \sqrt{-7}}{2}$.

41. Given $\frac{x - \sqrt{(x+1)}}{x + \sqrt{(x+1)}} = \frac{5}{11}$, to find the values of x .

Ans. $x = 8$, or $-\frac{8}{9}$.

42. Given $5 \cdot \frac{3x-1}{1+5\sqrt{x}} + \frac{2}{\sqrt{x}} = 3\sqrt{x}$, to find the values of x .

Ans. $x = 4$, or $\frac{1}{9}$.

43. Given $\sqrt{x^3} - \frac{40}{\sqrt{x}} = 3x$, to find the values of x .

Ans. $x = 4$, or $\sqrt[3]{-5}$.

44. Given $x^4 + 7x^4 = 44$, to find the values of x .

Ans. $x = \pm 8$, or $\pm \sqrt[4]{-11}$.

45. Given $4x^4 + x^4 = 39$, to find the values of x .

Ans. $x = 729$, or $\sqrt[4]{\frac{13}{4}}$.

46. Given $3x^6 + 42x^3 = 3321$, to find the values of x .

Ans. $x = 3$, or $-\sqrt[3]{41}$.

47. Given $\frac{8}{x^3} + 2 = \frac{17}{x^4}$, to find the values of x .

Ans. $x = 4$, or $\sqrt[3]{\frac{1}{4}}$.

48. Given $x^{\frac{1}{2}} + \frac{41\sqrt[3]{x}}{x} = \frac{97}{\sqrt[3]{x^2}} + x^{\frac{1}{2}}$, to find the values of x .

Ans. $x = 4$, or $-7\frac{1}{2}$.

49. Given $\sqrt{\frac{1}{x^4}} + \sqrt[3]{\frac{1}{x}} = \frac{3 - \sqrt[3]{x^3}}{x}$, to find the values of x .

Ans. $x = 1$, or $-\frac{27}{8}$.

50. Given $3x^{\frac{1}{2}}\sqrt[3]{x^{\frac{1}{2}}} - \frac{4x^{\frac{1}{2}}}{\sqrt[3]{x^{\frac{1}{2}}}} = 4$, to find the values of x .

Ans. $x = 8\frac{1}{27}$, or $-\frac{8}{27}\frac{1}{27}$.

51. Given $\frac{3\sqrt{x-x^{\frac{1}{2}}}}{x+2} = \frac{1\frac{1}{2} + 3\sqrt{x-2x}}{2\sqrt{x-3}}$, to find the values of x .

Ans. $x = 4$, or $\frac{1}{3}$.

52. Given $2x^{\frac{1}{2}} \cdot (x^{\frac{1}{2}} + a^{\frac{1}{2}})^{\frac{1}{2}} = 2x^{\frac{1}{2}} \cdot (x + 2a) + a^{\frac{1}{2}} \cdot (x - a)$, to find the values of x .

Ans. $x = \frac{a}{2}$, or $-a$.

53. Given $adx - acx^2 = bcx - bd$, to find the values of x .

Ans. $x = \frac{d}{c}$, or $-\frac{b}{a}$.

54. Given $\frac{a^2x^2}{b^2} - \frac{2ax}{c} + \frac{d^2}{c^2} = 0$, to find the values of x .

Ans. $x = \frac{b}{a} \times \frac{b \pm \sqrt{(b^2 - d^2)}}{c}$.

55. Given $9a^4b^4x^2 - 6a^3b^3x = b^3$, to find the values of x .

$$\text{Ans. } x = \frac{a^2 \pm \sqrt{(a^2 + b^2)}}{3a^2b^2}.$$

56. Given $(a + b) \cdot x^2 = cx + \frac{ac}{a+b}$, to find the values of x .

$$\text{Ans. } x = \frac{c \pm \sqrt{(c^2 + 4ac)}}{2 \cdot (a + b)}.$$

57. Given $3\sqrt{(112 - 8x)} = 19 + \sqrt{(3x + 7)}$, to find the values of x .

$$\text{Ans. } x = 6, \text{ or } \frac{7398}{625}.$$

58. Given $\sqrt{(2x + 7)} + \sqrt{(3x - 18)} = \sqrt{(7x + 1)}$, to find the values of x .

$$\text{Ans. } x = 9, \text{ or } -\frac{18}{5}.$$

59. Given $7 \cdot \sqrt{\left(\frac{3x}{2} - 5\right)} - \sqrt{\left(\frac{x}{5} + 45\right)} = \frac{7}{4}\sqrt{(10x + 56)}$,
to find the values of x .

$$\text{Ans. } x = 20, \text{ or } \frac{14568980}{2874649}.$$

60. Given

$$\frac{16 - 4\sqrt{x}}{8 - 3\sqrt{x}} = \frac{88 + 33\sqrt{x}}{4 + \sqrt{x}} + \frac{x^2 - 5x + 11}{(8 - 3\sqrt{x}) \cdot (4 + \sqrt{x})},$$

to find the values of x .

$$\text{Ans. } x = 93, \text{ or } 7.$$

61. Given

$$\frac{54 - 9\sqrt{x}}{x + 2\sqrt{x}} = \frac{23x - 46\sqrt{x}}{6 + \sqrt{x}} + \frac{7x^2 - 3x + 4}{(x + 2\sqrt{x}) \times (6 + \sqrt{x})}$$

to find the values of x .

$$\text{Ans. } x = 5, \text{ or } -\frac{32}{15}.$$

62. Given $x + \sqrt{x} : x - \sqrt{x} :: 3\sqrt{x} + 6 : 2\sqrt{x}$, to find the values of x .

$$\text{Ans. } x = 9, \text{ or } 4.$$

63. Given $x^2 + 11 + \sqrt{(x^2 + 11)} = 42$, to find the values of x .

$$\text{Ans. } x = \pm 5, \text{ or } \pm \sqrt{38}.$$

64. Given $(x - 5)^2 - 3 \cdot (x - 5)^{\frac{1}{2}} = 40$, to find the values of x .

$$\text{Ans. } x = 9, \text{ or, } -5^{\frac{1}{2}} + 5.$$

65. Given $x + \sqrt{(x + 6)} = 2 + 3\sqrt{(x + 6)}$, to find the values of x .

$$\text{Ans. } x = 10, \text{ or } -2.$$

66. Given $(x^2 + 5)^2 - 4x^2 = 160$, to find the values of x .

$$\text{Ans. } x = \pm 3, \text{ or } \pm \sqrt{-15}.$$

67. Given $x^2 - 7x + \sqrt{(x^2 - 7x + 18)} = 24$, to find the values of x .

$$\text{Ans. } x = 9, \text{ or } -2, \text{ or } \frac{7 \pm \sqrt{(173)}}{2}.$$

68. Given $9x - 4x^2 + \sqrt{(4x^2 - 9x + 11)} = 5$, to find the values of x .

$$\text{Ans. } x = 2, \text{ or } \frac{1}{4}, \text{ or } \frac{9 \pm \sqrt{(-31)}}{8}.$$

69. Given $x^2 + \sqrt{5x + x^2} = 42 - 5x$, to find the values of x .

$$\text{Ans. } x = 4, \text{ or } -9, \text{ or } \frac{-5 \pm \sqrt{221}}{2}.$$

70. Given $\frac{2}{(x+2)^{\frac{1}{2}}} + \frac{\sqrt{x+2}}{2} = \frac{17}{4\sqrt{x+2}}$, to find the values of x .

$$\text{Ans. } x = 6, \text{ or } -\frac{3}{2}.$$

71. Given $\frac{x}{x+4} + \frac{4}{\sqrt{x+4}} = \frac{21}{x}$, to find the values of x .

$$\text{Ans. } x = 12, \text{ or } -3, \text{ or } \frac{49 \pm \sqrt{3185}}{2}.$$

72. Given $\frac{7+x}{7-x} + \frac{7-x}{7+x} = \frac{29}{10}$, to find the values of x .

$$\text{Ans. } x = 7, \text{ or } -3.$$

73. Given $\frac{3x+5}{3x-5} - \frac{3x-5}{3x+5} = \frac{135}{176}$, to find the values of x .

$$\text{Ans. } x = 9, \text{ or } -\frac{25}{81}.$$

74. Given $x + \sqrt{x} + 2 = \frac{x^2 + x - 4}{\sqrt{x}}$, to find the values of x .

$$\text{Ans. } x = 4, \text{ or } 1.$$

75. Given $\frac{x^2}{(x^2-4)^{\frac{1}{2}}} + \frac{6}{x^2-4} = \frac{351}{25x^2}$, to find the values of x .

$$\text{Ans. } x = \pm 3, \text{ or } \pm \sqrt{\frac{39}{11}}.$$

76. Given $\left(x + \frac{8}{x}\right)^2 + x = 42 - \frac{8}{x}$, to find the values of x .

$$\text{Ans. } x = 4, \text{ or } 2, \text{ or } \frac{-7 \pm \sqrt{17}}{2}.$$

77. Given $x + 4 - 2\sqrt{\left(\frac{x+4}{x-4}\right)} = \frac{3}{x-4}$, to find the values of x .

$$\text{Ans. } x = \pm 5, \text{ or } \pm \sqrt{17}.$$

78. Given $\sqrt{\left(12 - \frac{12}{x^2}\right)} + \sqrt{\left(x^2 - \frac{12}{x^2}\right)} = x^2$, to find the values of x .

$$\text{Ans. } x = \pm 2, \text{ or } \pm \sqrt{-3}.$$

79. Given $x^4 \cdot \left(1 + \frac{1}{3x}\right)^3 - (3x^2 + x) = 70$, to find the values of x .

$$\text{Ans. } x = 3, \text{ or } -\frac{10}{3}, \text{ or } \frac{-1 \pm \sqrt{(-251)}}{6}.$$

80. Given $x^2 - \frac{5x}{2} + 15 = \frac{25x^2}{16} - \frac{64}{x^2}$, to find the values of x .

$$\text{Ans. } x = 4, \text{ or } -8, \text{ or } \frac{-2 \pm 2\sqrt{(-71)}}{9}.$$

81. Given $\frac{35\frac{1}{2}}{\sqrt{(x^4 - 9x^2)}} + \frac{\sqrt{(x^2 - 9)}}{7x} = \frac{19}{2x}$, to find the values of x .

$$\text{Ans. } x = \pm 5, \text{ or } \pm \frac{\sqrt{(15661)}}{2}.$$

82. Given $3 \cdot \{(x-1)^2 - x\}^2 + 2x = 341 + 2 \cdot (x-1)^2$, to find the values of x .

$$\text{Ans. } x = 5, \text{ or } -2; \text{ or } \frac{3\sqrt{3} \pm \sqrt{(-109)}}{2\sqrt{3}}.$$

83. Given $x^2 - 2x^{\frac{1}{2}} + 2x - \sqrt{x} = 6$, to find the values of x .

$$\text{Ans. } x = 4, \text{ or } 1, \text{ or } \frac{-5 \pm \sqrt{(-11)}}{2}.$$

84. Given $x^4 + \frac{13x^3}{3} - 39x = 81$, to find the values of x .

$$\text{Ans. } x = \pm 3, \text{ or } \frac{-13 \pm \sqrt{(-155)}}{6}.$$

85. Given $\frac{x^2 + 8}{6} - \frac{4x + 6}{x^2} = \frac{8 - \frac{2}{x^2}}{3}$, to find the values of x .

$$\text{Ans. } x = 4, \text{ or } -2; \text{ or } -1 \pm \sqrt{(-3)}.$$

86. Given $\sqrt{x} - \frac{8}{x} = \frac{7}{\sqrt{x-2}}$, to find the values of x .

$$\text{Ans. } x = 16, \text{ or } 1, \text{ or } \frac{1 \pm 3\sqrt{(-7)}}{2}.$$

87. Given $4x^4 + \frac{x}{2} = 4x^3 + 33$, to find the values of x .

$$\text{Ans. } x = 2, \text{ or } -\frac{3}{2}; \text{ or } \frac{1 \pm \sqrt{(-43)}}{4}.$$

88. Given $(x-2)^2 - 6x^{\frac{1}{2}} \cdot (x-2) = 24 - 14x + 15x^{\frac{1}{2}}$, to find the values of x .

$$\text{Ans. } x = 16, \text{ or } 1; \text{ or } \frac{\pm 3\sqrt{(-11)} - 1}{2}.$$

89. Given $(4x + 1)^2 + 4x^{\frac{1}{2}} \cdot (4x + 1) = 1912 - (10x + 3x^{\frac{1}{2}})$, to find the values of x .

$$\text{Ans. } x = 9, \text{ or } \frac{49}{4}; \text{ or } \frac{-90 \mp \sqrt{(-181)}}{8}.$$

90. Given $8x^3 - 13 = \frac{3x}{2} + \sqrt{(6x^3 + 52x^2)}$, to find the values of x .

$$\text{Ans. } x = 2, \text{ or } -\frac{13}{8}; \text{ or } \frac{3 \pm \sqrt{(3337)}}{64}.$$

91. Given $4x^3 + 21x + 8x^{\frac{1}{2}}\sqrt{(7x^2 - 5x)} = 207 - \frac{4x^3}{3}$, to find the values of x .

$$\text{Ans. } x = 3, \text{ or } \frac{207}{16}, \text{ or } \frac{-129 \pm 3\sqrt{(-2567)}}{32}.$$

92. Given $\frac{2x + \sqrt{x}}{2x - \sqrt{x}} = 3\frac{7}{5} - 3 \cdot \frac{2x - \sqrt{x}}{2x + \sqrt{x}}$, to find the values of x .

$$\text{Ans. } x = 4, \text{ or } \frac{49}{16}.$$

93. Given $a^2 b^3 x^{\frac{1}{n}} - 4 \cdot (ab)^{\frac{1}{2}} \cdot x^{\frac{m+n}{2mn}} = (a-b)^2 \cdot x^{\frac{1}{m}}$, to find the values of x .

$$\text{Ans. } x = \left\{ \frac{(\sqrt{a} + \sqrt{b})^2}{ab} \right\}^{\frac{2mn}{m-n}}, \text{ or } \left\{ \frac{-(\sqrt{a} - \sqrt{b})^2}{ab} \right\}^{\frac{2mn}{m-n}}.$$

94. Given $\sqrt[pq]{x^{p+q}} - \frac{1}{2} \cdot \frac{a^3 - b^3}{a^2 + b^2} \cdot \{\sqrt[p]{x} + \sqrt[q]{x}\} = 0$, to find the values of x .

$$\text{Ans. } x = \left(\frac{a \pm b}{a \mp b} \right)^{\frac{2pq}{p-q}}.$$

V. *Adfected Quadratics involving two unknown Quantities.*

1. GIVEN $x + 4y = 14,$
and $y^2 + 4x = 2y + 11,$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = -46, \text{ or } 2, \\ y = 15, \text{ or } 3. \end{cases}$$

2. Given $2x + 3y = 118,$
and $5x^2 - 7y^2 = 4333,$ } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 35, \text{ or } -\frac{3899}{17}, \\ y = 16, \text{ or } \frac{3268}{17}. \end{cases}$$

3. Given $\frac{2x + 7y}{4x} = 2y - \frac{51 + 2x}{10}$
and $\frac{4x + 3y}{16} = y - 2,$ } to find the values of x
and y .

$$\text{Ans. } \begin{cases} x = 5, \text{ or } -\frac{56}{27} \\ y = 4, \text{ or } \frac{640}{351}. \end{cases}$$

4. Given $\frac{4xy + 3y - 3}{5x} - 1 = \frac{4y + 3x - 2}{5} - \frac{18 - x}{3},$
and $\frac{3x + y}{7} = \frac{3x - 5y}{3} + 2,$ }
to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 6, \text{ or } -\frac{3}{266}, \\ y = 3, \text{ or } \frac{2784}{2527}. \end{cases}$$

5. Given $x^2 - y^2 = a^2$, } to find the values
 $(x + y + b)^2 + (x - y + b)^2 = 2c^2$ } of x and y .

$$\text{Ans. } \begin{cases} x = \frac{-b \pm \sqrt{(2a^2 - b^2 + 2c^2)}}{2}, \\ y = \pm \sqrt{\left(\frac{c^2 - a^2 \mp \sqrt{(2a^2 - b^2 + 2c^2)}}{2}\right)} \end{cases}$$

6. Given $x^2 + y : x^2 - y :: 9 : 7$, } to find the values of x
 and $1 + x^2 : y + 4 :: 5y + 7 : 3y$, } and y .

$$\text{Ans. } \begin{cases} x = \pm 4, \text{ or } \pm 4 \sqrt{\left(-\frac{7}{19}\right)}, \\ y = 2, \text{ or } -\frac{14}{19}. \end{cases}$$

7. Given $x^2 + 2x^2y = 441 - x^4y^2$, } to find the values of x
 and $xy = 3 + x$, } and y .

$$\text{Ans. } \begin{cases} x = 3, \text{ or } -7, \text{ or } -2 \pm \sqrt{(-17)}, \\ y = 2, \text{ or } \frac{4}{7}, \text{ or } \frac{5 \mp \sqrt{(-17)}}{7}. \end{cases}$$

8. Given $x^2 + 4y^2 = 256 - 4xy$, } to find the values of x
 and $3y^2 - x^2 = 39$, } and y .

$$\text{Ans. } \begin{cases} x = \pm 6, \text{ or } \pm 102, \\ y = \pm 5, \text{ or } \pm 59. \end{cases}$$

9. Given $(x + y)^2 - 3y = 28 + 3x$, } to find the values of x
 and $2xy + 3x = 35$, } and y .

$$\text{Ans. } \begin{cases} x = 5, \text{ or } \frac{7}{2}, \text{ or } \frac{-5 \pm \sqrt{(-255)}}{4} \\ y = 2, \text{ or } \frac{7}{2}, \text{ or } \frac{-11 \mp \sqrt{(-255)}}{4}. \end{cases}$$

10. Given $(2x - 4y)^2 + x - 2y = 5$, } to find the values of x
and $x^2 - y^2 = 8$, } and y .

$$\text{Ans. } \begin{cases} x = 3, \text{ or } -\frac{11}{3}, \\ y = 1, \text{ or } -\frac{7}{3}. \end{cases}$$

11. Given $\frac{3}{2}\sqrt{x-y} = 1 + \frac{1}{\sqrt{x-y}}$, } to find the values
and $\sqrt{x+y} + \sqrt{x-y} = 5$, } of x and y .

$$\text{Ans. } \begin{cases} x = \frac{13}{2}, \\ y = \frac{5}{2}. \end{cases}$$

12. Given $x^2 + 10x + y = 119 - 2\sqrt{y} \times (x+5)$, }
and $x + 2y = 13$, }
to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 5, \text{ or } \frac{17}{2}, \text{ or } \frac{-69 \pm \sqrt{(241)}}{4}, \\ y = 4, \text{ or } \frac{9}{4}, \text{ or } \frac{121 \mp \sqrt{(241)}}{8}. \end{cases}$$

13. Given $\frac{x^4}{y^2} + \frac{2x^2}{y} = 9\frac{39}{49}$, } to find the values of x and y .
and $x^2 + y^2 = 65$, }

ANSWER,

$$x = \pm 4, \text{ or } \pm \frac{4\sqrt{(-65)}}{7}; \text{ or } \pm \frac{\sqrt{(-450 \mp 30\sqrt{3410})}}{\sqrt{7}}$$

$$y = 7, \text{ or } -\frac{65}{7}; \text{ or } \frac{15 \pm \sqrt{(3410)}}{7}.$$

14. Given $x + y + \sqrt{(x + y)} = 6$, } to find the values of x
and $x^2 + y^2 = 10$, } and y .

$$\text{Ans. } \begin{cases} x = 3, \text{ or } 1; \text{ or } \frac{9 \pm \sqrt{-61}}{2} \\ y = 1, \text{ or } 3; \text{ or } \frac{9 \mp \sqrt{-61}}{2} \end{cases}$$

15. Given $x^2 + 4\sqrt{(x^2 + 3y + 5)} = 55 - 3y$, } to find the values
and $6x - 7y = 16$, } of x and y .

$$\text{Ans. } \begin{cases} x = 5, \text{ or } \frac{-53}{7}; \text{ or } \frac{-9 \pm \sqrt{(3895)}}{7} \\ y = 2, \text{ or } -\frac{46}{7}; \text{ or } \frac{-70 \pm \sqrt{(3895)}}{49} \end{cases}$$

16. Given $x^2 + 3x + y = 73 - 2xy$, } to find the values of x
and $y^2 + 3y + x = 44$, } and y .

$$\text{Ans. } \begin{cases} x = 4, \text{ or } 16; \text{ or } -12 \pm \sqrt{58} \\ y = 5, \text{ or } -7; \text{ or } -1 \mp \sqrt{58} \end{cases}$$

17. Given $\frac{y}{(x + y)^{\frac{1}{2}}} + \frac{\sqrt{(x + y)}}{y} = \frac{17}{4\sqrt{(x + y)}}$, } to find the
and $x = y^2 + 2$, } values of
 x and y .

$$\text{Ans. } \begin{cases} x = 6, \text{ or } 3, \text{ or } \frac{9 \mp 3\sqrt{(-119)}}{32} \\ y = 2, \text{ or } 1, \text{ or } \frac{-3 \pm \sqrt{(-119)}}{8} \end{cases}$$

18. Given $y - y^{\frac{1}{2}} = 16 - x$, } to find the values of x and y .
and $28 - y = x + 4x^{\frac{1}{2}}$, }

$$\text{Ans. } \begin{cases} x = 4, \text{ or } \frac{58}{17} \\ y = 16, \text{ or } \frac{784}{289} \end{cases}$$

19. Given $x^4 + y^4 = 97$,
and $x + y = 5$, } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 3, \text{ or } 2; \text{ or } \frac{5 \pm \sqrt{(-151)}}{2}, \\ y = 2, \text{ or } 3; \text{ or } \frac{5 \mp \sqrt{(-151)}}{2}. \end{cases}$$

20. Given $\sqrt{\left(\frac{3x-2y}{2x}\right)} + \sqrt{\left(\frac{2x}{3x-2y}\right)} = 2$, } to find the
and $x^2 - 18 = x \cdot (4y - 9)$, } values of
 x and y .

$$\text{Ans. } \begin{cases} x = 6, \text{ or } 3, \\ y = 3, \text{ or } \frac{3}{2}. \end{cases}$$

21. Given $x + 4\sqrt{x} + 4y = 21 + 8\sqrt{y} + 4\sqrt{xy}$, } to find the
and $\sqrt{x} + \sqrt{y} = 6$, } values of
values of x and y .

$$\text{Ans. } \begin{cases} x = 25, \text{ or } \frac{25}{9}, \\ y = 1, \text{ or } \frac{169}{9}. \end{cases}$$

22. Given $3x + \frac{4}{3}\sqrt{xy^3 + 9x^3y} = (x - \frac{1}{3}) \cdot y$, } to find the
and $6x + y : y :: x + 5 : 3$, } values of
values of x and y .

$$\text{Ans. } \begin{cases} x = 4, \\ y = 12. \end{cases}$$

23. Given $x + y = 5$, } to find the values of
and $(x^3 + y^3) \times (x^3 + y^3) = 455$, } x and y .

$$\text{Ans. } \begin{cases} x = 3, \text{ or } 2; \text{ or } \frac{5}{2} \pm \frac{1}{2}\sqrt{\left(-\frac{103}{3}\right)}, \\ y = 2, \text{ or } 3; \text{ or } \frac{5}{2} \mp \frac{1}{2}\sqrt{\left(-\frac{103}{3}\right)}. \end{cases}$$

24. Given $x + y - \sqrt{\left(\frac{x+y}{x-y}\right)} = \frac{6}{x-y}$, } to find the values
and $x^2 + y^2 = 41$, of x and y .

$$\text{Ans. } \begin{cases} x = \pm 5, \text{ or } \pm 3\sqrt{\frac{5}{2}}, \\ y = \pm 4, \text{ or } \pm \sqrt{\frac{37}{2}}. \end{cases}$$

25. Given $\frac{x^4}{y^2} + \frac{y^4}{x^2} = 136\frac{1}{9} - 2xy$, } to find the values of x
and $x + 4 = 14 - y$, and y .

$$\text{Ans. } \begin{cases} x = 6, \text{ or } 4; \text{ or } 5 \pm 5\sqrt{\left(-\frac{13}{11}\right)}, \\ y = 4, \text{ or } 6; \text{ or } 5 \mp 5\sqrt{\left(-\frac{13}{11}\right)}. \end{cases}$$

26. Given $\frac{x+y}{x-y} - \frac{x-y}{x+y} = 4\frac{4}{5}$, } to find the values of
and $\sqrt{\left(\frac{x-y}{x^4}\right)} + \frac{1}{x} = \frac{4}{9\sqrt{(x-y)}}$, } x and y .

$$\text{Ans. } \begin{cases} x = 3, \text{ or } \frac{45}{2}; \text{ or } \frac{3}{16}, \text{ or } \frac{45}{32}, \\ y = 2, \text{ or } -\frac{135}{4}; \text{ or } \frac{1}{8}, \text{ or } -\frac{135}{64}. \end{cases}$$

27. Given $\sqrt{(6\sqrt{x} + 6\sqrt{y})} + \frac{1}{2}\sqrt{x} = 9 - \frac{1}{2}\sqrt{y}$, } to find
and $x - y = 12$,
the values of x and y .

$$\text{Ans. } \begin{cases} x = 16, \text{ or } \frac{59536}{81}, \\ y = 4, \text{ or } \frac{58564}{81}. \end{cases}$$

28. Given $y^4 - 432 = 12xy^2$,
and $y^3 = 12 + 2xy$, } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 2, \\ y = 6. \end{cases}$$

29. Given $\frac{4}{y^2} + \frac{4+y}{y} = \frac{8+4y}{x} + \frac{12y^2}{x^2}$,
and $4y^2 - xy = x$, } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 2, \text{ or } -\frac{50}{3}, \\ y = 1, \text{ or } -\frac{5}{3}. \end{cases}$$

30. Given $\sqrt{\{(1+x)^2 + y^2\}} + \sqrt{\{(1-x)^2 + y^2\}} = 4$,
and $(4-x^2)^2 = 18 - 4y^2$,
to find the values of x and y .

$$\text{Ans. } \begin{cases} x = \pm 1, \text{ or } \pm \sqrt{10}, \\ y = \pm \frac{3}{2}, \text{ or } \pm 3\sqrt{-\frac{1}{2}}. \end{cases}$$

31. Given $\frac{x + \sqrt{x+y}}{x - \sqrt{x+y}} - \frac{\sqrt{x-x-y}}{\sqrt{x+x+y}} = 2\frac{9}{40}$,
and $y^2 - \sqrt{xy^2} = \frac{4x}{9}$, } to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 9, \text{ or } \frac{196}{9}, \text{ or } \frac{289}{9}, \text{ or } 16, \\ y = 4, \text{ or } -\frac{14}{9}, \text{ or } -\frac{68}{9}, \text{ or } \frac{4}{3}. \end{cases}$$

32. Given $\frac{x + \sqrt{(x^2 - y^2)}}{x - \sqrt{(x^2 - y^2)}} = 4\frac{1}{2} - \frac{x - \sqrt{(x^2 - y^2)}}{x + \sqrt{(x^2 - y^2)}}$,
and $x \cdot (x+y) = 52 - \sqrt{(x^2 + xy + 4)}$,
to find the values of x and y .

$$\text{Ans. } \begin{cases} x = \pm 5, \text{ or } \pm \frac{10}{\sqrt{3}}, \\ y = \pm 4, \text{ or } \pm \frac{8}{\sqrt{3}} \end{cases}$$

$$33. \text{ Given } 5y + \frac{\sqrt{(x^2 - 15y - 14)}}{5} = \frac{x^2}{3} - 36, \quad \left. \begin{array}{l} \text{and } \frac{x^2}{8y} + \frac{2x}{3} = \sqrt{\left(\frac{x^2}{3y} + \frac{x^2}{4}\right)} - \frac{y}{2}, \end{array} \right\} \begin{array}{l} \text{to find the} \\ \text{values of} \\ x \text{ and } y. \end{array}$$

ANSWER,

$$x = 12, \text{ or } -\frac{19}{2}; \text{ or } \frac{25 \pm \sqrt{(41569)}}{20}; \text{ or } \frac{45 \pm \sqrt{(3849)}}{4},$$

$$y = 2, \text{ or } -\frac{19}{12}; \text{ or } \frac{25 \pm \sqrt{(41569)}}{120}; \text{ or } \frac{135 \pm 3\sqrt{(3849)}}{8}.$$

$$34. \text{ Given } \sqrt{\left(\frac{x+y^2}{4x}\right)} + \frac{y}{\sqrt{(y^2+x)}} = \frac{y^2}{4} \cdot \sqrt{\left(\frac{4x}{y^2+x}\right)}, \quad \left\{ \begin{array}{l} \text{and } \frac{\sqrt{x} + \sqrt{(x-y-1)}}{\sqrt{x} - \sqrt{(x-y-1)}} = y + 1, \end{array} \right. \\ \text{to find the values of } x \text{ and } y.$$

$$\text{Ans. } \begin{cases} x = 4, \text{ or } \frac{1}{4}; \text{ or } \frac{1 \mp 3\sqrt{(-7)}}{8}, \\ y = 2, \text{ or } -1; \text{ or } \frac{-7 \pm \sqrt{(-7)}}{2}. \end{cases}$$

$$35. \text{ Given } \frac{x+y+\sqrt{(x^2-y^2)}}{x+y-\sqrt{(x^2-y^2)}} = \frac{9}{8y} \cdot (x+y), \quad \left\{ \begin{array}{l} \text{and } (x^2+y)^2 + x - y = 2x \cdot (x^2+y) + 506, \end{array} \right. \\ \text{to find the values of } x \text{ and } y.$$

$$\text{Ans. } \begin{cases} x = 5, \text{ or } -\frac{23}{5}; \text{ or } \frac{1 \pm \sqrt{(-1209)}}{5}, \\ y = 3, \text{ or } -\frac{69}{25}; \text{ or } \frac{3 \pm 3\sqrt{(-1209)}}{25}. \end{cases}$$

36. Given $\frac{y}{x} \cdot \sqrt{\frac{x}{y}} + \frac{1}{2} \cdot \sqrt{\frac{x}{y}} \cdot \sqrt[4]{\frac{y^3}{x^3}} = 5,$ } to find the values
and $\frac{2x^2}{y} - \frac{x}{3\sqrt{y}} = \frac{1}{3},$ } of x and $y.$

Ans. $\begin{cases} x = 4, \text{ or } \frac{16}{9}, \text{ or } \frac{625}{64}, \text{ or } \frac{625}{144}, \\ y = 64, \text{ or } \frac{256}{9}, \text{ or } \left(\frac{625}{32}\right)^2, \text{ or } \left(\frac{-625}{48}\right)^2. \end{cases}$

37. Given $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{61}{\sqrt{xy}} + 1,$ } to find the values of x
and $\sqrt[4]{x^3y} + \sqrt[4]{y^3x} = 78,$ } and $y.$

Ans. $\begin{cases} x = 81, \text{ or } 16; \text{ or } -\frac{371}{12} \pm 3\sqrt{47}; \\ y = 16, \text{ or } 81; \text{ or } -\frac{371}{12} \mp 3\sqrt{47}. \end{cases}$

38. Given $x + \sqrt{(3y^2 - 11 + 2x)} = 7 + 2y - y^2,$ } to find the
and $\sqrt{(3y - x + 7)} = \frac{x+y}{x-y},$ } values of
 x and $y.$

Ans. $\begin{cases} x = 4, \\ y = 2. \end{cases}$

39. Given $x^4 + y^4 = 1 + 2xy + 3x^2y^2,$ } to find the values of
and $x^3 + y^3 = 2y^2x + 2y^2 + x + 1,$ } x and $y.$

Ans. $\begin{cases} x = 2, \text{ or } -1, \\ y = 1. \end{cases}$

40. Given $x^2y - 4 = 4xy - \frac{y^2}{4},$ } to find the values of x and $y.$
 $x^2 - 3 = x^2y^2 \cdot (x^2 - y^2),$ }

Ans. $\begin{cases} x = 1, \\ y = 4. \end{cases}$

$$41. \quad \left. \begin{aligned} \text{Given } 5 - 2\sqrt{y+2} &= \frac{9x^2}{64} - \{\sqrt{x} - 3\sqrt{y}\}^2, \\ \text{and } \frac{7}{y} - 10\sqrt{\frac{x}{y}} &= x - 16, \end{aligned} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 4, \\ y = \frac{1}{4}. \end{cases}$$

$$42. \quad \left. \begin{aligned} \text{Given } \frac{y}{2x} + \frac{2}{3} \cdot \frac{y - \sqrt{(x-1)}}{y^2 - 2\sqrt{(x^2-1)}} &= \frac{\sqrt{(x+1)}}{x}, \\ \text{and } \frac{1}{4} \cdot y^4 &= y^2x - 1, \end{aligned} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = \frac{5}{4}, \text{ or } -\frac{85}{36}, \\ y = \pm 2, \text{ or } \pm \sqrt{\left(-\frac{85}{18} + \frac{\sqrt{5929}}{648}\right)}. \end{cases}$$

$$43. \quad \left. \begin{aligned} \text{Given } (x-2) \cdot y - \sqrt{xy} \cdot (y^2-1) &= 2y^2 - x, \\ \text{and } \frac{1}{4}xy &= \frac{\sqrt{xy} - 12}{xy - 18}, \end{aligned} \right\} \begin{array}{l} \text{to find the} \\ \text{values of} \\ x \text{ and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 8, \text{ or } -12, \\ y = 2, \text{ or } -\frac{4}{3}. \end{cases}$$

$$44. \quad \left. \begin{aligned} \text{Given } 3x - x\sqrt{\left(\frac{5x^2}{4} - 2y + 8\right)} &= 2 - y, \\ \text{and } \frac{\sqrt{(x+y)}}{2x} - \frac{3}{4}x &= \frac{2x-3}{\sqrt{(x+y)}} - \frac{3y}{2x}, \end{aligned} \right\} \begin{array}{l} \text{to find the} \\ \text{values of} \\ x \text{ and } y. \end{array}$$

$$\text{Ans. } \begin{cases} x = 2, \text{ or } \frac{34}{31}, \\ y = 2, \text{ or } \frac{1446}{961}. \end{cases}$$

45. Given $x^2 - y^2 = 3$,

and $(x^4 + y^4)^2 + x^2 y^2 \cdot (x^2 - y^2)^2 + x^2 - y^2 = 328$,

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = \pm 2, \text{ or } \pm \sqrt{-1}, \text{ or } \pm \sqrt{\left(\frac{3 \pm 2\sqrt{-13}}{2}\right)}, \\ y = \pm 1, \text{ or } \pm 2\sqrt{-1}, \text{ or } \pm \sqrt{\left(\frac{-3 \mp \sqrt{-13}}{2}\right)}. \end{cases}$$

46. Given $\frac{x^2 y^2}{2} + 4 - 40y^2 = 140 - y^2 \cdot \sqrt{\left(x^2 - \frac{272}{y^2}\right)},$ }
 and $x^2 - \frac{2}{y} \cdot \left(\frac{3}{y} + 15x\right) = \frac{30}{y^2} + \frac{5x}{y}.$ }

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 9, \text{ or } -\frac{45}{4}, \\ y = 4, \text{ or } -\frac{16}{5}. \end{cases}$$

47. Given $\frac{2y^2 - 8\sqrt{x}}{\sqrt{x}} + \sqrt{(4y^2 - 16\sqrt{x})} = \frac{3\sqrt{x}}{2},$ }
 and $\sqrt{x} + \sqrt{\{8 \cdot (y - \sqrt{x}) - 4\}} = y + 1,$ }

to find the values of x and y .

ANSWER,

$$x = 4, \text{ or } \frac{4}{9}, \text{ or } \frac{4}{25}, \text{ or } -\frac{4}{3} \mp 16\sqrt{-\frac{13}{3}}, \text{ or } \frac{788 \pm 24\sqrt{644}}{25},$$

$$y = 3, \text{ or } \frac{5}{3}, \text{ or } \frac{7}{5}, \text{ or } -1; \text{ or } 1 \pm 2\sqrt{-\frac{13}{3}}, \text{ or } \frac{37 \pm \sqrt{644}}{5}.$$

$$\left. \begin{aligned}
 48. \quad \text{Given } \frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}} \cdot (x^{\frac{1}{2}} - 1) + \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} \cdot (2x^{\frac{1}{2}} - 1) &= \frac{4y^{\frac{1}{2}}}{x^{\frac{1}{2}}} \cdot (y^{\frac{1}{2}} + x^{\frac{1}{2}}) \\
 &+ \frac{3y}{x^{\frac{1}{2}}} + 2, \\
 \text{and } \frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}} - \frac{2x^{\frac{1}{2}}}{y} - \frac{2x^{\frac{1}{2}}}{y^{\frac{1}{2}}} &= \frac{133}{36} \cdot \frac{1}{y^{\frac{1}{2}}} - \frac{2}{x^{\frac{1}{2}}} - \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}},
 \end{aligned} \right\}$$

to find the values of x and y .

$$\text{Ans. } \begin{cases} x = 27, \\ y = 8. \end{cases}$$

VI. Problems producing Simple Equations, involving only one unknown Quantity.

1. WHAT number is that, from the treble of which if 18 be subtracted, the remainder is 6? *Ans.* 8.

2. What number is that, the double of which exceeds four-fifths of its half by 40? *Ans.* 25.

3. In fencing the side of a field, whose length was 450 yards, two workmen were employed; one of whom fenced 9 yards, and the other 6 *per* day. How many days did they work? *Ans.* 30.

4. A Mercer bought 4 pieces of silk, which together measured 50 yards; the second was twice, the third three times, and the fourth four times as long as the first. What were the respective lengths of the pieces?

Ans. 5, 10, 15, 20 yards.

5. A Farmer sold 13 bushels of barley at a certain price; and afterwards 17 bushels at the same rate; and at the second time received 36 shillings more than at the first. What was the price of a bushel?

Ans. 9 shillings.

6. A person bought 198 gallons of beer, which exactly filled 4 casks; the first held twice as much as the second, the second twice as much as the third, and the third three times as much as the fourth. How many gallons did each hold?

Ans. 108, 54, 27, and 9 gallons.

7. A Silversmith has 3 pieces of metal, which together weigh 48 ounces. The second weighs 12 ounces more than the first, and the third 9 ounces more than the second. What are their respective weights?

Ans. 5, 17, and 26 ounces.

8. A Vintner fills a cask, containing 96 gallons, with a mixture of brandy, wine, and water. There are 20 gallons of water more than of brandy, and 17 more of wine than of water. How many are there of each?

Ans. 13 gallons of brandy, 33 of water, and 50 of wine.

9. A Gentleman buys 4 horses; for the second of which he gives £12. more than for the first; for the third £6. more than for the second; and for the fourth £2. more than for the third. The sum paid for all was £230. How much did each cost?

Ans. 45, 57, 63, and 65 pounds.

10. A poor man had 6 children, the eldest of which could earn 7*d.* a week more than the second; the second 8*d.* more than the third; the third 6*d.* more than the fourth; the fourth 4*d.* more than the fifth; and the fifth 5*d.* more than the youngest. They altogether earned 10*s.* 10*d.* a week. How much could each earn a week?

Ans. 38, 31, 23, 17, 13, and 8 pence *per* week.

11. An express set out to travel 240 miles in 4 days, but in consequence of the badness of the roads, he found

that he must go 5 miles the second day, 9 the third, and 14 the fourth day, less than the first. How many miles must he travel each day?

Ans. 67, 62, 58, and 53 miles.

12. There are 5 towns, in the order of the letters, *A, B, C, D, E*. From *A* to *E* is 80 miles. The distance between *B* and *C* is 10 miles more, between *C* and *D* is 15 miles less, and between *D* and *E* 17 miles more than the distance between *A* and *B*. What are the respective distances?

Ans. From *A* to *B* 17; from *B* to *C* 27; from *C* to *D* 2; and from *D* to *E* 34 miles.

13. A gentleman gave 27 shillings to two poor persons; but he gave 5 shillings more to one than to the other. What did he give to each?

Ans. 11, and 16 shillings.

14. What number is that, the treble of which is as much above 40, as its half is below 51?

Ans. 26.

15. Two workmen received the same sum for their labour; but if one had received 15 shillings more, and the other 9 shillings less, then one would have had just three times as much as the other. What did they receive?

Ans. 21 shillings each.

16. Two merchants entered into a speculation, by which one gained £54. more than the other. The whole gain was £49. less than three times the gain of the less. What were the gains?

Ans. £103, and £157.

17. The perimeter of a triangle is 75 feet, and the base is 11 feet longer than one of the sides, and 16 feet longer than the other. Required their respective lengths.

Ans. 34, 23, and 18 feet.

18. A company settling their reckoning at a tavern, pay 8 shillings each ; but observe, that if there had been 4 more, they should only have paid 7 shillings each. How many were there?

Ans. 28.

19. Divide the number 46 into two such parts, that one of them being divided by 7 and the other by 3, the quotients may together be equal to 10.

Ans. 28 and 18.

20. A certain sum is to be raised upon two estates, one of which pays 19 shillings less than the other ; and if 5 shillings be added to treble the less payment, it will be equal to twice the greater. What are the sums paid ?

Ans. 33, and 52 shillings.

21. Having bought a certain quantity of brandy at 19 shillings a gallon, and a quantity of rum exceeding that of the brandy by 9 gallons, at 15 shillings a gallon, I find that I paid one shilling more for the brandy than for the rum. How many gallons were there of each ?

Ans. 34 of brandy, and 43 of rum.

22. Two persons, *A* and *B*, have each an annual income of £400. *A* spends every year £40. more than *B*, and at the end of 4 years, the amount of their savings is equal to one year's income of either. What does each spend annually ?

Ans. £370, and £330, respectively.

23. A Draper sold two pieces of cloth, by one of which he lost £6. more than by the other ; and his whole loss was £5. less than treble the less loss. What were the losses sustained by each piece ?

Ans. £11, and £17.

24. A person engaged to reap a field of corn for 5 shillings an acre, but leaving 6 acres not reaped, he received £2. 10s. Of how many acres did the field consist?

Ans. 16.

25. In a naval engagement, the number of ships taken was 7 more, and the number burnt 2 fewer, than the number sunk. Fifteen escaped, and the fleet consisted of 8 times the number sunk. Of how many did the fleet consist?

Ans. 32.

26. A Farmer hires a farm for £175. 15s. a year; part of which he is not allowed to plough: he gives £2. *per* acre for the arable, and for the rest, which was 5 acres less, he gives £1. 5s. *per* acre. How many acres were arable, and how many not?

Ans. 56 acres arable, 51 not.

27. A Cistern is filled in twenty minutes by three pipes, one of which conveys 10 gallons more, and the other 5 gallons less, than the third, *per* minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

Ans. 22, 7, and 12 gallons.

28. A Fortress is garrisoned by 2600 men; and there are 9 times as many infantry, and three times as many artillery as cavalry. How many are there of each?

Ans. 200 cavalry, 600 artillery, and 1800 infantry.

29. *A* and *B* began to play; *A* with exactly $\frac{4}{9}$ of the sum which *B* had. After winning £10, he found that they had each the same sum. What had each at first?

Ans. *A* had £16, and *B* £36.

30. A person has a certain number of horses at livery stables, and three times as many at grass. He keeps 15 in

constant employment; and his whole number is seven times the number in the stables. Required the whole number.

Ans. 35.

31. Two men at the distance of 150 miles set out to meet each other, one goes three miles in the time the other goes 7. What part of the distance does each travel?

Ans. One 45, and the other 105 miles.

32. A Farm of 864 acres is divided between 3 persons. *C* has as many acres as *A* and *B* together; and the portions of *A* and *B* are in the proportion of 5 : 11. How many acres has each?

Ans. *A* has 135, *B* 297, *C* 432.

33. A charitable person distributed £5. 14s. amongst some poor women and children, giving to each woman 6 shillings, and to each child two; and the number of women was to the number of children as 4 : 7. How many were relieved?

Ans. 12 women, and 21 children.

34. *A* and *B* begin trade, *A* with triple the stock of *B*. They each gain £50, which makes their stocks in the proportion of 7 to 3. What were their original stocks?

Ans. *A*'s was £300, and *B*'s £100.

35. There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, which being increased respectively by 6 and 5, are in the proportion of $\frac{2}{5}$ to $\frac{1}{2}$. Required the numbers.

Ans. 30 and 40.

36. A Farmer has a stack of hay, from which he sells a quantity which is to the quantity remaining in the proportion of 4 to 5. He then uses 15 loads, and finds that

he has a quantity left which is to the quantity sold as 1 to 2. How many loads did the stack at first contain?

Ans. 45.

37. There are three pieces of cloth, whose lengths are in the proportion of 3, 5, and 7; and 6 yards being cut off from each, the whole quantity is diminished in the proportion of 20 to 17. Required the length of each piece at first.

Ans. 24, 40, and 56 yards.

38. The number of days that 4 workmen were employed were severally as the numbers 4, 5, 6, 7; their daily wages were the same, viz. 3 shillings, and the sum received by the first and second was 36 shillings less than that received by the third and fourth. How much did each receive?

Ans. 36, 45, 54, and 63 shillings.

39. From two casks of equal size are drawn quantities, which are in the proportion of 6 to 7; and it appears that if 16 gallons less had been drawn from that which is now the emptier, only half as much would have been drawn from it as from the other. How many gallons were drawn from each?

Ans. 24 and 28.

40. On the enclosure of a parish, a proprietor had for his allotment two pieces of land, which were of the form of rectangular parallelograms. The longer sides of the parallelograms were in the ratio of 6 to 11, and the adjacent sides of the less as 3 to 2. The periphery of the less was 135 yards more than the longer side of the greater. Required the sides of the less, and the longer side of the greater.

Ans. The sides of the less were 90 and 60; and the longer side of the greater was 165 yards.

41. Two persons *A* and *B* travelling, each with £80, meet with robbers, who take from *A* twice as much as from *B* and £5 over, and leave *A* within £13 half as much as *B*. How much is taken from each?

Ans. 69, and 32 pounds.

42. A person distributes forty shillings amongst 50 people; giving to some nine-pence each, and to the rest fifteen pence. How many were there of each?

Ans. 45, and 5.

43. A person put out a certain sum to interest for $6\frac{1}{4}$ years, at 5 *per cent.* simple interest; and found that if he had put out the same sum for 12 years and 9 months at 4 *per cent.* he would have received £185 more. What was the sum put out?

Ans. £1000.

44. A regiment of militia containing 594 men, is to be raised from three towns, *A*, *B*, *C*. The contingents of *A* and *B* are in the proportion of three to five; and of *B* and *C* in the proportion of eight to seven. Required the numbers raised by each.

Ans. 144 by *A*, 240 by *B*, 210 by *C*.

45. Two persons, *A* and *B*, were partners. *A*'s money remained in the firm 6 years, and his gain was one-fourth of his principal; and *B*'s money, which was £50 less than *A*'s, had been in the firm 9 years, when they dissolved partnership, and it appeared that if *B* had gained £6. 5*s.* less, his gain and principal would have been to *A*'s gain and principal as 4 to 5. What was the principal of each?

Ans. £200, and £150.

46. The estate of a Bankrupt, valued at £21000, is to be divided amongst four creditors proportionably to what

is due to them. The debts due to *A* and *B* are as 2 : 3; *B*'s claims and *C*'s are in the proportion of 4 : 5; and *C*'s and *D*'s in the proportion of 6 : 7. What sum must each receive?

Ans. *A* £3200, *B* £4800, *C* £6000, *D* £7000.

47. A Merchant bought wheat at the rate of £3. 10s. for 5 bushels. He afterwards bought some inferior, which was in quantity to the former as 3 to 4, at the rate of £3. 12s. for 8 bushels; and sold the whole for 10 shillings a bushel; in consequence of which, he lost £7. 16s. by the bargain. How much of each did he buy?

Ans. 48 bushels of the better, and 36 of the worse.

48. Three persons, *A*, *B*, and *C*, spent equal sums at a tavern. *C* having no money, the reckoning was paid by *A* and *B*. When *C* came to reimburse them, he paid 4 times as much to *A* as to *B*; and observed, that if *B* had paid 3 shillings more of his reckoning, their demands would have been equal. Required the sum each spent, and the respective parts of *C*'s reckoning that *A* and *B* paid.

Ans. Each spent 10 shillings; *A* paid 8, and *B* 2.

49. A certain sum is divided among three persons: *A* receives £3000 more than the half, *B* £1000 less than the third part, and *C* £800 more than the fourth part of the whole. What is the sum divided, and what does each receive?

Ans. The whole is £38400; *A* receives £16200, *B* £11800, *C* £10400.

50. A Farmer had two flocks of sheep, one of which contained 40, and the other was sold for £30; but one sheep of the latter was worth four of the other; and the value of the first flock was only £4 more than the price of eight sheep of the second. How many sheep did the

second flock contain, and what was the value of a sheep of each?

Ans. The number was 15, and the prices £2, and 10 shillings.

51. *A* and *B* playing at billiards, *A* bet 5 shillings to 4 on every game, and found that after a certain number of games he had won 10 shillings. Had *B* won one game more, the number won by him would have been to the number won by *A* as 3 to 4. How many did each win?

Ans. *A* won 20, and *B* 14.

52. A besieged garrison had such a quantity of bread, as would, if distributed to each at 10 ounces a day, last 6 weeks; but having lost 1200 men in a sally, the governor was enabled to increase the allowance to 12 ounces *per* day for 8 weeks. Required the number of men at first in the garrison.

Ans. 3200.

53. A composition of copper and tin containing 100 cubic inches weighed 505 ounces. How many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces, and a cubic inch of tin to weigh $4\frac{1}{4}$ ounces?

Ans. 420 of copper, and 85 of tin.

54. There are two towns, *A* and *B*, which are 131 miles distant from each other. A coach sets out from *A* at 6 o'clock in the morning, and travels at the rate of 4 miles an hour without intermission, in the direct road towards *B*. At 2 o'clock in the afternoon of the same day a coach sets out from *B* to go to *A*, and goes at the rate of 5 miles an hour constantly. Where will they meet?

Ans. 76 miles from *A*, and 55 from *B*.

55. Out of a certain sum, a man paid his creditors £96; half of the remainder he lent his friend; he then

spent one-fifth of what now remained; and after all these deductions had one-tenth of his money left. How much had he at first?

Ans. £128.

56. At the review of an army, the troops were drawn up in a solid mass, 40 deep, when there were just one-fourth as many men in front as there were spectators. Had the depth however been increased by 5, and the spectators drawn up in the mass with the army, the number of men in front would have been 100 fewer than before. Of what number of men did the army consist?

Ans. 180000.

57. *A* and *B*, in order to keep up the price of copper to £86 *per* ton, agree for a certain time to sell all the copper they raise, jointly; yet so that each shall be paid proportionably to the quantity he raises. Now the whole quantity raised in the stipulated time was 235 tons; and *A* receives £4214 more than *B*. Required the quantity raised by each.

Ans. 142 tons by *A*, and 93 by *B*.

58. Bought two pieces of linen, one of which wanted 12 yards of being four times as long as the other. The longer cost 5 shillings, and the shorter 4 shillings a yard; 23 yards being cut off from the longer, and 5 from the shorter, and the remainders being sold for one shilling a yard more than they cost, I received £7. 2s. How many yards of each were there?

Ans. 40, and 13.

59. On the first of January 1799, a certain beggar received from *A* as many groats as *A* was years old, who repeated a similar donation every January during the 7 following years, during the last of which *A* died, his alms to the poor

man having in all amounted to £7. 18s. 8d. Determine in what year he was born, and his age at his death.

Ans. He was born in 1743, and had completed 63 years at his death.

60. A Courier, passing through a certain place *A*, travels at the rate of 13 miles in 2 hours; 12 hours afterwards another passes through the same place, travelling the same road, at the rate of 26 miles in 3 hours. How long, and how far must he travel before he overtakes the first?

Ans. 36 hours, and 312 miles.

61. During a panic, there was a run on two Bankers *A* and *B*. *B* stopped payment at the end of three days, in consequence of which the alarm increased, and the daily demand for cash on *A* being trebled, *A* failed at the end of two more days. But if *A* and *B* had joined their capitals, they might both have stood the run, as it was at first, for seven days, at the end of which time *B* would have been indebted to *A* £4000. What was the daily demand for cash on *A*'s Bank at first?

Ans. £2000.

62. As *A* and *B* were going to school, *A* first shot an arrow in the direction in which they were going, which *B* took up and shot forward; and so on alternately till the arrow had passed exactly from one mile-stone to another; when it appeared that *A* had shot the arrow 8 times, and *B* 7 times. Some time afterwards, *A* and *B* were on the opposite banks of a river, the breadth of which they wished to ascertain; *A* first shot the arrow across the river, and it flew 13 yards beyond the bank on which *B* stood; *B* then took it up, and from the place where it had fallen, shot it back across the river; it now fell 97 yards beyond the bank upon which *A* stood. Required the breadth of the river.

Ans. 100 yards.

63. Three Merchants, *A*, *B*, and *C*, enter into a speculation; *B* subscribes £10 more than four-fifths of what *A* does; and *C* £30 more than half of what *B* does. *A*'s gain is two-fifths of his subscription, and *B*'s is £148. What are the respective sums subscribed, and whole gain?

Ans. The sums subscribed are 450, 370, and 215 pounds; and the whole gain is £414.

64. A Tenant agreed to pay his Landlord two-thirds of the profit of a farm after deducting the expense of cultivation, and upon this agreement found that his own share was one-sixth of the whole produce. Afterwards the expense of cultivation having fallen in the ratio of 3 : 2, and the value of the produce in the ratio of 5 : 3, he found that adhering to his agreement he must pay his Landlord £400. Determine the original value of the produce.

Ans. £2250.

65. A Gentleman left legacies to his four servants, *A*, *B*, *C*, *D*, proportional to the time each had been in his service, but in case any of them should die before the expiration of the year, their share to be divided equally among the others. Accordingly *B* died; and his share so divided made the share of *C* a mean proportional between those of *A* and *D*; whereas before *A* was to have had £78, *C* £30, and *D* £6. What did each receive?

Ans. *A* received £96, *C* £48, and *D* £24.

66. The Colonel of a regiment at Derby sent a detachment to Leeds to protect the Machinery. The detachment was a third of the regiment; but the commanding officer at Leeds thinking his numbers insufficient, and suspecting that 50 of his men had been corrupted by the Luddites, ordered the suspected men back to Derby, with a request that the Colonel would let him have half the regiment. To comply with his wishes, the Colonel had every

third man drafted off and sent to Leeds. Of how many men did the regiment consist?

Ans. 900.

67. There are two places, 154 miles distant, from which two persons set out at the same time to meet, one travelling at the rate of 3 miles in two hours, and the other at the rate of 5 miles in four hours. How long, and how far did each travel before they met?

Ans. 56 hours; and 84, and 70 miles.

68. *A* sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and eight hours afterwards *B* sets out from the same place, and travels the same road at the rate of 5 miles in three hours. How long, and how far must *A* travel before he is overtaken by *B*?

Ans. 50 hours, and 70 miles.

69. A Farmer's rent was £50 a year, and his annual expenditure (including the assessed taxes, which amounted to one-sixth of his expenses) was such, that he was able to pay his landlord only £30. The year following his rent was lowered 20 *per cent.*; the taxes also were reduced one half, and agricultural produce increased in value one-third; in consequence he was enabled to pay his rent and former debt, and to lay by £5. What was his expenditure and the value of his produce each year?

Ans. His expenditure was £60 the first year and £55 the second. The value of his produce £90, and £120, respectively.

70. A man lent out a certain sum to interest at £s *per cent. per annum.* He suffered this to accumulate at simple interest for 12 years; and then putting out the principal and interest at the same rate, found that the present

annual interest exceeded the former by £38. 8s. Required the sum put out each time.

Ans. £500 the first time, and £980 the second.

71. A Waterman finds by experience that he can with the advantage of a common tide row down a river from *A* to *B*, which is 18 miles, in an hour and a half, and that to return from *B* to *A* against an equal tide, though he rows back along the shore, where the stream is only three-fifths as strong as in the middle, takes him just two hours and a quarter. It is required from hence to find at what rate *per* hour the tide runs in the middle, where it is strongest.

Ans. At the rate of two miles and a half *per* hour.

72. The ingredients of a loaf of bread are rice, flour, and water, and the weight of the whole is 15lbs. The weight of the rice augmented by 5lbs. is two-thirds of the weight of the flour, and the weight of the water is one-fifth of the weight of the flour and rice together. Required the weight of each.

Ans. Rice 2lbs., flour $10\frac{1}{2}$ lbs., water $2\frac{1}{2}$ lbs.

73. In a battery two cannon were employed, the first of which had been fired 36 times before the second began to play; and afterwards was fired eight times whilst the second was fired seven. But the quantity of powder used for each shot of the first was less than what was used for the second in the proportion of 3 : 4. How many times was the second fired, before it had consumed as much powder as the first?

Ans. 189 times.

74. Suppose two fingers of a watch (*a*) and (*b*) were together on Sunday noon at 12 o'clock, and that the motion of each was such that (*a*) moved round the horary circle in one hour, and (*b*) in $1\frac{1}{80}$ hour. When will they be together again for the first time?

Ans. 61 hours.

75. A Draper bought a piece of cloth for £29, from which he cut off 11 yards. He then met with another piece of equal goodness, for which he gave £21, and found that if it had been one yard longer, its length would have been to the length of the remainder of the first as 2 to 3. How many yards were there in each piece, and what was the price of a yard?

Ans. 23 in the first, and 7 yards in the second; and the price £3 *per* yard.

76. A Fruiterer sells for 19s. 6d. a certain number of oranges and apples, of which the latter exceeded the former by 180. He sells the apples at the rate of five for 3d., and fifteen oranges bring him in $1\frac{1}{2}$ d. more than 35 apples. How many are there of each sort, and what are the oranges worth apiece?

Ans. 240 apples, and 60 oranges, which are worth $1\frac{1}{2}$ d. each.

77. Divide the number 198 into five such parts, that the first increased by one, the second increased by two, the third diminished by three, the fourth multiplied by four, and the fifth divided by 5, may be all equal.

Ans. 23, 22, 27, 6, and 120, are the numbers.

78. A person has four casks, the second of which being filled from the first, leaves the first four-sevenths full. The third being filled from the second leaves it one-fourth full; and when the third is emptied into the fourth, it is found to fill only nine-sixteenths of it. But the first will fill the third and fourth, and have fifteen quarts remaining. How many quarts does each hold?

Ans. 140, 60, 45 and 80 respectively.

79. A packet sailing from Dover with a fair wind, arrives at Calais in two hours; and on its return the wind being con-

trary, it proceeds six miles an hour slower than it went. Now when it is half way over, the wind changing, it sails two miles an hour faster, and reaches Dover sooner than it would have done had the wind not changed, in the proportion of 6 : 7. Required the rates of sailing and the distance between Dover and Calais.

Ans. The distance is 22 miles, and in returning it sails 5 and 7 miles an hour.

80. Six hundred persons voted upon a disputed question, which was lost by a certain number. The same number of persons having voted again upon the same question, it was from some change in circumstances carried by twice as many as it was before lost by; and the new majority was to the former one as 8 : 7. How many changed their minds?

Ans. 150.

81. The Gas Contractors engage to light a shop with 5 large and 3 small burners; but having by them only one large burner, supply the deficiency with five small ones. The shop-keeper not finding this light sufficient, procures 2 more small burners, and at the same time agrees for the light to burn double the usual time on Saturday nights, for which additional gas he was required to pay £1. 11s. How much did he pay altogether?

Ans. 5 guineas.

82. *A* and *B* set out from two places, *C* and *D*, at the same time, towards *E*; the road from *C* to *E* being through *D*. *A* travels 7 miles an hour, and at that rate of travelling would have overtaken *B* 5 miles before he got to *E*; but after arriving at *D*, he travels $6\frac{2}{3}$ miles an hour, in consequence of which he overtakes *B* just as he enters *E*. Supposing *B* to travel 5 miles an hour, what are the distances between *C*, *D*, and *E*?

Ans. From *C* to *D* 14 miles, and from *D* to *E* 40.

83. A Gentleman wishing his two daughters to receive equal portions when they became of age, bequeathed to the elder the accumulated interest of a certain sum of money, bought at the time of his death into the 4 *per* cent. stock at 88; and to the younger the accumulated interest of a sum less than the former by £3500, bought at the same time into the 3 *per* cents. at 63. Supposing their ages at the time of their father's death to have been 17 and 14, what would be the sum bought into the stocks in each case, and what would be the fortune of each?

Ans. The sums would be £7700, and £4200; and fortune £1400.

84. Two persons *A* and *B* start at the same time for a race which lasted six minutes. Now after galloping four minutes at the same uniform pace at which each started, the distance between them is $\frac{1}{10}$ th part of the whole length of the course. They continue to run for one minute more at the same speed as at first; and then *B*, who is last, quickens the speed of his horse 20 yards a minute, and comes in exactly two yards before *A*, whose horse had run at the same uniform pace throughout. What was the length of the course?

Ans. 3 miles.

85. Out of a common pack of cards, a certain number, including the ten of diamonds, was dealt equally amongst four persons, the dealer turning up the last card, which was the ten of spades, which he gave himself. Now if twice the number of cards had been dealt to each, the ten of spades being turned up by the dealer, and the ten of diamonds being still dealt out, the chance of the dealer's having the ten of diamonds would be to the chance against him as 3 : 10. Required the number of cards dealt to each the second time.

Ans. 10.

86. Two companies of soldiers, consisting of equal numbers, were sent out under *A* and *B*, from two hostile camps, to

reconnoitre. Falling in with each other, a skirmish ensued, in which *A* lost 50 killed and prisoners, and *B* had 20 killed. *A* however having been reinforced by a party equal to five-sevenths of the number which *B* had remaining, and *B* having been reinforced by a number greater by 46 than three-fifths of the number which *A* had remaining, they renewed the engagement, when *A* was forced to retire with the additional loss of 30 men. When the returns were made, *B* found he had again lost 20 men, but that he had then twice as many men remaining as *A* had. How many had each at first?

Ans. 90.

87. A Sportsman, who kept an account of the number of birds which he killed, found that each succeeding season he wanted 50, in order that the number killed might bear the proportion of 3 : 2 to the number killed in the preceding year. In the fourth year he found that he had killed 170 fewer than three times the number killed in the first year. How many did he kill the first year?

Ans. 180.

88. Several detachments of artillery divided a certain number of cannon balls. The first took 72, and one-ninth of the remainder; the next 144, and one-ninth of the remainder; the third 216, and one-ninth of the remainder; the fourth 288, and one-ninth of those that were left; and so on; when it was found that the balls had been equally divided. Determine the number of detachments and balls.

Ans. 4608 balls, and 8 detachments.

89. *A* entered into a Canal speculation with 14 others, and the profits of this concern amounted in all to £595 more than five times the price of an original share. Seven of his former partners in this affair joined with him in a scheme for navigating the said Canals with steam boats, each venturing

a sum of money less than his former gains by £173. But the steam boats unexpectedly blowing up, *A* found he had lost £419 by them, for the company not only never recovered the money advanced, but had lost all they had gained by digging the Canals and £368 besides. What were the prices of shares in the two concerns originally?

Ans. £700 in the first speculation, and £100 in the second.

90. A Merchant wishing to buy a certain quantity of pimento, the price of which he calculates at the rate of 5 bags for £8, transmits to his foreign agent the requisite sum of money. Before the order arrives, pimento has risen in value; and the money is sufficient only to buy a quantity less by 18 bags than that which the Merchant intended. It appears also that as many bags as exceed one-third of the intended quantity by $5\frac{1}{2}$, will now cost £10. 7s. more than they would have done, had the price not varied. What is the quantity purchased?

Ans. 432 bags.

91. Four men walking abroad found a purse containing shillings only, out of which every one of them took a number at a venture. Afterwards comparing their numbers together, they found that if the first took 25 shillings from the second, it would make his number equal to what the second had left. If the second took 30 shillings from the third, his money would then be triple what the third had left. And if the third took 40 shillings from the fourth, his money would then be double of what the fourth had left. Lastly, the fourth taking 50 shillings from the first, he would then have three times as much as the first had left, and five shillings over. What had each?

Ans. 100, 150, 90, and 105 shillings, respectively.

92. Fifteen current guineas should weigh 4 ounces; but a parcel of light gold being weighed and counted, was found to

contain 9 more guineas, than was supposed from the weight; and a part of the whole, exceeding the half by 10 guineas and a half, was found to be $1\frac{1}{2}$ oz. deficient in weight. What was the number of guineas?

Ans. 189.

93. A Merchant bought a quantity of wheat for £200, half of which he reserved for his private use. He then sold 5 bushels more than $\frac{1}{4}$ of the remaining quantity at such a price as to gain £40 *per cent*. But the price of wheat having advanced, he sold the remainder at such a price as to gain £67 *per cent*. by what he sold. And had the whole been sold at this latter price he would have gained £160 *per cent*. How much did he buy, and how did he sell it?

Ans. He bought 400 bushels; and sold the first portion at 14*s.*, and the second at 26*s.* *per bushel*.

94. A Brewer, from a certain quantity of ingredients which cost £20, brews 500 gallons of ale (on which there is a duty of 6*d.* a gallon), and sells it at 2*s.* a gallon. Afterwards from the same quantity of ingredients, he brews a certain number of gallons of strong beer (on which he pays the ale duty) and the remainder small beer, making together the same number of gallons as before;—when by mixing them together, and selling the mixture as ale, he finds his gains increased in the proportion of 10 : 7. Determine the number of gallons of strong beer, supposing the duty on small beer one-fourth of that on ale.

Ans. 100 gallons.

VII. *Problems producing Simple Equations, involving two unknown Quantities.*

1. A DRAPER bought two pieces of cloth for £12. 13*s.*; one being 8*s.*, and the other 9*s.* *per yard*. He sold them each

at an advanced price of *2s. per yard*, and gained by the whole £3. What were the lengths of the pieces?

Ans. 17 yards the first, and 13 the second.

2. A bill of £26. 5*s.* was paid with half guineas and crowns, and twice the number of half guineas exceeded three times the number of crowns by 17. How many were there of each?

Ans. 40 half guineas, and 21 crowns.

3. Two labourers, *A* and *B*, received £5. 17*s.* for their wages; *A* having been employed 15, and *B* 14 days; and *A* received for working four days 11*s.* more than *B* did for three days. What were their daily wages?

Ans. *A* had 5*s.*, and *B* 3*s.* a day.

4. A person had two casks, the larger of which he filled with ale, and the smaller with cyder. Ale being half a crown, and cyder 11*s. per gallon*, he paid £8. 6*s.*; but had he filled the larger with cyder, and the smaller with ale, he would have paid £11. 5*s.* 6*d.* How many gallons did each hold?

Ans. The larger contained 18, and the smaller 11 gallons.

5. A person expends half a crown in apples and pears, buying his apples at 4, and his pears at 5 a penny; and afterwards accommodates his neighbour with half his apples and one-third of his pears for 13 pence. How many did he buy of each?

Ans. 72 apples, and 60 pears.

6. Two persons, *A* and *B*, played cards, each with a different sum. After a certain number of games, *A* had won half as much as he had at first, and found that if he had 15*s.* more,

he would have had just three times as much as *B*. But *B* afterwards won 10*s.* back, and he had then twice as much as *A*. What had each at first?

Ans. *A* had 14, and *B* 19 shillings.

7. A certain sum of money put out to interest, amounts in 8 months to £297. 12*s.*; and in 15 months its amount is £306 at simple interest. What is the sum, and the rate *per* cent.?

Ans. £288, at 5 *per* cent.

8. A Farmer being asked how many quarters of wheat he had sold in the market, answered, if he had sold 8 quarters more, and got 7*s.* *per* quarter more than he did, he should have received £11. 15*s.* more than he had: but if he had sold 7 quarters more at 8*s.* *per* quarter more, he should have had £11. 17*s.* more. How many quarters did he sell, and what was the price?

Ans. 13 quarters, at 11*s.* *per* quarter.

9. There is a number consisting of two digits, the second of which is greater than the first; and if the number be divided by the sum of its digits, the quotient is 4; but if the digits be inverted, and that number divided by a number greater by 2 than the difference of the digits, the quotient becomes 14. Required the number.

Ans. 48.

10. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

Ans. $\frac{4}{5}$.

11. A Farmer parting with his stock, sells to one person 9 horses and 7 cows for £300; and to another, at the same prices, 6 horses and 13 cows for the same sum. What was the price of each?

Ans. The price of a cow was £12, and of a horse £24.

12. A Farmer hires a farm for £245 *per ann.*, the arable land being valued at £2 an acre, and the pasture at 28 shillings; now the number of acres of arable is to half the excess of the arable above the pasture as 28 : 9. How many acres were there of each?

Ans. 98 acres of arable, and 35 of pasture.

13. A person owes a certain sum to two creditors. At one time he pays them £53, giving to one four-elevenths of the sum which is due, and to the other £3 more than one-sixth of his debt to him. At a second time he pays them £42, giving to the first three-sevenths of what remains due to him, and to the other one-third of what is due to him. What were the debts?

Ans. £121, and £36.

14. *A* and *B* playing at backgammon, *A* bet 3s. to 2s. on every game, and after a certain number of games found that he had lost 17 shillings. Now had *A* won 3 more from *B*, the number he would then have won, would have been to the number *B* would have won as 5 to 4. How many games did they play?

Ans. 9.

15. A Vintner has 2 casks of wine, from the greater of which he draws 15 gallons, and from the less 11; and finds the quantities remaining in the proportion of 8 to 3. After they become half empty, he puts 10 gallons of water into

each, and finds that the quantities of liquor now in them are as 9 to 5. How many gallons will each hold?

Ans. The larger 79, and the smaller 35 gallons.

16. A person having laid out a rectangular bowling-green, observed that if each side had been 4 yards longer, the adjacent sides would have been in the ratio of 5 to 4; but if each had been 4 yards shorter, the ratio would have been 4 to 3. What are the lengths of the sides?

Ans. 36, and 28 yards.

17. At an election for two members of parliament, three men offer themselves as candidates, and all the electors give single votes. The numbers of voters for the two successful ones are in the ratio of 9 to 8; and if the first had had 7 more, his majority over the second would have been to the majority of the second over the third as 12 : 7. Now if the first and third had formed a coalition, and had one more voter, they would each have succeeded by a majority of 7. How many voted for each?

Ans. 369, 328, and 300, respectively.

18. Determine three numbers such that if 6 be added to the first and second, the sums will be in the proportion of 2 : 3; if 5 be added to the first and third, the sums will be in the proportion of 7 : 11; but if 36 be subtracted from the second and third, the remainders will be as 6 : 7.

Ans. 30, 48, 50.

19. Two shepherds, *A* and *B*, are intrusted with the charge of two flocks of sheep. *A*'s consisting chiefly of ewes, many of which produced lambs, is at the end of the year increased by 30; but *B* finds his stock diminished by 20; when their numbers are in the proportion of 8 to 3. Now had *A* lost 20 of his sheep, and *B* had an increase of 90, the

numbers would have been in the proportion of 7 to 10. What were the numbers?

Ans. *A*'s 160, and *B*'s 110.

20. Two persons, *A* and *B*, can perform a piece of work in 16 days. They work together for 4 days, when *A* being called off, *B* is left to finish it, which he does in 36 days more. In what time would each do it separately?

Ans. *A* in 24 days, and *B* in 48 days.

21. There is a cistern, into which water is admitted by three cocks, two of which are exactly of the same dimensions. When they are all open, five-twelfths of the cistern is filled in four hours; and if one of the equal cocks be stopped, seven-ninths of the cistern is filled in ten hours and forty minutes. In how many hours would each cock fill the cistern?

Ans. Each of the equal ones in 32 hours, and the other in 24.

22. Some hours after a courier had been sent from *A* to *B*, which are 147 miles distant, a second was sent, who wished to overtake him just as he entered *B*; in order to which he found he must perform the journey in 28 hours less than the first did. Now the time in which the first travels 17 miles added to the time in which the second travels 56 miles is 13 hours and 40 minutes. How many miles does each go per hour?

Ans. The first goes 3, and the second 7 miles an hour.

23. Two loaded waggons were weighed, and their weights were found to be in the ratio of 4 to 5. Parts of their loads, which were in the proportion of 6 to 7 being taken out, their weights were then found to be in the ratio of 2 to 3; and the sum of their weights was then 10 tons. What were the weights at first?

Ans. 16, and 20 tons.

24. A Gentleman gave away a certain sum in charity to 14 men and 15 women. Had the sum been less by 12 shillings, and only half the number of men relieved, the rest being divided amongst the women, each woman would have received two shillings more than each man did. But if there had been only 8 women, and the rest had been divided amongst the men, each man would have received twice as much as each woman. How much money was given away?

Ans. 24 guineas.

25. When wheat was 5 shillings a bushel, and rye 3 shillings, a man wanted to fill his sack with a mixture of rye and wheat for the money he had in his purse. If he bought 7 bushels of rye, and laid out the rest of his money in wheat, he would want 2 bushels to fill his sack; but if he bought 6 bushels of wheat, and filled his sack with rye, he would have 6 shillings left. How must he lay out his money, and fill his sack?

Ans. He must buy 9 bushels of wheat, and 12 bushels of rye.

26. A Stage Coach carries six inside, the fare outside is 13s. and one-third of the sum of the outside fares exceeds one-fifth of those inside by £1. 1s. 8½d. An opposition arising, the coachman loses three outside and two inside passengers, and also reduces the inside fare by 5s. and halves the outside; and then the whole loss is £7. 0s. 6d. Find the number of outside places, and the fare inside.

Ans. 10 outside places, and 18s. fare inside.

27. A Draper bought two pieces of cloth of different kinds for £37. 4s.: there were 6 yards of the coarser more than there were of the finer; and had the coarser cost 2 shillings a yard more than it did, 6 yards of the coarser would have cost just as much as 5 yards of the finer. He afterwards bought 4 yards of the finer, and 12 of the coarser at the same

prices *per* yard, and found their value less than that of the former pieces in the ratio of 20 : 31. How many yards did he buy the first time, and what did he give *per* yard for each ?

Ans. 9 yards of the finer, and 15 of the coarser ; and the prices were 36, and 28 shillings *per* yard.

28. A Mercer bought two pieces of silk of different lengths for £50 ; the price of two yards of the shorter was 6*s.* 8*d.* more than the price of 3 yards of the longer ; and each piece cost the same sum. He cut off two yards from each, and sold the rest for £53. 12*s.* Now if he had sold the whole at that rate, he would have gained £5 by each piece. How many yards did each piece contain ?

Ans. 25, and 15 yards.

29. *A* sets out express from *C* towards *D*, and three hours afterwards *B* sets out from *D* towards *C*, travelling 2 miles an hour more than *A*. When they meet, it appears that the distances they have travelled are in the proportion of 13 to 15 ; but had *A* travelled five hours less, and *B* gone 2 miles an hour more, they would have been in the proportion of 2 : 5. How many miles did each go *per* hour, and how many hours did they travel before they met ?

Ans. *A* went 4, and *B* 6 miles an hour, and they travelled 10 hours after *B* set out.

30. The revenue of a state was increased to provide for a war in the ratio of $2\frac{1}{2} : 1$; and after deducting the expence of collecting, and the interest of the National Debt, the available income was augmented in the ratio of $3\frac{1}{3} : 1$. Now it was found upon calculation, that had circumstances on the contrary permitted the revenue to be reduced in the ratio of $1\frac{1}{2} : 1$, the sum remaining after the specified deductions would have been diminished in the ratio of $7\frac{1}{3} : 1$, and would in fact have only amounted to four millions. Required the

amount of the revenue, and the interest of the debt; on supposition that the expence of collecting varies as the square root of the amount collected.

Ans. The revenue before the increase was 64 millions, and the interest of the National Debt 28 millions.

31. *A* and *B* engaged to reap a field of corn in 12 days. The times in which they could severally reap an acre are as 2 : 3. After some time, finding themselves unable to finish it in the stipulated time, they called in *C* to help them; whose rate of working was such, that if he had wrought with them from the beginning, it would have been finished in 9 days. Also the times in which he could severally have reaped the field with *A* alone, and with *B* alone, are in the proportion of 7 to 8. When was *C* called in?

Ans. After 6 days.

32. Two mixtures are made of brandy and sherry; the quantities of brandy in each being as 4 to 3; and the difference of the quantities of sherry being greater by 25 gallons than the difference of the quantities of brandy. Also if three times the quantity of brandy had been put into the first mixture, and twice the quantity into the second, the quantities of brandy would have been proportional to the quantities of sherry. But if the sherry in the second mixture had been mixed with the brandy in the first, and the sherry in the first with the brandy in the second, the whole mixtures would then have been in the ratio of 5 to 6. Required the quantities of brandy and sherry in each mixture.

Ans. The quantities of brandy are 80, and 60 gallons, and the quantities of sherry are 90, and 45 gallons.

33. During a winter, when fuel was scarce, two men, *A* and *B*, went in quest of coals and turf, which they agreed to use in common. *A* met with three bushels of coals, and *B* two, at the same price *per* bushel, and also seven baskets of turf. *A* stipulated that he should consume twice as many

coals as *B*. *B* assented, but demanded of him 2*s*. 10*d*. When this stock was exhausted, *B* purchased one bushel of coals, and *A* five, together with 6 baskets of turf, at the same rates respectively as before; but now *B* consumed three times as many coals as *A*, and paid him 1*s*. 6*d*. What was the price of a bushel of coals, and of a basket of turf; equal quantities of turf having been consumed by each person?

Ans. The price of a bushel of coals was 5*s*., and of a basket of turf 4*d*.

34. Two Spanish muleteers, *A* and *B*, were seated under a tree in order to dine; and on examining, found their stock of provisions to consist of 5 small loaves of bread, three of which were *A*'s property, and a bottle of wine, which was *B*'s. A stranger, who happened to come up at the time, was invited to partake of their fare, which was just sufficient for three persons; and at parting, being pleased with their behaviour, he gave them what Spanish money he had about him, which amounted to 6*s*. 5½*d*., to be equitably shared between them. Now as many shillings as a loaf cost pence would, with four pence more, at the next town have bought six such loaves and four bottles of the same wine; and when the money was divided, *B* received 1*s*. 10½*d*. more than *A*. What was the price of each loaf, and a bottle of wine?

Ans. A loaf cost 7 pence, and a bottle of wine 11½ pence.

VIII. *Problems producing Pure Equations.*

1. FIND two numbers, which are in the proportion of 8 to 5, and whose product is equal to 360.

Ans. ± 24 , and ± 15 .

2. There are two numbers, whose sum is to their difference as 8 to 1, and the difference of whose squares is 128. What are the numbers?

Ans. ± 18 , and ± 14 .

3. In a court there are two square grass-plots; a side of one of which is 10 yards longer than the side of the other; and their areas are as 25 to 9. What are the lengths of the sides?

Ans. 25, and 15 yards.

4. A person bought two pieces of linen, which together measured 36 yards. Each of them cost as many shillings *per* yard, as there were yards in the piece; and their whole prices were in the proportion of 4 to 1. What were the lengths of the pieces?

Ans. 24, and 12 yards.

5. There are two numbers, whose sum is to the less as 5 to 2; and whose difference, multiplied by the difference of their squares, is 135. Required the numbers.

Ans. 9, and 6.

6. There are two numbers, which are in the proportion of 3 to 2; the difference of whose fourth powers is to the sum of their cubes as 26 to 7. Required the numbers.

Ans. 6, and 4.

7. There is a field in the form of a rectangular parallelogram, whose length is to its breadth in the proportion of 6 to 5. A part of this, equal to one-sixth of the whole, being planted, there remain for ploughing 625 square yards. What are the dimensions of the field?

Ans. The sides are 30, and 25 yards.

8. Some Gentlemen made an excursion; and every one took the same sum. Each gentleman had as many servants

attending him as there were gentlemen; and the number of pounds which each had was double the number of all the servants; and the whole sum of money taken out was £3456. How many gentlemen were there?

Ans. 12.

9. Divide the number 49 into two such parts, that the quotient of the greater divided by the less may be to the quotient of the less divided by the greater as $\frac{4}{3}$ to $\frac{3}{4}$.

Ans. 28, and 21.

10. A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished four times as many men as there were companies in the regiment; but these being found to be insufficient, each company furnished 3 more men; when their number was found to be increased in the ratio of 17 to 16. How many companies were there in the regiment?

Ans. 12.

11. A charitable person distributed a certain sum amongst some poor men and women, the numbers of whom were in the proportion of 4 to 5. Each man received one-third of as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. Now the men received all together 18s. more than the women. How many were there of each?

Ans. 12 men, and 15 women.

12. A Gentleman who had a certain number of horses, kept part of them at livery stables, for which he paid £4. 10s. *per* week. The rest he kept at home, and their number was to the number kept at the livery stables as 7 to 3. He found that the expense of keeping 5 at home was just equal to that of keeping 4 at the stables; and the number of shillings that

one horse cost him at home was to the number of horses kept at home as 6 to 7. How many horses had he?

Ans. 6 at the livery stables, and 14 at home.

13. A city barge, with chairs for the company and benches for the rowers, went a summer excursion, with two bargemen on every bench. The number of gentlemen on board was equal to the square of the number of bargemen, and the number of ladies was equal to the number of gentlemen, twice the number of bargemen, and one over. Among other provisions, there were a number of turtles equal to the square root of the number of ladies; and a number of bottles of wine less than the cube of the number of turtles by 361. The turtles in dressing consumed a great quantity of wine, and the party having stayed out till the turtles were all eaten, and the wine all gone, it was computed, that supposing them all to have consumed an equal quantity, (viz. gentlemen, ladies, bargemen, and turtles,) each individual would have consumed as many bottles as there were benches in the barge. Required the number of turtles.

Ans. 19.

14. From two towns, *C* and *D*, two travellers, *A* and *B*, set out to meet each other; and it appeared that when they met, *B* had gone 35 miles more than three-fifths of the distance that *A* had travelled; but from their rate of travelling, *A* expected to reach *C* in 20 hours and 50 minutes; and *B* to reach *D* in 30 hours. Required the distance of *C* from *D*.

Ans. 275 miles.

15. A Farmer bought two flocks of sheep, the first of which contained 18 fewer than the second. If he had given for the first flock as many pounds as there were sheep in the second, and for the second as many pounds as there were sheep in the first, then the price of 6 sheep of the

first flock would have been to the price of 7 sheep of the second in the proportion of 7 to 6. Required the numbers in each flock.

Ans. 108, and 126.

16. A Poulterer bought a number of ducks and turkeys, the number of ducks exceeding the number of turkeys by 8. For each duck he gave half as many shillings as there were turkeys, and for each turkey half as many shillings as there were ducks. He afterwards bought another small flock of turkeys, containing 4 fewer than the number of turkeys he bought before; and having given for each of them as many shillings as there were turkeys in the flock, he found, that if his former purchase had cost 16 shillings more, it would have cost exactly four times as much as the present one. How many ducks and turkeys did he buy at first?

Ans. 12 turkeys, and 20 ducks.

17. Two men, *A* and *B*, entered into partnership with stocks, which are in the proportion of 9 to 8; and after trading one year, *A* found his share of their gain to amount to one-third of his stock. They continued to trade for as many years as are equal to three-fourths of the number of pounds which *B* contributed to the stock, and found their whole gain amount to £1666. What did each contribute to the stock; and how many years did they trade?

Ans. *A* contributed £63, and *B* £56; and the number of years is 42.

18. A person wishing to ascertain the area of a certain quadrilateral field, found that he could determine it the most readily by dividing it into two portions, one of which was of the form of a rectangular parallelogram, the shorter side of which measured 60 yards. The other was of the form of a right-angled triangle, whose shortest side was equal to the shorter side of the parallelogram, and the other side

containing the right angle, was equal to the diagonal of the parallelogram; and the area of the triangle was to the area of the parallelogram as 5 to 8. What was the area of the field?

Ans. 7800 square yards.

19. A Merchant laid out a certain sum upon a speculation, and found at the end of a year that he had gained £69. This he added to his stock, and at the end of another year found that he had gained exactly as much *per cent.* as in the year preceding. Proceeding in the same manner, and each year adding to his stock the gain of the year preceding, he found at the beginning of the fifth year that his stock was to the original stock as 81 to 16. What was the sum he first laid out?

Ans. £138.

20. There is a number consisting of two digits, which being multiplied by the digit on the left hand, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10. Required the number.

Ans. 23.

21. From two towns, *C* and *D*, which were at the distance of 396 miles, two persons, *A* and *B*, set out at the same time, and met each other, after travelling as many days as are equal to the difference of the number of miles they travelled *per day*; when it appears that *A* has travelled 216 miles. How many miles did each travel *per day*?

Ans. *A* went 36, and *B* 30.

22. There are two numbers, whose sum is to the greater as 40 is to the less, and whose sum is to the less as 90 is to the greater. What are the numbers?

Ans. 36, and 24.

23. It is required to find two numbers such, that the product of the greater and the cube of the less may be to

the product of the less and the cube of the greater as 4 to 9 ; and the sum of the cubes of the numbers may be 35.

Ans. 3, and 2.

24. The paving of two square court-yards cost £205 ; a yard of each costing one-fourth of as many shillings as there were yards in a side of the other. And a side of the greater and less together measure 41 yards. Required the length of a side of each.

Ans. 25, and 16 yards.

25. A person bought a number of apples and pears, amounting together to 80. Now the apples cost twice as much as the pears : but had he bought as many apples as he did pears, and as many pears as he did apples, his apples would have cost 10*d.*, and his pears 3*s.* 9*d.* How many did he buy of each ?

Ans. 60 apples, and 20 pears.

26. A person exchanged a quantity of brandy for a quantity of rum and £11. 5*s.* ; the brandy and rum being each valued at as many shillings *per* gallon as there were gallons of that liquor. Now had the rum been worth as many shillings *per* gallon as the brandy was, the whole value of the rum and brandy would have been £56. 5*s.* How many gallons were there of each ?

Ans. 25 gallons of brandy, and 20 of rum.

27. There are two rectangular vats, the greater of which contains 20 solid feet more than the other. Their capacities are in the ratio of 4 to 5 ; and their bases are squares, a side of each of which is equal to the depth of the other. What are the depths ?

Ans. 5 feet, and 4 feet.

28. Bought two square carpets for £62. 1*s.* ; for each of which I paid as many shillings *per* yard as there were

yards in its side. Now had each of them cost as many shillings *per* yard as there were yards in the side of the other, I should have paid 17*s.* less. What was the size of each?

Ans. One contained 81, and the other 64 square yards.

29. The number of men in both fronts of two columns of troops, *A* and *B*, when each consisted of as many ranks as it had men in front, was 84; but when the columns changed ground, and *A* was drawn up with the front *B* had, and *B* with the front *A* had, the number of ranks in both columns was 91. Required the number of men in each column.

Ans. 2304, and 1296.

30. A field in the form of a rectangular parallelogram was planted with trees placed at such distances as to have four on every square yard. The expense of planting was such, that every 40 trees cost one-third of as many shillings as there were yards in the diagonal of the parallelogram. But had they been planted at such a price as that every hundred should have cost as many shillings as there were yards in the shorter side of the parallelogram, the expense would have been less by £224. Now a square described upon the diagonal of the parallelogram would be equal to eight-thirds of the square described on the less side, together with the square described on a line which is equal to the difference of the sides. Required the dimensions of the parallelogram.

Ans. The longer side is 80, and the shorter 60 yards.

31. *A* and *B* are two towns, situated on the bank of a river, which runs at the rate of 4 miles an hour. A Waterman rows from *A* to *B*, and back again, and finds that he is 39 minutes longer upon the water than he would have been, had there been no stream. The next day he repeats

his voyage with another waterman, with whose assistance he can row half as fast again : and they find that they are only eight minutes longer in performing their voyage than they would have been, had there been no stream. Determine the rate at which the waterman would row by himself.

Ans. 6 miles *per* hour.

32. There are three towns, *A*, *B*, *C*, the straight lines joining which form a right-angled triangle ; *B* being situated at the right angle, and the distance from *A* to *B* being the least of the three. A pedestrian making a circuit of them, at an uniform rate, finds that the time of his going from *A* to *B*, together with the time of going from *B* to *C*, exceeds the time from *C* to *A* by two hours and forty minutes. A coach, which left *A*, to make the same circuit, four hours after the pedestrian, overtakes him at the end of the eighth mile from *B* to *C* ; the rate of the coach's travelling being three times that of the pedestrian ; and after reaching *A*, and waiting there six hours and forty minutes, it sets out again to make the same circuit, and arrives again at *A* exactly at the same time with the pedestrian, who had rested four hours at *C*. Find the distances of the towns from each other, and the rates of travelling of the pedestrian and the coach.

Ans. The distances are 10, 24, and 26 miles respectively ; and the rates of travelling of the pedestrian and the coach are 3 and 9 miles *per* hour.

IX. Problems producing Affected Quadratics.

1. WHAT two numbers are those, whose sum is 19, and whose difference multiplied by the greater is 60 ?

Ans. 12, and 7.

2. If the square of a certain number be taken from 40, and the square root of this difference be increased by 10, and the sum multiplied by 2, and the product divided by the number itself, the quotient will be 4. Required the number.

Ans. 6.

3. There is a field in the form of a rectangular parallelogram, whose length exceeds the breadth by 16 yards; and it contains 960 square yards. Required the length and breadth.

Ans. 40, and 24 yards.

4. A person being asked his age, answered, If you add the square root of it to half of it, and subtract 12, there will remain nothing. Required his age.

Ans. 16.

5. Two casks of ale were bought for £2. 18s., one of which contained 5 gallons more than the other, and the price *per* gallon was 2 shillings less than one-third of the number of gallons in the less. Required the number of gallons in each, and the price *per* gallon.

Ans. The numbers were 12, and 17, and the price *per* gallon 2 shillings.

6. From two places, at the distance of 320 miles, two persons, *A* and *B*, set out at the same time to meet each other. *A* travelled 8 miles a day more than *B*, and the number of days in which they met was equal to half the number of miles *B* went in a day. How many miles did each travel *per* day, and how far did each travel?

Ans. *A* went 24, and *B* 16 miles *per* day; *A* went 192, and *B* 128 miles.

7. The difference between the hypotenuse and base of a right-angled triangle is $= 6$, and the difference between the hypotenuse and the perpendicular is $= 3$. What are the sides?

Ans. 15, 12, and 9.

8. In a parcel which contains 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins, and the whole is worth 18 shillings. How many are there of each?

Ans. 6 of one, and 18 of the other.

9. A Farmer received £7. 4s. for a certain quantity of wheat, and an equal sum at a price less by 1s. 6d. *per* bushel for a quantity of barley, which exceeded the quantity of wheat by 16 bushels. How many bushels were there of each?

Ans. 32 bushels of wheat, and 48 of barley.

10. Two messengers, *A* and *B*, were dispatched at the same time to a place 90 miles distant; the former of whom riding one mile an hour more than the other, arrived at the end of his journey an hour before him. At what rate did each travel *per* hour?

Ans. *A* went 10, and *B* 9 miles *per* hour.

11. Bought a number of books, consisting of folios, quartos, and octavos, for £96. 12s. Fourteen folios (which was the whole number) cost 3 times as much as all the quartos; and one quarto cost as many shillings as there were quartos. The number of octavos was 32, and their value was such, that 4 of them cost as much as one quarto. Required the value of each, and the number of quartos.

Ans. There were 21 quartos, each folio cost $4\frac{1}{2}$ guineas, each quarto one guinea, and each octavo 5s. 3d.

12. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he should have been 6 hours longer in performing the same journey. How many miles did he go *per* hour?

Ans. 7 miles.

13. Bought two flocks of sheep for £65. 13s., one containing 5 more than the other. Each sheep cost as many shillings as there were sheep in the flock. Required the numbers in each flock.

Ans. 23, and 28.

14. A regiment of soldiers, consisting of 1066 men, is formed into two squares, one of which has four men more in a side than the other. What number of men are in a side of each of the squares?

Ans. 21, and 25.

15. What number is that, to which if 24 be added, and the square root of the sum extracted, this root shall be less than the original quantity by 18?

Ans. 25.

16. After taking the kings, queens, and knaves out of a pack of cards, the rest were divided into three heaps. The number of pips contained in the second heap was found to be 4 times the square of the number in the first heap; and had the third heap contained 5 more pips than it did, the number in it would have been exactly half of what the first and second heap contained. Required the number of pips in each heap.

Ans. 6, 144, and 70.

17. A Tailor bought a piece of cloth for £147, from which he cut off 12 yards for his own use, and sold the re-

mainder for £120. 5s., gaining 5 shillings *per* yard. How many yards were there, and what did it cost him *per* yard?

Ans. 49 yards, at £3 *per* yard.

18. A regiment of foot was ordered to send 216 men on garrison duty, each company being to furnish an equal number; but before the detachment marched, 3 of the companies were sent on another service, when it was found that each company that remained was obliged to furnish 12 additional men, in order to make up the complement 216. How many companies were there in the regiment, and what number of men was each company ordered to send at first?

Ans. There were 9 companies; and each was to send 24 men.

19. A Poulterer bought 15 ducks and 12 turkeys for five guineas. He had two ducks more for 18 shillings than he had of turkeys for 20 shillings. What was the price of each?

Ans. The price of a duck was 3s. and of a turkey 5s.

20. Two men, *A* and *B*, entered into a speculation, to which *B* subscribed £15 more than *A*. After 4 months, *C* was admitted, who added £50 to the stock; and at the end of 12 months from *C*'s admission they found they had gained £159; when *A* withdrawing received for principal and gain £88. What did he originally subscribe?

Ans. £40.

21. A wall was built round a rectangular court to a certain height. Now the length of one side of the court was two yards less than 8 times the height of the wall, and the length of the adjacent side was 5 yards less than 6 times the height of the wall; and the number of square yards in the court was greater

than the number in the wall by 178. Required the dimensions of the court, and the height of the wall.

Ans. The sides were 30, and 19, and the height 4 yards.

22. A ship containing 74 sailors, and a certain number of soldiers besides officers, took a prize. The sailors received each one-third as many pounds as there were soldiers, and the soldiers received £3 a piece less, and £768 fell to the share of the officers. Had the officers however received nothing, the soldiers and sailors might have received half as many pounds *per* man, as there were soldiers. How many soldiers were there, and how much did each receive?

Ans. There were 36 soldiers, each soldier received £9, and each sailor £12.

23. A Poulterer going to market to buy turkeys, met with four flocks. In the second were 6 more than three times the square root of double the number in the first. The third contained three times as many as the first and second; and the fourth contained 6 more than the square of one-third of the number in the third; and the whole number was 1938. How many were there in each flock?

Ans. The numbers were 18, 24, 126, 1770, respectively.

24. A body of men are just sufficient to form a hollow equilateral wedge, three deep; and if 597 be taken away, the remainder will form a hollow square, four deep, the front of which contains one man more than the square root of the number contained in a front of the wedge. What is the number of men?

Ans. 693.

25. Two men, *A* and *B*, undertake to perform a piece of work in four days, for which they are to receive a certain

number of shillings; but after some time, finding that they shall not be able to finish it in the time proposed, they call in *C* to assist them, and upon an equitable division of the money, *C* receives a sum equal to the square root of the whole number of shillings; but had they been obliged to call in *C* to their assistance $1\frac{1}{2}$ day sooner, his share of the money would have been two-fifths more. How long did *C* work, and what did he receive?

Ans. He worked 2 days, and received 5 shillings.

26. A cask, whose content is 20 gallons, is filled with brandy, a certain quantity of which is then drawn off into another cask of equal size; this last cask is then filled with water; after which the first cask is filled with the mixture, and it appears, that if $6\frac{2}{3}$ gallons of the mixture be drawn off from the first into the second cask, there will be equal quantities of brandy in each. Required the quantity of brandy first drawn off.

Ans. 10 gallons.

27. There are three numbers, the difference of whose differences is 5; their sum is 20; and their continual product 130. Required the numbers.

Ans. 2, 5, and 13.

28. There are three numbers, the difference of whose differences is 3; their sum is 21; and the sum of the squares of the greatest and least is 137. Required the numbers.

Ans. 4, 6, 11.

29. There is a number consisting of 2 digits, which when divided by the sum of its digits gives a quotient greater by 2 than the first digit. But if the digits be inverted, and then divided by a number greater by unity than the sum of the

digits, the quotient is greater by 2 than the preceding quotient. Required the number.

Ans. 24.

30. *A* and *B* gained by trading £100. Half of *A*'s stock was less than *B*'s by £100; and *A*'s gain was three-twentieths of *B*'s stock. What did each put into stock, and what are the respective shares of the gain?

Ans. *A*'s stock was £600 and *B*'s £400. *A*'s gain was £60 and *B*'s £40.

31. *A*, *B*, and *C* were three Architects. *A* and *B* built four warehouses with flat roofs, each a large one, and each a small one; the linear width of the two large ones being the same, and also that of the two small ones. *A* built his as long and as high as they were wide; but *B* made the length and height of his large one equal to the width of his small one, and the length and height of his small one equal to the width of his large one, in such a manner that the difference between the solid content of those built by *A* and those built by *B* was 73728 cubic feet. *C* also built a warehouse upon a square plot of ground, which was equal to the difference between the ground-plots occupied by those which *A* built, and found that it would have stood on 2688 square feet, if he had added eight times as many square feet to the ground-plot as there were linear feet in its width. How many feet wide were the several buildings erected by *A*, *B*, and *C*?

Ans. The width of *A*'s and *B*'s large warehouse was 52 feet, and of their small one 20; and the width of *C*'s 48.

32. A certain sum was to be raised on three estates belonging to *A*, *B*, and *C*, at the rate of one shilling *per* acre. Now the number of acres *A* and *B* had, were as 3 to 7; and if the number of acres in the whole were divided by one-third of the product of the numbers in the first

and third, the quotient would be $\frac{3}{4}$. Also the sum paid by *A* and *C* was 36 shillings less than the sum of three times the money paid by *C*, and two-sevenths of the money paid by *B*. Of how many acres did each estate consist; and what was the whole sum to be raised?

Ans. *A* had 12, *B* 28, and *C* 20 acres; and the sum was £3.

33. A Butcher bought a certain number of calves and sheep, and for each of the former gave as many shillings as there were sheep, and for each of the latter one-fourth as much. Now had he given 4 shillings more for each of the former, and 2 shillings more for each of the latter, he would have paid seven pounds more. But had a sheep cost as much as a calf, he would have expended £56. 8s. How many did he buy of each; and what were their prices?

Ans. 23 calves, and 24 sheep; and their prices were 24, and 6 shillings, respectively.

34. A Farmer at a fair found the price of an ox equal to that of three sheep, and that he could just dispose of £100, buying twice as many sheep as oxen. But waiting till the evening when the price of an ox fell £1, and of a sheep 6s. 8d., he got for £100 three times as many sheep as oxen, and increased his whole stock by ten more than he would have done in the former case. How many sheep and oxen did he buy, and what was the price of each?

Ans. 10 oxen and 30 sheep; and the prices were £5, and £1. 13s. 4d.

35. Two persons, *A* and *B*, comparing their wages, observe that if *A* had received *per* day in addition to what he does receive, a sum equal to one-fourth of what *B* received *per* week, and had worked as many days as *B* received shillings *per* day, he would have received £2. 8s.; and had *B* received

2 shillings a day more than *A* did, and worked for a number of days equal to half the number of shillings he received *per* week, he would have received £4. 18s. What were their daily wages?

Ans. *A*'s 5 shillings, and *B*'s 4.

36. There are four towns in the order of the letters, *A*, *B*, *C*, *D*. The difference between the distances from *A* to *B* and from *B* to *C* is greater by four miles than the distance from *B* to *D*. Also the number of miles between *B* and *D* is equal to two-thirds of the number between *A* and *C*. And the number between *A* and *B* is to the number between *C* and *D* as seven times the number between *B* and *C* : 26. Required the respective distances.

Ans. $AB = 42$, $BC = 6$, $CD = 26$ miles.

37. A person bought a quantity of cloth of two sorts for £7. 18s. For every yard of the better sort he gave as many shillings as he had yards in all; and for every yard of the worse as many shillings as there were yards of the better sort more than of the worse. And the whole price of the better sort was to the whole price of the worse as 72 to 7. How many yards had he of each?

Ans. 9 yards of the better, and 7 of the worse.

38. From each of two bags containing a certain number of balls respectively, a person draws out a handful, and finds that the number remaining in the greater is exactly the cube of that remaining in the lesser, and exactly the square of one handful. He then draws out of the greater, until he finds that the number remaining in it is exactly the square of that remaining in the lesser, and also that if he now empties the greater into the lesser, its original number will be increased by two-thirds. Determine the number of balls in each bag.

Ans. 72 and 12.

39. A Farmer sold a certain number of bushels of barley, and ten bushels of wheat for £7. 19s. Now each bushel of wheat cost within 3 shillings as much as two bushels of barley. He afterwards sold as many bushels of barley and four more, and fifteen bushels of wheat, and received two shillings *per* bushel more for his wheat and barley than he did before; when he found that if he had received £1. 4s. more, he should just have received twice as much as he did before. How many bushels of barley did he sell the first time; and what were the prices *per* bushel of the wheat and barley?

Ans. 7 bushels of barley; and the prices of wheat and barley were 11s. and 7s. *per* bushel.

40. A Farmer laid up a stock of corn, expecting to sell it in six months at three shillings *per* bushel more than he gave for it. But the price of corn falling one shilling *per* bushel, he found that by selling it he should lose the price of five bushels. He therefore kept it till the end of the year, and selling it at two shillings *per* bushel under prime cost, found his loss to be ten shillings less than his expected gain. Required the quantity of corn laid up, and the price *per* bushel, allowing 5 *per cent.* simple interest.

Ans. 40 bushels, and the price was 10s. *per* bushel.

41. In digging among some ruins the workmen found 9 urns, together containing 60 gold coins; the second and eighth containing 8 and 4 respectively. They secreted a certain number of these, greater than the number they left; which being afterwards recovered, it was found that the number of urns secreted was to the number left as the number of coins secreted was to the number remaining. Now if instead of taking the second urn they had carried off the eighth, then the number of coins taken away would have been to the number remaining as the square of the number of urns secreted to the difference between that square and

20 times the number of urns remaining. Required the numbers of urns and coins secreted.

Ans. 6 urns, and 40 coins.

42. Two men, *A* and *B*, set out from the same place to travel. *A* goes in 6 days twice as many miles as *B* goes in 5 days, but does not arrive at the end of his journey till 5 days after *B* has arrived at the end of his, when he finds that he has travelled 259 miles more than *B*. But had *B* gone 2 miles *per* day more than he did, and *A* stopped 6 days sooner, *A* would then have gone only 37 miles more than *B*. How many miles did each travel *per* day, and how many days did they travel?

Ans. *A* travelled 11 days, and 35 miles *per* day; *B* travelled 6 days, and 21 miles *per* day.

43. Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for two-thirds of the time that Silenus would have taken to empty the whole cask. After that Silenus awoke, and drank what Bacchus had left. Had they drunk both together, it would have been emptied two hours sooner, and Bacchus would have drunk only half what he left Silenus. Required the time in which they would empty the cask separately.

Ans. Silenus in 3 hours, and Bacchus in 6.

44. Two persons, *A* and *B*, comparing the distances they have travelled, found that the square of the number of miles which *A* usually walked *per* hour, exceeded the square of the number which *B* usually walked by 5; and that if to the square of the product of those numbers there be added the square of the sum of their fourth powers, augmented by the product of the square of the difference of their squares into the square of the product of the numbers themselves, the aggregate amount would be 10345. How many miles did each walk *per* hour?

Ans. *A* walked 3, and *B* 2 miles.

45. From the middle of a town two streets branched off, and crossed a river that ran in a straight course, by two bridges *A* and *B*. From their junction a sewer equally inclined to both streets led to a point in the river at the distance of 6 chains from the bridge *A*, and a distance from *B* less by 11 chains than the length of the sewer: the expense of making it amounting to as many pounds *per* chain, as there were chains in the street leading to *A*. The sewer however being insufficient to carry off the water, an additional drain was made from a point in this street, distant 4 chains from the bridge *A*, which entered the river at the same point with the sewer, and was equally inclined to the river and sewer. Now it was found that a drain down the middle of each street, at the rate of £9 *per* chain would have cost only £54 more than the expense of the sewer. Required the lengths of the streets and the sewer; and the distance of its mouth from the bridge *B*.

Ans. The lengths of the streets were 18 and 30 chains, of the sewer 21, and the distance from *B* 10.

46. Two plantations, one of an oblong, and the other of a square form, contain the same number of trees, and they have one fence common to both, viz. that which bounds the end of the oblong one. Upon every pole in the square are planted as many trees as there are poles in the square, and upon every pole in the oblong four times as many trees as there are poles in the breadth, besides 144 in the hedges. Also the area of the oblong wants 6 poles to be to the area of the square as 3 to 2. Required the number of trees.

Ans. 1296.

47. The roof of a storehouse is formed of two squares terminated by two equal and parallel isosceles triangles; the height of the walls being equal to the base of either of these triangles. The quantity of wood which the storehouse will hold, increased by six cubical piles each of the same length as the building, is to the quantity which the same storehouse

would hold if its roof were flat, in the proportion of 11 : 2. The roof cost as many pence *per* square foot, as there are feet in its ridge, and the flooring was laid at the same rate. Both together cost £208. 6s. 8d. Required the dimensions of the storehouse.

Ans. The length is 25, and the height 15 feet.

48. There are two sorts of metal, each being a mixture of gold and silver, but in different proportions. Two coins from these metals of the same weight are to each other in value as 11 to 17; but if to the same quantities of silver as before in each mixture double the former quantities of gold had been added, the values of two coins from them of equal weights would have been to each other as 7 to 11. Determine the proportion of gold to silver in each mixture, the values of equal weights of gold and silver being as 13 to 1.

Ans. The proportion of gold to silver is 1 : 9 in the first mixture, and 1 : 4 in the second.

49. A Mason has two cubical pieces of white marble of exactly the same size, and two cubical equal pieces of black, larger than the other. The number of solid yards in the four pieces is 9 more than 11 times the number of yards in a side of a white one, together with 12 times the number in a side of a black one. He afterwards finds another block, the length of which is two yards longer than a side of one of the white pieces, and the width 4 times the length of a side of the black one; and this when laid on its largest side occupies a space greater by 3 yards than the difference between 4 times the space occupied by a black, and 3 times the space occupied by a white one. Required the dimensions of the blocks.

Ans. The side of a white block is 1 yard, and of a black one 3 yards; the length of the other is 3 yards, and the width 12 yards.

50. *A* and *B* travelled on the same road and at the same rate from Huntingdon to London. At the 50th milestone

from London, *A* overtook a drove of geese which were proceeding at the rate of three miles in 2 hours; and two hours afterwards met a stage waggon, which was moving at the rate of nine miles in four hours. *B* overtook the same drove of geese at the 45th milestone, and met the same stage waggon exactly forty minutes before he came to the 31st milestone. Where was *B* when *A* reached London?

Ans. 25 miles from London.

51. The hold of a vessel partly full of water (which is uniformly increased by a leak) is furnished with two pumps worked by *A* and *B*, of whom *A* takes three strokes to two of *B*'s; but four of *B*'s throw out as much water as five of *A*'s. Now *B* works for the time in which *A* alone would have emptied the hold; *A* then pumps out the remainder, and the hold is cleared in 13 hours and 20 minutes. Had they worked together, the hold would have been emptied in 3 hours and 45 minutes; and *A* would have pumped out 100 gallons more than he did. Required the quantity of water in the hold at first, and the horary influx at the leak.

Ans. The quantity in the hold was 1200 gallons, and the horary influx 120 gallons.

X. Problems in Arithmetical and Geometrical Progressions.

1. THERE are three numbers in arithmetical progression, whose sum is 21; and the sum of the first and second is to the sum of the second and third as 3 to 4. Required the numbers.

Ans. 5, 7, 9.

2. There are six towns in the order of the letters, *A*, *B*, *C*, *D*, *E*, *F*, whose distances from each other are in an in-

creasing arithmetical progression. The distance from *A* to *C* is 16 miles, and from *C* to *E* is 24 miles. Required their respective distances.

Ans. From *A* to *B* is 7, from *B* to *C* 9, from *C* to *D* 11, from *D* to *E* 13, and from *E* to *F* 15 miles.

3. A person makes a mixture of 51 gallons, consisting of brandy, rum, and water, the quantities of which are in arithmetical progression. The number of gallons of brandy and rum together is to the number of gallons of rum and water together as 8 to 9. Required the quantities of each.

Ans. 15 gallons of brandy, 17 of rum, and 19 of water.

4. A number consisting of three digits which are in arithmetical progression, being divided by the sum of its digits, gives a quotient 48; and if 198 be subtracted from it, the digits will be inverted. Required the number.

Ans. 432.

5. During a scarcity, a person wished to make a mixture of 24 bushels, consisting of wheat, oats, and barley, the quantities of each forming an increasing arithmetical progression. Not being able however to procure any barley, he mixed additional quantities of wheat and oats in the proportion of 2 to 3, so as to complete his 24 bushels, when he found the whole quantities of wheat and oats to be in the proportion of 5 to 7. How many bushels of each did he originally intend to mix?

Ans. 6 of wheat, 8 of oats, and 10 of barley.

6. The difference between the first and second of four numbers in geometrical progression is 36, and the difference between the third and fourth is 4. What are the numbers?

Ans. 54, 18, 6, and 2.

7. A person employed three workmen, whose daily wages were in arithmetical progression. The number of days they worked was equal to the number of shillings that the second received *per* day. The whole amount of their wages was seven guineas, and the best workman received 28 shillings more than the worst. What were their daily wages?

Ans. 5, 7, and 9 shillings.

8. There are three numbers in geometrical progression; the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

Ans. 3, 6, 12.

9. There are three numbers in geometrical progression; whose sum is 14; and the sum of the first and second is to the sum of the second and third as 1 to 2. Required the numbers.

Ans. 2, 4, 8.

10. There are three numbers in geometrical progression, whose continual product is 64, and the sum of their cubes is 584. Required the numbers.

Ans. 2, 4, 8.

11. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers.

Ans. 1, 3, 9, 27.

12. From two towns which were 168 miles distant, two persons, *A* and *B*, set out to meet each other; *A* went 3 miles the first day, 5 the next, 7 the third, and so on; *B* went 4 miles the first day, 6 the next, and so on. In how many days did they meet?

Ans. 8.

13. A traveller set out from a certain place, and went 1 mile the first day, 3 the second, 5 the next, and so on, going every day 2 miles more than he had gone the preceding day. After he had been gone three days, a second sets out, and travels 12 miles the first day, 13 the second, and so on. In how many days will the second overtake the first?

Ans. In 2, and 9 days.

14. A person has two pieces of ground, one of which is in the form of an equilateral triangle, and the other of a rectangular parallelogram, one side of which is equal to a side of the triangle, and the other side is 8 yards less. These he plants with trees at the distance of two yards from each other, and finds that there are 5 more on the rectangle than on the triangle. What are the lengths of the sides?

Ans. A side of the triangle is 20 yards, and the sides of the parallelogram are 20 and 12 yards.

15. There are four numbers in arithmetical progression, whose sum is 28; and their continual product is 585. Required the numbers.

Ans. 1, 5, 9, 13.

16. There are four numbers in arithmetical progression; the sum of the squares of the first and second is 34; and the sum of the squares of the third and fourth is 130. Required the numbers.

Ans. 3, 5, 7, 9.

17. The sum of £700 was divided among four persons, whose shares were in geometrical progression; and the difference between the greatest and least was to the difference between the means as 37 to 12. What were their respective shares?

Ans. £108, £144, £192, £256.

18. Five persons undertake to reap a field of 87 acres. The five terms of an arithmetical progression, whose sum is 20, will express the times in which they can severally reap an acre; and they altogether can finish the undertaking in 60 days. In how many days can each separately reap an acre?

Ans. 2, 3, 4, 5, 6 days.

19. Out of a vessel containing 24 gallons of pure spirit, a vintner drew off at three successive times a certain number of gallons, which formed an increasing arithmetical progression, in which the difference between the squares of the extremes was equal to 16 times the mean, and filled up the vessel with water after each draught, till he found what he last drew off reduced to one-sixth of its original strength. Required the number of gallons of pure spirit drawn off each time.

Ans. 12, 8, $3\frac{1}{2}$.

20. A number of persons purchased a field for £345. The youngest contributed a certain sum, the next £5 more, the third £5 more than the second, and so on to the oldest. For the greater accommodation of the seniors, the field was divided into two parts, the younger half taking a portion proportional to the sum they had subscribed; and in order that each might have an equal share in this portion, they agreed to equalize their contributions, and each to pay £22. Required the number of persons and the sums paid by each.

Ans. The number of persons was 10; and the sum paid by the youngest £12.

21. The number of deaths in a besieged garrison amounted to 6 daily; and allowing for this diminution their stock of provisions was sufficient to last for 8 days. But on the evening of the sixth day 100 men were killed in a sally, and afterwards the mortality increased to 10 daily. Supposing the stock of provisions unconsumed at the end of the sixth day to support 6 men for 61 days; it is required to find how long it would

support the garrison; and the number of men alive when the provisions were exhausted.

Ans. 6 days, and 26 men remained alive when the provisions were exhausted.

22. A Ship with a crew of 175 men set sail with a store of water sufficient to last to the end of the voyage. But in 30 days the scurvy made its appearance, and carried off three men every day, and at the same time a storm arose, which protracted the voyage three weeks. They were however just enabled to arrive in port, without any diminution in each man's daily allowance of water. Required the time of the passage, and the number of men alive when the vessel reached harbour.

Ans. The voyage lasted 79 days, and the number of men alive was 28.

23. Three persons, *A*, *B*, and *C*, went into a gaming house; the sums which they severally had, were in a decreasing geometrical progression. Upon quitting it they found that the sums which they then had, were in a decreasing arithmetical progression; that what *B* had remaining was to what he had lost in proportion of the sum to the difference of what he and *C* had at first; and that *C* had neither won nor lost. If *C* had won what *A* lost, he would then have had £64 more than *A* had remaining; also the whole sum which they had remaining was to that they had lost as 6 : 7. Required the sums which they had at first.

Ans. 144, 48 and 16 pounds respectively.

24. The Fly starts 10 miles before the Telegraph; but the Fly coachman having made an appointment with the driver of the Telegraph, walks his horses so as to be overtaken at the end of the second mile. Now it is observed, that the number of revolutions made in a given time by the

hinder wheel of the Fly, its fore wheel, and the hinder wheel of the Telegraph increase in arithmetical progression, and that the circumference of these wheels, viz. of the fore wheel of the Fly, its hinder wheel, and the hinder wheel of the Telegraph, increase in a geometrical progression, whose common ratio is the same as the common difference of the arithmetical progression. It is required to find the ratio that the wheels bear to each other.

Ans. 1, 2, 4 are their proportional lengths.

25. A company of Merchants fitted out a privateer, each subscribing £100. The captain subscribed nothing, but was entitled to a £100 share at the end of every certain number of months. In the course of 25 months he captured three prizes, which were in geometrical progression, the middle term being one-fourth of the cost of the equipment, the common ratio the number of months which entitled the captain to his £100 share, and their sum £1375 more than the cost of the equipment. After deducting £875 for prize-money to the crew, the captain's share of the remainder amounted to one-fifth of that of the company. Required the number of merchants, and the captain's pay.

Ans. The number of merchants was 25, and the captain was entitled to a £100 share at the end of every 5 months.

26. On the institution of Savings Banks, an industrious labourer with his wife and children saved each a certain number of pence in a decreasing arithmetical progression. The sum saved monthly, was less by 3s. 3d. than would have purchased one-sixth of as many bushels of wheat as the seventh child saved pence: the price of wheat being such that the sum saved by the eldest and fifth child augmented by 10s. would buy two bushels. But wheat rising 2s. *per* bushel, and work being scarce, the family find the sum saved would not buy as much wheat as their former savings by two bushels; when it appears that at this rate the sum annually

saved would be less by five guineas than by the former. Now the two youngest dying, it is found that if the remaining members of the family saved each one shilling less than the oldest child had done before the rise of wheat, their monthly account with the bank would not be affected by the deaths of the two youngest: but if they saved only $2d.$ less than the oldest had done, their monthly account would be $2s. 1d.$ less than it was at the first institution. Of how many did the family consist? What were the sums saved by each? and what was the price of wheat?

Ans. The family at first consisted of 10. The labourer saved $4s.$, each member saving $3d.$ less than the preceding. And the price of wheat was $8s.$ *per* bushel.

APPENDIX I.

I. *Problems in Arithmetic Progression.*

1. DETERMINE the 28th term of the series $13, 12\frac{2}{3}, 12\frac{1}{3}, \&c.$

2. Having given the first and last terms of an arithmetic progression, and their common difference; determine their number of terms.

3. Having given the first and last terms of an arithmetic progression, and the number of terms; determine the progression.

4. In an arithmetic progression it is observed that the fifth and ninth terms are 13 and 25 : what is the 7th term?

5. If three quantities are in an increasing arithmetic progression; show that the second will have to the first a greater ratio than the third to the second.

6. Find the sums of the following series :

$1 + 3 + 5 + 7 + \&c.$ to n terms.

$1 + 5 + 9 + 13 + \&c.$ to n terms.

$1 + 4 + 7 + 10 + \&c.$ to 12 terms.

$5 + 7 + 9 + 11 + \&c.$ to 50 terms.

$2 + 2\frac{1}{3} + 2\frac{2}{3} + 3 + \&c.$ to 13 terms.

$\frac{1}{4} + \frac{8}{8} + \frac{1}{2} + \&c.$ to 16 terms.

$$\frac{1}{3} + \frac{7}{6} + 2 + \frac{17}{6} + \frac{11}{3} + \&c. \text{ to 12 terms.}$$

$$\frac{5}{7} + 1 + 1\frac{1}{7} + \&c. \text{ to 8 terms.}$$

$$-9 - 7 - 5 - \&c. \text{ to 20 terms.}$$

$$-5 - 3 - 1, \&c. \text{ to 8 terms.}$$

$$11 + 8 + 5 + \&c. \text{ to 8 terms.}$$

$$\frac{1}{2} - 1 - \frac{5}{2} - \&c. \text{ to 29 terms.}$$

$$15 + \frac{44}{3} + \frac{43}{3} + \&c. \text{ to 16 terms.}$$

$$\frac{5}{6} + \frac{1}{2} + \frac{1}{6} + \&c. \text{ to 19 terms.}$$

$$\frac{27}{16} + \frac{18}{15} + \frac{57}{80} + \&c. \text{ to 8 terms.}$$

$$\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \&c. \text{ to } n \text{ terms.}$$

$$na - b + (n-1) \cdot a + (n-2) \cdot a + b + \&c. \text{ to } n \text{ terms.}$$

$$(a+x)^2 + (a^2 - x^2) + (a-x)^2 + \&c. \text{ to } n \text{ terms.}$$

7. Show that $\frac{a-b}{a+b} + \frac{3a-2b}{a+b} + \&c. \text{ to } n \text{ terms, is}$

$$= \frac{n}{a+b} \cdot \left(na - \frac{n+1}{2} \cdot b \right).$$

8. Having given the first and last terms, and the sum of an arithmetic series; determine the common difference.

9. Having given the first term = 1, the number of terms = n , and the sum = s , of an arithmetic series; determine the common difference.

10. Having given the n^{th} term of an arithmetic series, and also the sum of n terms; determine the series.

11. If the first term of an arithmetic series be $= 1$, and the common difference $= m$, the sum of n terms of the series is $\frac{1}{2} \cdot \{mn^2 - (m - 2) \cdot n\}$.

12. If the first term of an arithmetic series be $= 1$, the common difference $= 4$, and the sum $= 120$; determine the number of terms.

13. If the first term of an arithmetic series be $= 3\frac{1}{2}$, the common difference $= 1\frac{1}{2}$, and the sum $= 22$; determine the number of terms.

14. If the first term of an arithmetic series be $= 11$, the common difference $= -5$, and the sum $= 5$; determine the number of terms.

15. The sum of an arithmetic series is $= 1455$, the first term $= 5$, and the number of terms $= 30$; determine the common difference.

16. If a be the first term, b the second, l the last term of an arithmetic series, the sum $= \frac{a+l}{b-a} \cdot \frac{l-2a+b}{2}$.

17. In the expression $s = \{2a + (n - 1) \cdot d\} \cdot \frac{1}{2}n$; if n be *negative*, point out the form of the series which satisfies *that* condition; and determine the series in the case of $a = 7$, $d = 2$, $s = 40$.

18. The sum of n terms of any arithmetic series whose common difference is equal to the least term, will be equal to the sum of $(n + 1)$ magnitudes, each of which is half the greatest term of the progression.

19. If from any square number (n^2) there be subtracted the sum of an arithmetic progression beginning from unity, having a common difference unity, and continued to as many

terms as there are units in the root of the number (n); the remainder will be the sum of the progression continued to $(n - 1)$ terms.

20. The sum of an even number of terms of any arithmetic series whose common difference is equal to the least term, will be four times the sum of half that number of terms diminished by half the last term; the first term being the same in each case.

21. Prove that 1, 3, 5, 7, &c. is the only arithmetic progression beginning from 1, in which the sum of the first half of any even number of terms bears to the sum of the second half the same constant ratio; and determine that ratio.

22. The sum of n terms of the series 1, 3, 5, 7, &c. is to the sum of $(n - 1)$ terms of the series 2, 4, 6, &c. $\therefore n : n - 1$. Required a proof.

23. The two first terms of an arithmetic progression being together = 18, and the three next = 12; how many terms, beginning with the first, must be taken to make 28; and explain the reason of the double solution.

24. How many terms of the series 1, 3, 5, 7, &c. must be added together to produce the $(2m)^{\text{th}}$ power of a given quantity r .

25. Find n terms of the indefinite series 3, 5, 7, &c. whose sum may be the $(m)^{\text{th}}$ power of n .

26. Having given (a) and (b) , the $(m)^{\text{th}}$ and $(n)^{\text{th}}$ terms of an arithmetic series; determine the value of the $(x)^{\text{th}}$ term.

27. Having given as before; determine the sum of (p) terms of the series.

28. Having given (a) and (b) , the $(m)^{\text{th}}$ and $(n)^{\text{th}}$ terms of an arithmetic series, and its last term $(a + b)$; determine the first term, common difference, and number of terms.

29. In an arithmetic series, if the $(m + n)^{\text{th}}$ term $= p$, and the $(m - n)^{\text{th}}$ term $= q$; show that the $(m)^{\text{th}}$ term $= \frac{1}{2} \cdot (p + q)$, and the $(n)^{\text{th}}$ term $= p - (p - q) \cdot \frac{m}{2n}$.

30. If S, S', S'' be the sums of three arithmetic series, 1 = the first term of each, and the respective differences be 1, 2, 3; prove that $S + S'' = 2S'$.

31. If there be (p) arithmetical progressions, each beginning from unity, whose common differences are 1, 2, 3, ..., p ; show that the sum of their $(n)^{\text{th}}$ terms is

$$= \frac{1}{2} \cdot \{ (n - 1) \cdot p^2 + (n + 1) \cdot p \}.$$

32. If a and b are respectively the first term and common difference of an arithmetic series, S_n the sum of n terms, S_{n+1} the sum of $(n + 1)$ terms, &c. prove that $S_n + S_{n+1} + S_{n+2} + \&c.$ to n terms $= (3n - 1) \cdot n \cdot \frac{a}{2} + (7n - 2) \cdot (n - 1) \cdot n \cdot \frac{b}{6}$.

33. If $S_1, S_2, S_3, \dots, S_p$ be the sums of (p) arithmetic progressions continued to n terms, and their first terms be 1, 2, 3, 4, &c. and their common differences 1, 3, 5, 7, &c.; show that $S_1 + S_2 + S_3 + \dots + S_p = \frac{1}{2} \cdot (np + 1) np$.

34. If S_m denote generally the sum of m terms of any arithmetic progression; prove that

$$S_{n^2} = S_n \cdot n \cdot \frac{n-1}{2} - S_n \cdot n \cdot (n-2).$$

35. Find three arithmetic means between 1 and 11; and seven between 1 and $-\frac{1}{2}$.

36. If between all the terms of an arithmetic progression the same number of arithmetic means be inserted; show that the new series will still form an arithmetic progression.

37. The sum of n arithmetic means between 1 and 19 is to the sum of the first $(n - 2)$ of them $:: 5 : 3$; determine the means.

38. If S be the sum of an arithmetic progression, a, b, c, d , &c. to n terms; determine the sum of the series

$$S \pm a, S \pm (a + b), S \pm (a + b + c), \&c.$$

39. Determine the sum of n terms of the triangular numbers 1, 3, 6, 10, 15, &c. the terms of which series are 1, $1 + 2$, $1 + 2 + 3$, &c. the successive sums of 1, 2, 3, &c.

40. Determine the sum of n terms of the pyramidal numbers, 1, 4, 10, 20, 35, &c. the successive sums of 1, 3, 6, 10, 15, &c.

41. Find the sum of n terms of the series $1^2, 2^2, 3^2$, &c.

42. Having given the first term and common difference of an arithmetic progression; find the sum of n terms and the sum of their squares.

43. Having given the sum of $(2n)$ quantities in arithmetic progression, and the sum of their squares; determine the quantities themselves.

44. In the series 1, 2, 3, 4.....100, determine the sum of the numbers which are not squares.

45. Prove that the sum of the series $1^2 + 3^2 + 5^2$, &c. to n terms $= \frac{n}{3} \cdot (4n^2 - 1)$.

46. The sum of the series 0, 1, 2, 3, &c. continued to an unknown number of terms, being $= 1225$; determine the sum of their squares.

47. The sum of 9 terms of the series $n^2 + (n + 1)^2 + (n + 2)^2 + \&c. = 501$; determine the value of n .

48. Determine the sum of 10 square numbers, whose roots are in an arithmetic progression, the least term of which is $= 3$, and the common difference $= 2$.

49. The square of any number of digits less than ten, each of which is unity, will when reckoned from either end, form the same arithmetic series whose common difference is unity, and greatest term the number of digits in the root. Required a proof.

50. A gentleman owed to each of two persons, *A* and *B*, an equal sum of money, which he discharged as follows: to *A* he paid £8 the first payment, £12 the second, £16 the third, and so continued increasing £4 each payment. Now *B* at his first payment received but £1, the second £4, the third £9, increasing according to the square of the number of payments. Determine what he owed each person, and the number of payments required to discharge the debt.

51. Compare the sum of the numbers 1, 2, 3, 4, &c. with the sum of their cubes.

52. Find the sum of n cube numbers, whose roots are in arithmetic progression, the least term of which is a , and common difference d .

53. Determine the arithmetic progression, the number of whose terms is 11, their sum 220, and the sum of their cubes 147400.

54. Find the sum of (n) terms of a series of polygonal numbers, which numbers are formed by assuming any arithmetic series that has its first term 1, and difference a whole number, and by making generally the $(m)^{\text{th}}$ polygonal number equal to the sum of (n) terms of the arithmetical series.

55. If the first term of an arithmetic series be (a) , the last term (l) , the common difference (d) ; and $S_1, S_2, S_3, \dots, S_{m-1}$ be the sums of the 1st, 2nd, 3rd, ... $(m-1)^{\text{th}}$ powers of the terms; prove that $(l+d)^m - a^m = mdS_{m-1} + m \cdot \frac{m-1}{2}$

$$d^2 S_{m-2} + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot d^3 S_{m-3} + \&c.$$

56. Let $a, a - r, a - 2r, \&c.$ and $b, b + r, b + 2r, \&c.$ be two arithmetical progressions, each to n terms: then if the sum of the first = 0, and the sum of the product of the terms of the first multiplied respectively by the corresponding terms of the second = $\frac{nr^2 \cdot (1 - a^2)}{12}$; determine the value of n .

57. If the quantities $a, b, c, d, \&c.$ be in arithmetic progression; prove that the terms of any order of the differences of the quantities $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \&c.$ increase or decrease according as the progression decreases or increases.

58. A number (n) of boys arrange themselves in a right line at equal intervals, the nearest being at a given distance (a) from a fixed station S . A person P walks from S to the first boy, and as soon as he begins to return, the remaining boys move from S at the same rate as P , and stop when he comes to S . P advances again to the second boy, and whilst he is returning, the remainder move onward, and halt as before. The same is repeated until P has reached the last boy and returned. Now if under the same circumstances the boys had moved each time towards S , P would have passed over only $\frac{1^{\text{th}}}{m}$ part of his former distance. Show that the distance (d) of the boys from each other is equal to

$$\frac{(2^n - m - 1) \cdot a}{n \cdot (m + 1) - (2^n + m - 1)}.$$

II. Problems in Geometric Progression.

1. If in any geometric progression four terms be taken, so that as many are wanting between the first and second, as between the third and fourth, these four terms will be in geometric progression.

2. In every geometric progression, the first term is to the second, as the sum of all the terms diminished by the last is to the sum of all diminished by the first.

3. Find the sums of the following series :

$$13 + 39 + 117 + \&c. \text{ to } 7 \text{ terms.}$$

$$8 + 20 + 50 + \&c. \text{ to } 15 \text{ terms.}$$

$$100 + 40 + 16 + \&c. \text{ to } 10 \text{ terms.}$$

$$1 + \frac{2}{3} + \frac{4}{9} + \&c. \text{ to } 10 \text{ terms.}$$

$$2 + 3 + \frac{9}{2} + \&c. \text{ to } 20 \text{ terms.}$$

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \&c. \text{ to } n \text{ terms.}$$

$$4 + 3 + \frac{9}{4} + \frac{27}{16} + \&c. \text{ to } 10 \text{ terms.}$$

$$3 + 4\frac{1}{2} + 6\frac{5}{2} + \&c. \text{ to } 5 \text{ terms.}$$

$$\frac{4}{9} + \frac{1}{3} + \frac{1}{4} + \frac{3}{16} + \&c. \text{ to } 5 \text{ terms.}$$

$$\frac{1}{3} + \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \&c. \text{ to } n \text{ terms.}$$

$$1 - 2 + 4 - 8 + \&c. \text{ to } n \text{ terms.}$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \&c. \text{ to } n \text{ terms.}$$

$$21 - 3 + \frac{3}{7} - \&c. \text{ to } 6 \text{ terms.}$$

$$\frac{1}{3} - \frac{1}{2} + \frac{3}{4} - \frac{9}{8} + \&c. \text{ to } n \text{ terms.}$$

$$\frac{2}{3} - \sqrt{\frac{2}{3}} + 1 - \&c. \text{ to } 8 \text{ terms.}$$

$$\sqrt{\frac{3}{5}} - \sqrt{6} + 2\sqrt{15} - \&c. \text{ to } 8 \text{ terms.}$$

$$3\frac{3}{8} + 2\frac{1}{4} + 1\frac{1}{2} + \&c. \text{ in } \textit{inf.}$$

$$2 - \frac{1}{3} + \frac{1}{18} - \frac{1}{108} + \&c. \text{ in } \textit{inf.}$$

$$1 + \frac{1}{3} + \frac{1}{9} + \&c. \text{ in } \textit{inf.}$$

$$1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \&c. \text{ in } \textit{inf.}$$

$$\frac{1}{3} - \frac{1}{6} + \frac{1}{12} - \frac{1}{24} + \&c. \text{ in } \textit{inf.}$$

$$\frac{1}{3} - \frac{1}{3 \cdot 2} + \frac{1}{3 \cdot 2^2} + \&c. \text{ in } \textit{inf.}$$

$$\frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{2 \cdot 2^3} - \&c. \text{ in } \textit{inf.}$$

$$\frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \&c. \text{ in } \textit{inf.}$$

$$1 + \frac{2}{3} - \frac{1}{4} + \frac{3}{32} - \&c. \text{ in } \textit{inf.}$$

$$\frac{a}{x} \sqrt{\frac{3}{2}} + \sqrt{\frac{a}{x}} + \sqrt{\frac{2x}{3a}} + \&c. \text{ to } n \text{ terms and } \textit{inf.}$$

$$r - \frac{1}{2}r^2 + \frac{1}{2}r^3 - \&c. \text{ to } n \text{ terms.}$$

$$x^2 - ax + \frac{a^2}{\sqrt{x}} - \&c. \text{ to } n \text{ terms.}$$

$$\frac{1}{\sqrt{x}} - \frac{\sqrt{2}}{x} + \frac{2}{x\sqrt{x}} - \&c. \text{ to } n \text{ terms.}$$

$$(a^2 - b^2) + (a + b) + \frac{a+b}{a-b} + \&c. \text{ to } n \text{ terms.}$$

$$\frac{a+x}{a-x} + \frac{a-x}{a+x} + \left(\frac{a-x}{a+x}\right)^2 + \&c. \text{ to } n \text{ terms.}$$

$$1 - x\sqrt{-1} - x^2 + x^3\sqrt{-1} + \&c. \text{ to } 2n \text{ terms.}$$

$$5\sqrt{x} + \frac{1}{\sqrt{x}} + 3\sqrt{x} + \frac{1}{x} + \sqrt{x} + \frac{1}{x\sqrt{x}} + \&c. \text{ to } n \text{ terms.}$$

4. Which is greater of the two series, and by what quantities,

$$\left\{ \begin{array}{l} \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \&c. \text{ in } \textit{inf.} \\ \text{or } \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \&c. \text{ in } \textit{inf.} \end{array} \right\}$$

$$\text{and also } \left\{ \begin{array}{l} 2 + 1 + \frac{1}{2} + \&c. \text{ in } \textit{inf.} \\ \text{or } \frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \&c. \text{ to } 12 \text{ terms.} \end{array} \right\}$$

5. Find an expression for the sum of n terms of the series $\frac{1}{5} - \frac{2}{15} + \frac{4}{45} - \&c.$ that may be applied according as n is an even or odd number.

6. If (a) and (b) be the two first terms of a decreasing geometric progression; prove that the sum of the series *in inf.* will be $= \frac{a^2}{a-b}$.

7. If a be the first term of a geometric series, and b the sum of the three first terms; find the common ratio.

8. The first term of a geometric series continued *in inf.* $= 1$, and any term is equal to the sum of all the succeeding terms. Determine the series.

9. The second and third terms of a geometric series are together $= 24$, and the two next $= 216$; determine the first term.

10. If four quantities be in geometric progression, the sum of the two extremes is greater than the sum of the two means.

11. If any quantities whose differences are inconsiderable in respect to the quantities themselves, be in arithmetic progression, the same quantities are also in geometric progression.

12. If there be an arithmetic and geometric series, each containing an odd number of terms, and the middle term of each be the same; the sum of any two terms of the arithmetic series equidistant from the middle term, multiplied by the middle term, is equal to twice the product of any two terms of the geometric series equidistant from the middle term.

13. The sum of a series of quantities in geometric progression wanting the first term, is equal to the sum of all the terms except the last, multiplied by the common ratio. Required proof.

14. Having given the sum (s) and the sum of the squares (S) of a geometric series continued *in inf.*; determine the series.

15. Having given the first term and common ratio of a geometric series, find the number of terms, when the sum is equal to m times the sum of the same series with its terms inverted.

16. In every geometric progression consisting of an odd number of terms, the sum of the squares of the terms is equal to the sum of all the terms multiplied by the excess of the odd terms above the even.

17. If P be the product, S the sum, and s the sum of the reciprocals of n quantities in geometric progression; prove that $P^n = \left(\frac{S}{s}\right)^n$.

18. Find four geometric means between 1 and 32; and two between 1 and 100.

19. If the arithmetic mean between (a) and (b) is double the geometric; prove that $a : b :: 2 + \sqrt{3} : 2 - \sqrt{3}$.

20. Between $n + 1$ quantities (x, y) , $(x, 2y)$, $(x, 4y)$, are inserted n geometric means, and M_1, M_2, M_3 , &c. are the n^{th} terms respectively, prove that

$$\frac{M_1}{M_2} + \frac{M_2}{M_3} + \frac{M_3}{M_4} + \dots = \sqrt[n+1]{\frac{n^{n+1}}{2^n}}.$$

21. Determine the ratio of two quantities, having given the ratio of the geometric and arithmetic means between them.

22. In a geometric series, having given (a) and (b) the $(p)^{\text{th}}$ and $(q)^{\text{th}}$ terms, determine the value of the $(x)^{\text{th}}$ term.

23. Having given the same; determine the value of the $(p + q)^{\text{th}}$ term.

24. Having given the sum of the n^{th} and $2n^{\text{th}}$ terms of a geometric series $= p$, and the sum of the $2n^{\text{th}}$ and $3n^{\text{th}}$ terms $= q$. Determine the first term and common ratio.

25. In a geometric series, if the $(p + q)^{\text{th}}$ term $= m$, and the $(p - q)^{\text{th}}$ term $= n$; prove that the $(p)^{\text{th}}$ term

$$= \sqrt{mn}, \text{ and the } (q)^{\text{th}} \text{ term} = m \left(\frac{n}{m} \right)^{\frac{p}{2q}}.$$

26. There are two geometric series S and S' continued *in inf.*, and $S : S' :: 4 : 9$, and the two first terms of S are 40 and 35; the second term of the second series being $46\frac{19}{36}$; determine the first term and the ratio of the second series.

27. If S , a , r , n , be respectively the sum, first term, common ratio, and number of terms of a geometric series; find the sum of the series

$$S \pm a, S \pm (a + ar), S \pm (a + ar + ar^2), \&c.$$

28. In a geometric progression, having given the first term and common ratio, find the sum of the series

$$S_n + S_{2n} + S_{3n} + \&c., \text{ to } r \text{ terms.}$$

29. If $S_1, S_2, S_3, \dots, S_n$ be the sums of n geometric series continued *in inf.*, the first term of which is 1, and the common ratio $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}, \dots, \frac{1}{r^n}$ respectively; determine the sum of the reciprocals $\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \dots + \frac{1}{S_n}$.

30. If there be an infinite number of decreasing geometric series, each continued *in inf.*; $a, a^2, a^3, \&c.$ the first terms; $r, 2r, 3r, \&c.$ the common ratios; $S_1, S_2, S_3, \&c.$ the sums; prove that $\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \&c. = \frac{a \cdot (1 - r) - 1}{(a - 1)^2}$.

31. Find the sum of an infinite series of quantities

$$\frac{a}{c} + \frac{a+b}{ce} + \frac{a+2b}{ce^2} + \frac{a+3b}{ce^3} + \&c.,$$

of which the numerators are in arithmetic progression, and the denominators in geometric. Also the sum of

$$\frac{1}{2} + \frac{5}{4} + \frac{9}{8} + \frac{13}{16} + \&c. \text{ in } \textit{inf}.$$

32. If the terms of an arithmetic progression, $a, a+r, a+2r, \&c.$ be multiplied by the corresponding terms of a geometric progression, $b, bd, bd^2, \&c.$ of the same number of terms; determine the sum of the resulting series.

33. Determine the sum of $7+5+\&c.$, an arithmetic series of 12 terms; also of $-\frac{7}{5}+\frac{7}{2}-\&c.$ a geometric series of 5 terms: and also of the infinite series, whose terms are the products of the corresponding terms of these series continued *in inf.*

34. There are two infinite geometric progressions, each beginning from 1, whose sums are σ and σ^1 ; prove that the sum of the series formed by multiplying their corresponding terms is $\frac{\sigma\sigma^1}{\sigma+\sigma^1-1}$.

$$35. \text{ If } S_1 = 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \&c. \text{ in } \textit{inf}.$$

$$\text{and } S_2 = 1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \&c. \text{ in } \textit{inf}.$$

prove that $S_1 : S_2 :: 27 : 1$.

36. If $S_1 = 1 + \frac{1}{r} + \frac{1}{r^2} + \&c. \text{ in } \text{inf.}$

$$S_2 = 1 - \frac{1}{r} + \frac{1}{r^2} - \&c. \dots\dots$$

$$S_3 = 1 + \frac{1}{r^3} + \frac{1}{r^4} + \frac{1}{r^5} \dots\dots$$

prove that $S_1 \times S_2 = S_3$.

37. Determine the sum of the series

$$ar + 3ar^2 + 6ar^3 + 10ar^4 + \&c.$$

which arises from multiplying the terms of a geometric progression by the corresponding terms of a series of triangular numbers.

38. Find the sums of the following series :

$1 + 2x + 3x^2 + \&c. \text{ to } n \text{ terms.}$

$1 + \frac{2}{x} + \frac{3}{x^2} + \frac{4}{x^3} + \&c. \text{ in } \text{inf.}$

$1 - \frac{2x^3}{a^2} + \frac{3x^4}{a^4} - \frac{4x^5}{a^6} + \&c. \text{ in } \text{inf.}$

$1.2.x + 2.3.x^2 + 3.4.x^3 + \&c. \text{ in } \text{inf. and to } n \text{ terms.}$

$1.2.3.x + 2.3.4.x^2 + 3.4.5.x^3 + \&c. \text{ in } \text{inf. and to } n \text{ terms.}$

$1.2.3.4.x + 2.3.4.5.x^2 + 3.4.5.6.x^3 + \&c. \text{ in } \text{inf. \& to } n \text{ terms.}$

$1.x + 3x^3 + 5x^5 + 7x^7 + \&c. \text{ in } \text{inf.}$

$1.x + 5x^3 + 9x^5 + \&c. \text{ in } \text{inf.}$

$1.2.x + 3.4.x^3 + 5.6.x^5 + \&c. \text{ in } \text{inf.}$

$1.2.3.x + 4.5.6.x^3 + 7.8.9.x^5 + \&c. \text{ in } \text{inf.}$

$1.3.x + 2.4.x^2 + 3.5.x^3 + \&c. \text{ in } \text{inf.}$

$1.4.x + 2.5.x^2 + 3.6.x^3 + \&c. \text{ in } \text{inf.}$

$1.4.x + 3.6.x^2 + 5.8.x^3 + \&c. \text{ in } \text{inf.}$

$1.3.x + 4.6.x^2 + 7.9.x^3 + \&c. \text{ in } \text{inf.}$

$1.3.x + 3.5.x^2 + 5.7.x^3 + \&c. \text{ in } \text{inf.}$

39. Prove that the sum of the reciprocals of the n^{th} powers of the odd numbers is to the sum of the reciprocals of the same powers of the even numbers $:: 1 : 2^n - 1$.

III. *Problems in Harmonic Progression.*

1. EXPLAIN the nature of harmonic progression; and continue in both directions the series, 2, 3, 6.

2. Continue the harmonic progression 3, 4, 6.... upwards and downwards. How far can it be continued either way?

3. Prove that the reciprocals of quantities in harmonic progression are in arithmetic progression.

4. In any harmonic progression, the product of the two first terms is to the product of any two adjacent terms as the difference between the two first is to the difference between the two others.

5. In any harmonic progression, the difference between the two first terms is to the difference between any two others as the second term diminished by (n) times the difference between the first and second is to the last; where n = the number of terms between the first and last.

6. In any harmonic progression, the second term diminished by (n) times the difference between the first and second is to the last as the product of the two first is to the product of the two last; n as before.

7. Any term of a harmonic progression is equal to the product of the two first terms divided by the difference between [the second] and [n times the difference between the first and second].

8. The sum of any two terms of a harmonic progression is greater than twice the intermediate mean term; and this excess is greater, as they are the more remote.

9. If the two extremes and the number of terms in a harmonic progression be known; the intervening series may be found.

10. Insert two harmonic means between 2 and 4; two between 6 and 24; four between 2 and 12; six between 1 and 20; and n between x and y .

11. Insert n harmonic means between a and b ; and if a_1 be the first harmonic mean, prove that

$$a : b :: a^{n+1} : a_1^{n+1}.$$

12. Prove that a geometric mean between two quantities is a mean proportional between an arithmetic and harmonic mean between the same two quantities.

13. If (a) be an arithmetic, (b) a geometric, and (c) a harmonic mean between two quantities; show that a is greater than b , and b greater than c .

14. If the geometric mean between two quantities x and y , be to the harmonic as $1 : n$; show that

$$x : y :: 1 + \sqrt{1 - n^2} : 1 - \sqrt{1 - n^2}.$$

15. If y be an harmonic mean between x and z , and x and z be respectively the arithmetic and geometric means between a and b , show that.

$$y = \frac{2 \cdot (a + b)}{\left\{ \left(\frac{a}{b} \right)^{\frac{1}{2}} + \left(\frac{b}{a} \right)^{\frac{1}{2}} \right\}^2}$$

16. Having given (a) the sum of three numbers in harmonic progression, and (b) their continual product; determine the numbers.

17. There are four numbers, the first three of which are in arithmetic and the last three in harmonic progression; prove that the first has to the second the same ratio which the third has to the fourth.

18. The sum of three terms of an harmonic progression, whose first term is $\frac{1}{2}$, is $= \frac{11}{12}$; determine the progression and continue it both ways.

19. If S and s be the sums of two infinite series, the common ratios of whose terms are R and r respectively; then S, s, R, r are in harmonical progression, the form of each series being $(r, r^2, r^3, \&c.)$ and r and R fractional.

20. Having given the two first terms of an harmonic progression; determine the $(n)^{\text{th}}$ term.

21. Having given the $(m)^{\text{th}}$ and $(n)^{\text{th}}$ terms of an harmonic progression; determine the $(m + n)^{\text{th}}$ term.

22. Compare the lengths of the sides of a right-angled triangle, when the squares described upon them are in harmonic progression.

APPENDIX II.

1. FORM the equation, whose roots are

$$2 + \sqrt{-3}, 2 - \sqrt{-3}, 1, \text{ and } -5.$$

2. Also whose roots are $\pm \sqrt{-2}, 3, \text{ and } 4.$

3. Also whose roots are $1 \pm \sqrt{-2}, \text{ and } 2 \pm \sqrt{-3}.$

4. Form the biquadratic equation, two of whose roots are

$$1 + \sqrt{a^2}, \text{ and } -\sqrt{-b}.$$

5. Also if two of the roots be $\sqrt{3}$, and $-\sqrt{-5}$.

6. Determine the equation whose roots are

$$\frac{1}{2}a + \sqrt{-\frac{3}{4}a^2}, \frac{1}{2}a - \sqrt{-\frac{3}{4}a^2}, \text{ and } -a.$$

7. Form the equation, of which the roots are the different values of $a + \sqrt[n]{b}$.
-

8. Given that an equation has *one* root, show that it will have as many roots as it has dimensions.

9. If any coefficient in an equation be changed, prove that all the roots will be changed.

10. If a be a root of the equation

$$x^n + px^{n-1} + \dots + Px^3 + Qx^2 + Rx + S = 0,$$

$$\text{and if } \frac{S}{a} + R = R_1, \frac{R_1}{a} + Q = Q_1, \&c.;$$

$$\text{show that } \frac{S}{a}, R_1, Q_1, \&c. \text{ are integers.}$$

11. If $\alpha, \beta, \gamma, \&c.$ be the roots of the equation

$$x^n + px^{n-1} + \dots + Qx + P = 0;$$

show that

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha}{\gamma} + \frac{\gamma}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \&c. = \frac{pQ - nP}{P}.$$

12. The roots of the equation

$x^n - px^{n-1} + qx^{n-2} \dots - Qx + R = 0$, being α, β, γ , &c.; show that

$$\begin{aligned} \frac{\alpha^2}{\beta} + \frac{\alpha^2}{\gamma} + \&c. + \frac{\beta^2}{\alpha} + \frac{\beta^2}{\gamma} + \&c. + \frac{\gamma^2}{\alpha} + \frac{\gamma^2}{\beta} + \&c. \\ &= (p^2 - 2q) \cdot \frac{Q}{R} - p. \end{aligned}$$

13. If $x^n - px^{n-1} + qx^{n-2} - \&c. \pm W = A$, and a be any root of the equation $x^n - px^{n-1} + qx^{n-2} - \&c. \pm W = 0$, prove that $x - a$ is a divisor of the expression

$$x^n - px^{n-1} + qx^{n-2} - \&c. \pm W.$$

14. Take away the second term of the following equations :

1. $x^2 - 9x^2 + 26x - 34 = 0.$

2. $x^2 - 3x^2 + 4x - 5 = 0.$

3. $x^4 + 24x^3 - 12x^2 + 4x - 30 = 0.$

4. $x^4 + 8x^3 + x^2 - x - 10 = 0.$

15. Take away the third term of the following equations :

1. $x^2 - 6x^2 + 9x - 20 = 0.$

2. $x^2 - 4x^2 + 5x - 2 = 0.$

16. Prove that the third term of the equation-

$$x^3 - px^2 + qx - r = 0,$$

cannot be taken away by the common method, if p^2 be less than $3q$. Show how it *may* be taken away in this case.

17. In an equation of n dimensions, show that the second and third terms may be taken away by the same transformation, when the square of the sum of the roots is to the sum of their squares $\therefore n : 1$.

18. Exterminate the last term but one of an equation of five dimensions by the solution of a simple equation.

19. Transform the equation $x^3 - \frac{ax^2}{m} + \frac{bx}{n} + \frac{c}{p} = 0$, into one whose coefficients shall be integral.

20. Transform the equation $y^3 - 2py^2 - 33p^2y + 14p^3 = 0$, into one whose coefficients shall be numerical.

21. Transform the equation

$$x^n - a^{\frac{1}{2}}px^{n-1} + qx^{n-2} - a^{\frac{1}{2}}rx^{n-3} + \&c. = 0,$$

into one whose coefficients are rational.

22. Transform the following equations into others whose terms shall be alternately positive and negative :

$$1. \ x^3 - x^2 + \frac{3x}{2} + 3 = 0.$$

$$2. \ x^4 + x^3 - 19x^2 + 11x + 30 = 0.$$

23. Transform the equation $x^3 + x^2 - 10x + 4 = 0$, into one whose roots shall be greater by 4 than the roots of the given equation.

24. Transform the equation $x^4 - 4x^3 + 6x^2 - 12 = 0$, into one whose roots shall be greater by 5 than the roots of the given equation.

25. Transform the equation $x^3 - 6x^2 + 9x - 12 = 0$, into one whose roots shall be less by 6 than the roots of the given equation.

26. Transform the equation $3x^3 - 12x^2 + 15x - 21 = 0$, into one whose roots shall be treble the roots of the given equation.

27. Transform the equation $x^4 - 2x^3 - 3x + 4 = 0$, into one whose roots shall be one-eighth of the roots of the given equation.

28. If a, b, c , &c. be the roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - \&c. = 0,$$

transform it into one whose roots are ma, mb, mc , &c.

29. If the roots of the equation $x^3 - px^2 + qx - r = 0$, be a, b, c ; transform it into another whose roots shall be

$$a + b, a + c, b + c.$$

30. Transform the equation $x^4 - 40x + 39 = 0$, into one whose roots shall be the sum of every two roots of the original equation.

31. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c , into another whose roots are

$$\frac{1}{a+b}, \frac{1}{a+c}, \frac{1}{b+c}.$$

32. If the roots of the equation

$$x^3 - px^2 + qx - r = 0, \text{ be } a, b, c;$$

determine the equation whose roots are ab, ac, bc .

33. Transform the equation $x^3 - px^2 + qx - r = 0$, into one whose roots shall be mean proportionals between the roots of the equation, and a given quantity (m).

34. If the roots of the equation $x^2 - px + q = 0$, be a and b ; determine the equation, of which $-\sqrt{a}$, and $-\sqrt{b}$ are roots.

35. If the roots of the equation

$$x^2 - px + q = 0, \text{ be } a \text{ and } b;$$

determine the equation whose roots are an arithmetic, a geometric, and a harmonic mean between a and b .

36. Transform the equation $x^3 - 2x^2 + 2x - 4 = 0$, into one whose roots are the squares of the roots of the original equation.

37. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c , into one whose roots are a^2, b^2, c^2 .

38. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c , into one whose roots are $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$.

39. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c , into one whose roots are

$$a^2 + b^2, a^2 + c^2, b^2 + c^2.$$

40. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c , into one whose roots are

$$\frac{1}{a^2} + \frac{1}{b^2}, \frac{1}{a^2} + \frac{1}{c^2}, \frac{1}{b^2} + \frac{1}{c^2}.$$

41. Transform the equation $x^3 - 6x^2 + 11x - 6 = 0$, into one whose roots are $\frac{1}{a^2 + b^2}, \frac{1}{a^2 + c^2}$ and $\frac{1}{b^2 + c^2}$.

42. Transform the equation $x^4 + x^3 + x^2 + x + 1 = 0$, into one whose roots shall be the squares of the roots of the given equation; and show from the roots themselves that the transformation is correct.

43. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c , into one whose roots are

$$\left(\frac{a}{b} + \frac{b}{a}\right), \left(\frac{a}{c} + \frac{c}{a}\right) \text{ and } \left(\frac{b}{c} + \frac{c}{b}\right).$$

44. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c , into one whose roots are

$$\frac{c}{a+b-c}, \frac{b}{a+c-b}, \frac{a}{b+c-a}.$$

45. Transform the equation

$$x^3 - px^2 + qx - r = 0,$$

whose roots are a, b, c , into another whose roots are

$$a + b + ab, a + c + ac, b + c + bc.$$

46. Transform an equation into one whose roots shall be the squares of the differences of the roots of the original equation; and show by means of this transformation how the number of impossible roots in an equation of five dimensions may be detected.

47. Transform the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, into one whose roots are the reciprocals of every $(n-1)$ roots of the original equation.

48. If the roots of the equation $x^3 - px^2 + qx - r = 0$, be a, b, c ; transform it into one whose roots are a^2, b^2, c^2 .

49. If $x^3 - 2x^2 + 1 = 0$; deduce the equation of which the roots are the cubes of the roots of the original equation.

50. Transform the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, into one whose m^{th} term shall be a given quantity.

51. Determine the roots of the equation

$$x^4 - 4x^3\sqrt[4]{2} + 6x^2\sqrt[4]{4} - 4x\sqrt[4]{8} + 2 = 0.$$

52. Solve the equation $x^3 - 4x^2 - 3x + 12 = 0$, one root of which is of the form $\sqrt{a}$.

53. One root of the equation $x^3 - 6x^2 + 6x + 8 = 0$, being $1 + \sqrt{3}$; find all the roots.

54. One root of the equation $x^3 - 11x^2 + 37x - 35 = 0$, being $3 + \sqrt{2}$; find all the roots.

55. One root of the equation $x^4 + x^3 - 8x^2 - 16x - 8 = 0$, being $1 - \sqrt{5}$; find all the roots.

56. Solve the following equations, two of whose roots are equal:

$$1. \quad x^3 - 7x^2 + 16x - 12 = 0.$$

$$2. \quad x^3 + 8x^2 + 20x + 16 = 0.$$

$$3. \quad x^3 - 5x^2 + 8x - 4 = 0.$$

$$4. \quad x^3 - 5x^2 - 8x + 48 = 0.$$

$$5. \quad x^3 - x^2 - 8x + 12 = 0.$$

$$6. \quad x^4 - \frac{1}{2}x + \frac{3}{16} = 0.$$

$$7. \quad x^3 + \frac{10}{7}x^2 - \frac{4000}{9261} = 0.$$

57. The equation $3x^3 - 10x^2 + 15x + 8 = 0$, has three equal roots; determine them.

58. Solve the equation $x^4 - 14x^3 + 61x^2 - 84x + 36 = 0$, whose roots are of the form a, a, b, b .

59. Solve the equation

$$x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0,$$

whose roots are of the form a, a, a, b, b .

60. The equation $x^5 - 2x^4 + 6x^3 - 8x^2 + 12x - 8 = 0$, has equal roots; determine them.

61. Solve the equation $x^4 + px^3 + qx^2 + rx + s = 0$, which has two pairs of equal roots.

62. Solve the following equations, which have two roots of the form $+a, -a$,

$$1. \quad 4x^3 - 32x^2 - x + 8 = 0.$$

$$2. \quad x^4 - 5x^3 - 5x^2 + 45x - 36 = 0.$$

$$3. \quad x^4 + 3x^3 - 7x^2 - 27x - 18 = 0.$$

$$4. \quad x^4 + x^3 - 11x^2 + 9x + 18 = 0.$$

63. Solve the equation

$$x^5 - 10x^4 + 29x^3 - 10x^2 - 62x + 60 = 0,$$

two of its roots being 3 and $\sqrt{2}$.

64. The equation $x^3 - 15x^2 + 66x - 80 = 0$, has two roots whose sum is 13; find all the roots.

65. The equation $x^4 - 45x^3 - 40x + 84 = 0$, has two roots whose difference is 3; determine all the roots.

66. The roots of the equation $x^3 - 16x^2 + 66x - 80 = 0$, have a common difference; determine them.

67. In the equation $x^3 - 6x^2 + 11x - 6 = 0$, one root is double another; determine all the roots.

68. The product of two roots of the equation

$$x^4 + x^3 - 62x^2 - 80x + 1200 = 0, \text{ is } 30;$$

determine all the roots.

69. Determine the roots of the equation

$$x^3 - 17x^2 + 94x - 168 = 0,$$

two of them being in the proportion of 2 : 3.

70. In the equation $x^3 - 10x^2 + 27x - 18 = 0$, the greatest root is double of the second, and the second treble of the third; determine all the roots.

71. One root of the equation $x^4 - 5x^3 - x + 5 = 0$ is 5; determine all the roots.

72. The equation $x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1 = 0$, has two roots of the form $a, \frac{1}{a}$; determine all the roots.

73. The roots of the equation

$$6x^4 - 43x^3 + 107x^2 - 108x + 36 = 0,$$
are of the form $a, b, \frac{b}{a}$ and $\frac{a}{b}$; determine them.

74. Determine the roots of the equation

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0,$$
they being of the form $a + 1, a - 1, b + 1, b - 1$.

75. Solve the following equations, whose roots are in arithmetical progression :

1. $x^3 - 6x^2 - 4x + 24 = 0.$

2. $x^3 - 9x^2 + 23x - 16 = 0.$

3. $x^3 - 6x^2 + 11x - 6 = 0.$

4. $x^3 - 3x^2 + 6x + 8 = 0.$

5. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$

6. $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0.$

7. $x^4 + x^3 - 11x^2 + 9x + 18 = 0.$

76. The roots of the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$ being in arithmetical progression; prove that the least root is

$$\frac{p}{n} - \frac{n-1}{n} \sqrt{\left\{ \frac{(n-1) \cdot 3p^2 - 6nq}{n^2 - 1} \right\}},$$

and the common difference

$$\frac{2}{n} \cdot \sqrt{\left\{ \frac{(n-1) \cdot 3p^2 - 6nq}{n^2 - 1} \right\}}.$$

77. Solve the following equations, whose roots are in geometrical progression :

1. $x^3 - 7x^2 + 14x - 8 = 0.$
2. $x^3 - 13x^2 + 39x - 27 = 0.$
3. $x^3 - 14x^2 + 56x - 64 = 0.$
4. $x^3 - 26x^2 + 156x - 216 = 0.$
5. $x^3 - px^2 + qx - r = 0.$
6. $x^4 + px^3 + qx^2 + rx + s = 0.$

78. If the roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - \&c. = 0,$$

be in geometrical progression ; having given $p = 15$, $q = 70$; find n , r , &c.

79. Solve the following equations, whose roots are in harmonic progression :

1. $x^3 - 11x^2 + 36x - 36 = 0.$
2. $x^3 - 13x^2 + 54x - 72 = 0.$
3. $x^3 - \frac{5}{2}x^2 + x - \frac{1}{6} = 0.$
4. $8x^3 - 6x^2 - 3x + 1 = 0.$
5. $x^3 - \frac{13}{12}x^2 + \frac{9}{24}x - \frac{1}{24} = 0.$
6. $ax^3 - bx^2 - cx + 1 = 0.$

80. In the common cubic $x^3 - px^2 + qx - r = 0$, if the roots are in harmonic progression, and p , q , r , integer numbers, then r is the square of the greatest root. Apply this to solve the equation $x^3 - 23x^2 + 135x - 225 = 0$.

81. If the roots of the equation $x^3 - px^2 + qx - r = 0$, be in harmonic progression ; show that

$$x^3 - \frac{pq - 3r}{q} \cdot x + \frac{p^3 - 3pqr + 9r^2}{q^3} = 0,$$

contains the greatest and least.

82. If the roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - \dots + Qx^2 - Px + L = 0,$$

be in harmonic progression, then will the greatest and least be

$$\frac{n\sqrt{(n+1)} \cdot L}{\sqrt{(n+1)} \cdot P - \sqrt{\{3 \cdot (n-1)^2 \cdot P^2 - 6n \cdot (n-1) \cdot QL\}}},$$

and $\frac{n \cdot \sqrt{(n+1)} \cdot L}{\sqrt{(n+1)} \cdot P + \sqrt{\{3 \cdot (n-1)^2 \cdot P^2 - 6n \cdot (n-1) \cdot QL\}}}.$

83. Solve the equation $x^3 - 31x^2 + 300x - 900 = 0$, whose roots are successive triangular numbers.

84. Explain the method of finding the equal roots of equations, and apply it to the equations

1. $x^4 - 9x^3 + 4x + 12 = 0.$

2. $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0.$

85. Having given the equation

$$2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0,$$

determine whether it has equal roots.

86. If the equation $x^4 + qx^3 - rx^2 - t = 0$ has two roots equal to each other; prove that one of them will be a root of

the quadratic $x^2 + \frac{2q^2}{5r} \cdot x + \frac{5t}{3r} - \frac{4q}{15} = 0.$

87. Show that if an equation have two equal roots, and the terms be multiplied by the terms of an arithmetic progression, the result will $= 0$.

88. If an equation have (n) equal roots, the equation formed by multiplying the terms by the terms of an arithmetic progression has $(n-1)$ of them.

89. Having given

$$\left. \begin{aligned} x^3 - px^2 + qx - r &= 0, \text{ whose roots are } a, b, c; \\ \text{and } x^3 - p^1x^2 + q^1x - r^1 &= 0, \text{ whose roots are } a, b, c^1; \end{aligned} \right\}$$

find c , and c^1 .

90. One root being common to the two equations

$$x^3 - 9x^2 + 26x - 24 = 0,$$

$$\text{and } x^3 - 7x^2 + 7x + 15 = 0,$$

find the remaining roots of each.

91. Determine all the roots of the two equations which have one common root

$$x^3 - 3x^2 + 11x - 9 = 0,$$

$$x^3 - 5x^2 + 11x - 7 = 0.$$

92. Solve the equation $x^3 - 1 = 0$; and show that its roots are of the form a, b, b^{-1} .

93. The roots of the equation $x^4 + px^2 + 1 = 0$, must be of the form $a, b, \frac{1}{a}, \frac{1}{b}$; exhibit them in that form.

94. Solve the following recurring equations:

1. $x^4 - 3x^3 + 2x^2 - 3x + 1 = 0.$

2. $x^4 + 6ax^3 - 20a^2x^2 - 6a^3x + a^4 = 0.$

3. $x^4 - \frac{5}{2}x^3 + 2x^2 - \frac{5}{2}x + 1 = 0.$

4. $x^4 \pm 1 = 0.$

5. $x^5 - 21x^4 + 37x^3 - 37x^2 + 21x - 1 = 0.$

6. $x^5 - \frac{15}{2}x^4 + \frac{37}{2}x^3 - \frac{37}{2}x^2 + \frac{15}{2}x - 1 = 0.$

7. $x^5 + 4ax^4 - 12a^2x^3 - 12a^3x^2 + 4a^4x + a^5 = 0.$

8. $x^{12} - a^{12} = 0.$

95. Reduce

$$x^8 - px^7 + qx^6 - rx^5 + sx^4 - sx^3 + rx^2 - qx + px - 1 = 0,$$

to an equation of four dimensions.

96. Exhibit the quadratic factors of the equation

$$x^m \pm 1 = 0.$$

97. Show that when m is a prime number, the roots both real and imaginary of the equation $x^m + 1 = 0$, are different powers of any one of its roots.

98. In any recurring equation $x^n - px^{n-1} + \&c. = 0$, whose roots are $a, b, c, \&c.$ prove that

$$\frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{a^2}{c^2} + \frac{c^2}{a^2} + \&c. = (p^2 - 2q + \sqrt{n}) \cdot (p^2 - 2q - \sqrt{n}).$$

99. The roots of the cubic equation $x^3 - qx + r = 0$, are real when $\frac{q^3}{27}$ exceeds $\frac{r^2}{4}$.

100. The solution of the cubic equation $x^3 + qx + r = 0$ is dependent on the solution of the equation $x^3 - 1 = 0$.

101. Having given, 1, α, β , the cube roots of 1,

$$\text{and } A = \left\{ -\frac{r}{2} + \left(\frac{r^2}{4} - \frac{q^3}{27} \right)^{\frac{1}{2}} \right\}^{\frac{1}{3}},$$

$$\text{and } B = \left\{ -\frac{r}{2} - \left(\frac{r^2}{4} - \frac{q^3}{27} \right)^{\frac{1}{2}} \right\}^{\frac{1}{3}},$$

prove synthetically that $A + B$, $\alpha A + \beta B$, and $\beta A + \alpha B$ are the three roots of the equation.

102. If two roots of the cubic equation

$$x^3 - qx + r = 0 \text{ be } a + b\sqrt{-3}, \text{ and } a - b\sqrt{-3};$$

$$\text{then will } -\frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)} = (b - a)^2.$$

103. Explain in what case, and for what reason, Cardan's formula for the solution of a cubic equation does not enable us to determine the roots.

104. Determine whether Cardan's rule is applicable to the solution of the equation $x^3 - 237x - 884 = 0$.

105. Show that Cardan's rule for the solution of a cubic equation is applicable when all the roots are possible, and two of them equal; and by means of it, find the roots of the equation $x^3 + 6x^2 - 32 = 0$.

106. Solve the following equations by Cardan's rule:

1. $x^3 - 9x - 14 = 0$.

2. $x^3 - 6x - 40 = 0$.

3. $x^3 - 9x + 28 = 0$.

4. $x^3 + 3x^2 + 9x - 13 = 0$.

5. $x^3 - 6x^2 + 3x - 18 = 0$.

6. $x^3 - 12x^2 + 57x - 94 = 0$.

7. $x^3 + 6x^2 + 20x + 15 = 0$.

8. $x^3 - 12x^2 + 36x - 7 = 0$.

9. $x^{3m} - px^{2m} + qx^m - r = 0$.

107. Find by the doctrine of permutations, the roots of the equation $x^3 - qx + r = 0$.

108. Assuming the quadratic factors in Des Cartes's solution of a biquadratic to be $x^2 + ax + b = 0$, and $x^2 - ax + c = 0$; find the reducing equation in (b) or (c) : and show that it may be depressed to a cubic.

109. If a be a root of Des Cartes's reducing cubic, the four roots of the equation $x^4 + qx^3 + rx + s = 0$,

$$\text{are } -\frac{\sqrt{a}}{2} \pm \sqrt{\left(-\frac{q}{2} - \frac{a}{4} + \frac{r}{2\sqrt{a}}\right)},$$

$$\text{and } +\frac{\sqrt{a}}{2} \pm \sqrt{\left(-\frac{q}{2} - \frac{a}{4} - \frac{r}{2\sqrt{a}}\right)}.$$

110. Prove that Des Cartes's solution of a biquadratic succeeds when all the roots are possible and two of them equal; and apply it to solve the equation

$$x^4 - 6x^3 + 8x^2 + 6x - 9 = 0.$$

111. Find the roots of the following equations by Des Cartes's method:

$$1. \quad x^4 - 4x^3 - 8x + 32 = 0.$$

$$2. \quad x^4 - 3x^3 - 4x - 3 = 0.$$

$$3. \quad x^4 - 6x^3 + 5x^2 + 2x - 10 = 0.$$

$$4. \quad x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$$

112. Give Euler's solution of a biquadratic; and show that the cubic involved in that method has its roots four times less than the roots of the cubic in Des Cartes's.

113. Solve the equation $x^4 = 12x + 5$, by the method attributed to Waring. One root of the reducing cubic is 2.

114. If $a + \beta\sqrt{-1}$ be a root of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

two roots are

$$\frac{1}{2} \left\{ -(p + 2a) \pm \sqrt{\left[p^2 - 2q - 2\left(a^2 - \beta^2 + \frac{s}{a^2 + \beta^2}\right)\right]} \right\}.$$

115. Prove that if (u) the last term of any equation be resolved into prime factors a, β, γ , so that $u = a^m \beta^n \gamma^p$, then the number of divisors of u will $= (m + 1) \cdot (n + 1) \cdot (p + 1)$.

116. Find by the method of divisors the roots of the following equations:

$$1. \quad x^3 - 6x^2 + 5x + 12 = 0.$$

$$2. \quad x^3 - 9x^2 + 22x - 24 = 0.$$

$$3. \quad x^4 - 6x^3 - 16x + 21 = 0.$$

$$4. \quad x^4 - 4x^3 - 8x + 32 = 0.$$

$$5. \quad x^4 + x^3 - 29x^2 - 9x + 180 = 0.$$

$$6. \quad 3x^3 - 26x^2 + 34x - 12 = 0.$$

$$7. \quad 8x^3 - 26x^2 + 11x + 10 = 0.$$

$$8. \quad 8x^3 - 45x^2 + 73x - 30 = 0.$$

117. In the method of divisors, show how the number of substitutions may be lessened:—and in the equation

$$x^4 - x^3 - 16x^2 + 55x - 75 = 0,$$

determine whether 3, 5, and -5 are roots.

118. Apply the rule for quadratic divisors to the equation

$$x^4 - 17x^3 + 88x^2 - 172x + 112 = 0.$$

119. Solve by the method of divisors, the equation

$$6x^4 + 53x^3 - 95x^2 - 25x + 42 = 0.$$

120. It is always possible to find those roots of numeral equations which are whole numbers or rational fractions without the aid of formulæ of approximation.

121. If a be an approximate root of the equation

$$x^3 + px^2 + qx = r, \text{ so that } a^3 + pa^2 + qa = r_1,$$

$$\text{prove that } x = a + \frac{a \cdot (r - r_1)}{r \text{ or } r_1 + 2a^2 + pa^2} \text{ very nearly ;}$$

r or r_1 being used according as a is greater or less than 1. Approximate by this formula to the value of x in the equation

$$x^3 - 2x = 5.$$

122. Three given quantities $(a + z)$, $(a + z) + h$, $(a + z) + h'$, approximations to the root a of an equation, being substituted for the unknown quantity, give results n , $n + \delta$, $n + \delta'$; show that z will be very nearly found from the equation

$$z^2 \cdot (h\delta' - h'\delta) + z \cdot (h^2\delta' - h'^2\delta) + nh h' \cdot (h' - h) = 0.$$

123. Approximate to a root of the following equations :

$$1. \quad x^3 - x - 50 = 0.$$

$$2. \quad x^3 - 2x - 5 = 0.$$

$$3. \quad x^3 + 2x - 30 = 0.$$

$$4. \quad x^3 + x^2 + x = 90.$$

$$5. \quad x^3 - 6x + 1 = 0.$$

$$6. \quad x^3 - 2x^2 + 3x - 4 = 0.$$

$$7. \quad x^4 + x = 3.$$

$$8. \quad x^4 - 12x + 7 = 0.$$

$$9. \quad 2x^4 - 16x^3 + 40x^2 - 30x + 1 = 0.$$

$$10. \quad \begin{cases} x^3 + xy = 5, \\ 2xy - y^2 = 2. \end{cases}$$

$$11. \quad \begin{cases} x^3 + y = 157, \\ y^3 - x = 6. \end{cases}$$

$$12. \quad \begin{cases} x^3 + y^3 = 12, \\ x^3 + y^3 = 8. \end{cases}$$

124. In the equation $x^3 + 9x^2 + 4x = 80$, approximate to the value of x by means of a series of converging fractions.

125. Express the roots of $x^3 - 7x + 7 = 0$, by continued fractions; and determine the accuracy of the approximation of any converging fraction deduced from these.

126. If a be an approximate value of x in any equation, and b , c be the results when a is substituted for x in the original and in the limiting equation; then will

$$x = a - \frac{b}{c}, \text{ nearly.}$$

127. Determine the number of positive and negative roots in the equation $x^5 + 4x^4 - 19x^3 - 34x^2 + 60x + 36 = 0$, of which all the roots are real.

128. Determine the same in the equation

$$x^5 - 5x^4 - 15x^3 + 85x^2 - 26x - 120 = 0.$$

129. Prove without resolving the equation into factors, that if two numbers (a) and (b) when substituted for the unknown quantity in the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, give results affected with contrary signs, there is at least one real root between (a) and (b).

130. If P represent the sum of the positive terms in any equation, and a series of quantities be successively substituted for the unknown quantity; prove that the successive increments of P may be made less than any assignable quantity.

131. If P and N be the greatest positive and negative coefficients in the equation

$$x^n - px^{n-1} \dots + Px^{n-m} \dots - Nx^{n-r} \dots \pm u = 0,$$

then a superior limit to the roots is $N + 1$; and an inferior

limit is $\frac{u}{u + N}$ or $\frac{u}{u + P}$ according as u is positive or negative.

132. If Mx^{n-1} be the first negative term of the equation

$$x^n + px^{n-1} \dots - Mx^{n-1} - \&c. = 0,$$

and if P be the greatest negative coefficient, then $1 + \sqrt[n]{P}$ is greater than the greatest root of the equation.

133. If $x^n + Ax^{n-1} \dots - Px^p \dots - Sx^s \dots + Tx + V = 0$, where P is the greatest and S the last negative coefficient, then

$$\frac{\sqrt[n]{V}}{\sqrt[n]{V} + \sqrt[n]{P}}$$

is an inferior limit of the positive roots.

134. If the terms of an equation be so arranged as to be alternately positive and negative; and if A, A', A'' be the coefficients of the 1st, 3rd, 5th B, B', B'' of the 2nd, 4th, 6th terms so arranged; show that the greatest of the ratios $\frac{B}{A}, \frac{B'}{A'}, \frac{B''}{A''}$ is greater than the greatest positive root. Apply this to determine a quantity greater than the greatest root of the equation $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0$.

135. If in any equation each negative coefficient be divided by the sum of the positive ones which precede it, and the greatest of these fractions be taken, then this fraction so taken increased by unity is greater than the greatest root of the equation.

136. Let the roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - \&c. = 0, \text{ be } a, b, c, \&c.;$$

and those of the equation

$$nx^{n-1} - (n-1) \cdot px^{n-2} + (n-2) \cdot qx^{n-3} - \&c. = 0, \text{ be } \alpha, \beta, \gamma, \&c.;$$

then if when $\alpha, \beta, \gamma, \&c.$ are successively substituted in the equation $x^n - px^{n-1} + \&c. = 0$, the results are $P, Q, R, \&c.$, and when a, b, c are substituted in the equation

$$nx^{n-1} - (n-1) \cdot px^{n-2} + \&c. = 0, \text{ the results are } p, q, r, \&c.;$$

then will $P \times Q \times R \times \&c. : p \times q \times r \times \&c. :: 1 : n$.

137. Find a limit greater than the greatest root of the following equations:

$$1. \quad x^3 - 6x^2 - 25x - 12 = 0.$$

$$2. \quad x^3 - 12x^2 + 41x - 43 = 0.$$

$$3. \quad x^4 - 5x^3 + 6x^2 - 7x + 8 = 0.$$

138. Find superior and inferior limits to the positive roots of the equation $x^5 + 2x^4 - 50x^3 - 100x^2 + 49x + 98 = 0$.

139. Determine a limit less than the least positive root of the equation $x^4 - 3x^3 - 5x^2 + 2x + 3 = 0$.

140. Find a limit less than the least root of the following equations:

$$1. \quad x^3 + 12x - 20 = 0.$$

$$2. \quad x^3 + 8x^2 - 8x - 64 = 0.$$

$$3. \quad x^4 - 5x^2 - 3 = 0.$$

141. Find a number greater than the greatest positive root, and also one less than the least negative root of the equation $x^3 - 4x^2 - x + 20 = 0$.

142. Find a limit less than the least difference of the roots of the equation $x^3 - 7x + 7 = 0$.

143. Find between which of the roots of the equation $x^3 - 7x^2 + 7x + 10 = 0$, the number 3 lies.

144. Prove that one of the roots of the equation

$$x^3 - qx - r = 0,$$

when squared, will lie between (q) and $\left(\frac{q}{3}\right)$.

145. Investigate by means of the limiting equation, whether $x^3 - 5x^2 + 3 = 0$, has a possible root.

146. Determine the rational roots of the equation

$$2x^4 + x^3 - 10x^2 - 2x + 12 = 0.$$

147. Determine the number of possible roots in the equation $x^7 - 14x^4 + 90 = 0$.

148. Determine the number of imaginary roots in the equation $x^5 - 6x^4 + 5x^3 - 30x^2 + 6x - 36 = 0$.

149. Apply Newton's rule for detecting impossible roots, to the equation $x^5 + 3x^4 - 4x^3 - 12 = 0$.

150. The number of impossible roots of any equation

$$x^n - px^{n-1} + \&c. = 0,$$

is not increased by multiplying its terms by the successive terms of the series 0, 1, 2, 3, 4, &c.

151. If several consecutive terms of an equation, whose roots are real, be wanting, and if the next terms on each side of those wanting have the same sign, prove that the equation cannot have as many roots as it has dimensions.

152. Let $\alpha, \beta, \gamma, \&c.$ be the roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - \&c. = 0, (m) \text{ of which are possible ;}$$

show that if the equation be transformed into one, whose roots are $(\alpha - \beta)^2, (\alpha - \gamma)^2, (\beta - \gamma)^2, \&c.$, the last term of the transformed equation will be positive or negative according as

$m \cdot \frac{m-1}{2}$ is an even or an odd number.

THE END.